The p-adic section conjecture for localisations of curves

DISSERTATION

zur Erlangung des Doktorgrades

der Naturwissenschaften

vorgelegt beim Fachbereich für Informatik und Mathematik

der Johann Wolfgang Goethe-Universität

in Frankfurt am Main

von

MARTIN LÜDTKE

aus BIELEFELD

Frankfurt (2020)

(D 30)

vom Fachbereich für Informatik und Mathematik der

Johann Wolfgang Goethe-Universität als Dissertation angenommen

Dekan:	Prof. Dr.-Ing. Lars Hedrich
Gutachter:	Prof. Dr. Jakob Stix Prof. Dr. Annette Werner
Datum der Disputation:	14. Dezember 2020

Contents

1 Introduction

This work is concerned with intermediate versions between the birational section conjecture and the full section conjecture for curves over p-adic fields. We are going to explain those and more general versions of the section conjecture in this introduction. Each of them predicts that the rational points of a curve can be described purely group-theoretically in terms of the étale fundamental group. The conjecture is part of a wider range of ideas, called *anabelian geometry*, in which one tries to recover arithmetic and geometric information from the associated fundamental groups. This area goes back to a letter of Grothendieck to Faltings [Gro97] and is still largely conjectural today.

1.1 The section conjecture

We want to motivate briefly the basic ideas.

1.1.1 Rational points as sections

Let k be a field. Suppose that we are searching elements $a_1, \ldots, a_n \in k$ which satisfy a list of polynomial equations with coefficients in k:

$$F_j(a_1, \ldots, a_n) = 0 \quad \text{for } j = 1, \ldots, m \tag{$*$}$$

with polynomials $F_1, \ldots, F_m \in k[T_1, \ldots, T_n]$. A tuple $(a_1, \ldots, a_n) \in k^n$ is equivalent, via the rule $T_i \mapsto a_i$, to a ring homomorphism $k[T_1, \ldots, T_n] \to k$ which is left inverse to the inclusion. The tuple satisfies the equations $(*)$ if and only if the corresponding homomorphism factors through the quotient

$$A := k[T_1, \ldots, T_n]/(F_1, \ldots, F_m).$$

In other words, a k-rational solution to the equations $(*)$ is equivalent to a retraction as follows:

$$\begin{tikzcd} A \arrow[d, dashed, bend right] \\ k \arrow[u] \end{tikzcd}$$

This arithmetic problem can be expressed more geometrically in the language of schemes. Elements of the field k are viewed as functions on a space $\mathrm{Spec}(k)$.

Similarly, elements of A are viewed as functions on $X := \mathrm{Spec}(A)$. The map $k \to A$ defines a morphism $X \to \mathrm{Spec}(k)$ in the opposite direction. A solution $(a_1, \ldots, a_n) \in k^n$ of the equations $(*)$ is equivalent to a retraction as above and thus translates under the Spec functor into a section:

$$\begin{tikzcd} X \arrow[d, shift right] \\ \mathrm{Spec}(k) \arrow[u, dashed, bend right, shift right] \end{tikzcd}$$

1.1.2 Sections on fundamental groups

Consider a continuous map of topological spaces $f\colon X \to B$. We view this as a family of spaces $X_b := f^{-1}(b)$ which is parametrised over the base B. Suppose that we want to choose for every $b \in B$ a point in the fibre $x_b \in X_b$ which varies continuously with b in the sense that the resulting map $B \to X$ given by $b \mapsto x_b$ is continuous. In other words, we are looking for a continuous section:

$$\begin{tikzcd} X \arrow[d, shift right] \\ B \arrow[u, dashed, bend right, shift right] \end{tikzcd}$$

An important invariant of a topological space is its fundamental group. It can be used to analyse the given situation. To this end, one has to choose compatible base points $b_0 \in B$ and $x_0 \in f^{-1}(b_0)$ and obtains a group homomorphism induced by f:

$$f_*\colon \pi_1(B, b_0) \to \pi_1(X, x_0).$$

Now let $x\colon B \to X$ be a given section of f. For any loop in the base B starting and ending at b_0 we get via x a loop in X starting and ending at $x(b_0)$. This defines a homomorphism $x_*\colon \pi_1(B, b_0) \to \pi_1(X, x(b_0))$. As the base points $x(b_0)$ and x_0 do not in general coincide, x_* is not yet a section of f_*. However, if we assume that the fibre $f^{-1}(b_0)$ is path-connected, then we can choose a path $\gamma\colon x(b_0) \rightsquigarrow x_0$ in $f^{-1}(b_0)$ and obtain a homomorphism

$$s_x\colon \pi_1(B, b_0) \xrightarrow{x_*} \pi_1(X, x(b_0)) \xrightarrow{\gamma(-)\gamma^{-1}} \pi_1(X, x_0).$$

This is now indeed a section of f_* since the path γ is mapped to the constant path at b_0 under f. Any other choice of path γ' differs from γ by a loop $\delta := \gamma' \circ \gamma^{-1}$ in $f^{-1}(b_0)$ which starts and ends at x_0. This changes the section s_x by a conjugation with $i_*([\delta])$, where $i_*\colon \pi_1(f^{-1}(b_0), x_0) \to \pi_1(X, x_0)$ is induced by the inclusion of the fibre. The section $x\colon B \to X$ yields therefore a well-defined $\pi_1(f^{-1}(b_0), x_0)$-conjugacy class of sections $[s_x]$. The association $x \mapsto [s_x]$ is thus a map

$$(\text{ sections } x\colon B \to X \text{ of } f\ ) \longrightarrow \begin{pmatrix} \text{conjugacy classes} \\ \text{of sections of } f_* \end{pmatrix}.$$

In particular, one can prove the non-existence of sections $x\colon B \to X$ of f by showing the non-existence of sections $\pi_1(B, b_0) \to \pi_1(X, x_0)$ of f_*. This latter point is also true if the space X (rather than the fibre $f^{-1}(b_0)$) is path-connected and $\pi_1(B, b_0)$ is abelian.

To illustrate this with an example, consider the squaring map $\mathbb{C}^\times \to \mathbb{C}^\times$, $z \mapsto z^2$. On fundamental groups it induces the multiplication by 2 on $\mathbb{Z}$. This does not admit a section, so one can conclude that it is impossible to choose for every nonzero complex number $z \in \mathbb{C}^\times$ a square root in a continuous way.

1.1.3 The étale fundamental group

We explained above that solutions to polynomial equations over a field k can be viewed geometrically as sections of a map of schemes $X \to \operatorname{Spec}(k)$. One is therefore in a similar situation as in the previous paragraph: X can be considered as a parametrised family of spaces over the base $\operatorname{Spec}(k)$. On first sight, this point of view might seem fruitless since the underlying topological space of $\operatorname{Spec}(k)$ consists only of a single point, corresponding to the zero ideal in k. However, the underlying Zariski space of a scheme is generally not suited to define the fundamental group via homotopy classes of loops. Following [SGA 1, Exp. V], one takes a different approach and defines the étale fundamental group of a scheme by transferring the role that the topological fundamental group plays in the theory of coverings into the world of schemes.

To explain this, consider a connected scheme X. As base point we choose a geometric point x_0 of X, i.e. a morphism $\operatorname{Spec}(\Omega) \to X$ with a separably closed field Ω. The role that (finite) covering maps play in topology is taken by the finite étale morphisms $f\colon Y \to X$. The fibre $f^{-1}(x_0) \to x_0$ is a finite disjoint union of copies of $\operatorname{Spec}(\Omega)$ and can thus be viewed simply as a finite set. In topology, one could define (under mild assumptions on the topological space X) a group action by $\pi_1(X, x_0)$ on the fibre $f^{-1}(x_0)$ via path-lifting: given a loop γ in X starting and ending at x_0, and given a point $y \in f^{-1}(x_0)$, one can lift γ uniquely to a path $\tilde{\gamma}$ in Y starting at y, and one defines $\gamma.y \in f^{-1}(x_0)$ as the endpoint of $\tilde{\gamma}$. In the algebraic world, an element of $\pi_1(X, x_0)$ is simply *defined* as a system of permutations of $f^{-1}(x_0)$ for every finite étale cover $f\colon Y \to X$. The system is only required to satisfy a naturality condition, i.e. a compatibility with morphisms between finite étale coverings: for any two finite étale coverings $f_i\colon Y_i \to X$ and a morphism $g\colon Y_1 \to Y_2$ over X between them, the following square has to commute for all $\gamma \in \pi_1(X, x_0)$:

$$\begin{tikzcd} f_1^{-1}(x_0) \arrow[r, "g"] \arrow[d, "\gamma"] & f_2^{-1}(x_0) \arrow[d, "\gamma"] \\ f_1^{-1}(x_0) \arrow[r, "g"] & f_2^{-1}(x_0). \end{tikzcd}$$

This is expressed more economically as follows: we have a category $\mathrm{Cov}(X)$ of finite étale coverings of X, we have a fibre functor $\mathrm{Fib}_{x_0}\colon \mathrm{Cov}(X) \to \mathrm{FinSet}$ to the category of finite sets given by $(f\colon Y \to X) \mapsto f^{-1}(x_0)$, and the étale fundamental group $\pi_1(X, x_0)$ is defined as the automorphism group of the fibre functor:

$$\pi_1(X, x_0) := \mathrm{Aut}(\mathrm{Fib}_{x_0}).$$

One endows $\pi_1(X, x_0)$ with the coarsest topology rendering the actions on all fibres $f^{-1}(x_0)$ continuous. In this way, $\pi_1(X, x_0)$ becomes a profinite group.

Let us consider the special case that $X = \mathrm{Spec}(k)$ is the spectrum of a field. The choice of a separable closure $\overline{k}/k$ defines a geometric base point $x_0\colon \mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(k)$. The connected finite étale coverings of $\mathrm{Spec}(k)$ are of the form $\mathrm{Spec}(\ell) \to \mathrm{Spec}(k)$ with ℓ/k a finite separable field extension. The fibre over x_0 can be identified with the set of k-embeddings $\mathrm{Hom}_k(\ell, \overline{k})$. The absolute Galois group $\mathrm{Gal}(\overline{k}/k)$ acts naturally on $\mathrm{Hom}_k(\ell, \overline{k})$, and it is easy to see that every compatible system of permutations of the sets $\mathrm{Hom}_k(\ell, \overline{k})$ is defined by an element of $\mathrm{Gal}(\overline{k}/k)$. It follows that the étale fundamental group of $\mathrm{Spec}(k)$ is precisely the absolute Galois group:

$$\pi_1(\mathrm{Spec}(k), \mathrm{Spec}(\overline{k})) = \mathrm{Gal}(\overline{k}/k).$$

In particular, the étale fundamental group of $\mathrm{Spec}(k)$ contains a lot more information than the underlying Zariski space suggests. This is an expression of the fact that the étale topology of a scheme (which is not a topology in the classical sense but a Grothendieck topology) is a lot finer than the Zariski topology.

1.1.4 The Section Conjecture for proper curves

Let X/k be a smooth, proper, geometrically connected curve over a field k of characteristic zero. For example, X might be defined as the vanishing set of a system of homogeneous polynomials in a projective space. Choosing an algebraic closure $\overline{k}$ of k and a geometric base point $\overline{x}_0$ on $X_{\overline{k}} = X \otimes_k \overline{k}$, we have the following *fundamental exact sequence* of étale fundamental groups

$$1 \longrightarrow \pi_1(X_{\overline{k}}, \overline{x}_0) \longrightarrow \pi_1(X, \overline{x}_0) \longrightarrow \mathrm{Gal}(\overline{k}/k) \longrightarrow 1. \tag{1.1.1}$$

A k-rational point $x \in X(k)$, being by definition a section of the structural morphism $X \to \mathrm{Spec}(k)$, induces a section s_x of $\pi_1(X, \overline{x}_0) \to \mathrm{Gal}(\overline{k}/k)$ by functoriality:

$$\begin{tikzcd} X \arrow[d] & & \pi_1(X,\overline{x}_0) \arrow[d, two heads] \\ \mathrm{Spec}(k) \arrow[u, dashed, bend right, "x"'] & \rightsquigarrow & \mathrm{Gal}(\overline{k}/k) \arrow[u, dashed, bend right, "s_x"'] \end{tikzcd}$$

As explained in the topological situation above, in order to account for differing base points, the construction of s_x depends on the choice of an étale path $\overline{x} \rightsquigarrow \overline{x}_0$ on the connected scheme $X_{\overline{k}}$, where $\overline{x}\colon \mathrm{Spec}(\overline{k}) \to X_{\overline{k}}$ is the canonical lift of x. The difference of any two such paths is a loop, i.e. an element of $\pi_1(X_{\overline{k}}, \overline{x}_0)$. As a consequence, the section s_x is well-defined up to $\pi_1(X_{\overline{k}}, \overline{x}_0)$-conjugacy. Over base fields k which are finitely generated over $\mathbb{Q}$, Grothendieck's *Section Conjecture* [Gro97] predicts that the map

$$X(k) \longrightarrow \begin{pmatrix} \text{conjugacy classes of} \\ \text{sections } s \text{ of (1.1.1)} \end{pmatrix} \tag{1.1.2}$$

given by $x \mapsto [s_x]$, is a bijection, provided that the genus of X is at least 2 (i.e. X is *hyperbolic*). The Section Conjecture as well as its analogue over p-adic fields (the *p-adic Section Conjecture*) are still open (see [Sti13] for partial results and evidence).

1.1.5 The Section Conjecture for open curves

Grothendieck formulated also a variant of the section conjecture for not necessarily proper curves. He observed that even the k-rational points at infinity (*cusps*), i.e. the points in $\overline{X} \setminus X$ where $\overline{X}$ is the smooth compactification of X, give rise to sections of the fundamental exact sequence (1.1.1) (*cuspidal sections*). The Section Conjecture for open curves predicts that every section $\mathrm{Gal}(\overline{k}/k) \to \pi_1(X, \overline{x}_0)$ is induced by a unique k-rational point of $\overline{X}$, i.e. either by a cusp or a point in X. For not necessarily proper curves, the hyperbolicity condition on X now takes the form $\chi(X) < 0$, i.e. the Euler characteristic of X should be negative. If g is the genus of $\overline{X}$ and $r = \#(\overline{X} \setminus X)(\overline{k})$ the geometric number of cusps, then the Euler characteristic is given by

$$\chi(X) = 2 - 2g - r. \tag{1.1.3}$$

Thus, X is hyperbolic if and only if $X_{\overline{k}}$ is either $\mathbb{P}^1$ minus at least three points, a genus one curve minus at least one point, or a curve of higher genus with an arbitrary number of points removed.

1.1.6 The Birational Section Conjecture

The formula (1.1.3) for the Euler characteristic suggests that X becomes "more hyperbolic" as we remove more and more closed points. This motivates a birational variant of the section conjecture where X is reduced to its generic point, i.e. the spectrum of the function field K. The fundamental exact sequence (1.1.1) in this case becomes a short exact sequence of absolute Galois groups:

$$1 \longrightarrow G_{K\bar{k}} \longrightarrow G_K \longrightarrow G_k \longrightarrow 1. \tag{1.1.4}$$

Every rational point $x \in X(k)$ induces birational sections as follows. Let $\widetilde{X} \to X$ be the normalisation of X in the algebraic closure $\overline{K}/K$ and choose a point $\tilde{x}$ over x in $\widetilde{X}$. The G_K-action on $\overline{K}$ induces an action on $\widetilde{X}$ and the stabiliser $\mathrm{D}_{\tilde{x}|x} \subseteq G_K$ of the point $\tilde{x}$ is called the *decomposition group* of $\tilde{x}|x$. The group $\mathrm{D}_{\tilde{x}|x}$ acts on the residue field $\kappa(\tilde{x})$ which is canonically isomorphic to $\bar{k}$. The resulting homomorphism $\mathrm{D}_{\tilde{x}|x} \to G_k$ is surjective and its kernel $\mathrm{I}_{\tilde{x}|x}$ is called the *inertia group* of $\tilde{x}|x$. A section $s\colon G_k \to G_K$ is called *section over* x if its image is contained in a decomposition group $\mathrm{D}_{\tilde{x}|x}$ for some $\tilde{x}$ over x.

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{I}_{\tilde{x}|x} \arrow[r] \arrow[d, hook] & \mathrm{D}_{\tilde{x}|x} \arrow[r] \arrow[d, hook] & G_k \arrow[r] \arrow[d, equal] \arrow[l, dashed, bend right, "s"'] & 1 \\ 1 \arrow[r] & G_{K\bar{k}} \arrow[r] & G_K \arrow[r] & G_k \arrow[r] & 1 \end{tikzcd} \tag{1.1.5}$$

To explain why sections over $x \in X(k)$ always exist, let K_x^{h} be the fixed field of $\mathrm{D}_{\tilde{x}|x}$ in $\overline{K}$, and set $U_x^{\mathrm{h}} := \mathrm{Spec}(K_x^{\mathrm{h}})$. The field K_x^{h} is a *henselisation* of K at x, and U_x^{h} plays the role of a small punctured neighbourhood of x in X. The top row of (1.1.5) can thus be viewed as the fundamental exact sequence (1.1.1) for U_x^{h}/k. Denote by $T_{X,x}^{\circ}$ the Zariski tangent space without origin of X at x. As a k-scheme, it is isomorphic to $\mathbb{G}_m$; more intrinsically, $\mathrm{T}_{X,x}^{\circ}$ is the spectrum of the graded algebra of K_x^{h} filtered by integer powers of the maximal ideal $\mathfrak{m}_x$. Deligne, in his theory of *tangential base points* [Del89, §15], has constructed an equivalence of categories between finite étale coverings of U_x^{h} and $\mathrm{T}_{X,x}^{\circ}$. Consequently, the top row of (1.1.5) is isomorphic to the fundamental exact sequence of $\mathrm{T}_{X,x}^{\circ}/k$. In particular, every k-rational point of $\mathrm{T}_{X,x}^{\circ}$, i.e. every nonzero tangent vector at x, induces an $\mathrm{I}_{\tilde{x}|x} \cong \hat{\mathbb{Z}}(1)$-conjugacy class of sections over x with image in the decomposition group $\mathrm{D}_{\tilde{x}|x}$.

The set of $G_{K\bar{k}}$-conjugacy classes of sections over x is traditionally called the *packet at* x. Unlike the case of complete curves, there can be many conjugacy classes of sections over the same point: the packet of sections at a cusp is uncountable under only mild assumptions on the base field [Sti12, §4].

The *Birational Section Conjecture* states that for proper X/k, every section of (1.1.4) lies over exactly one k-rational point of X. Over number fields, the conjecture is still open, but its p-adic analogue over a finite extension of $\mathbb{Q}_p$ has been proved by Königsmann, using model theory of p-adically closed fields [Koe05]. Pop has proved a "minimalistic" variant of this, where G_K is replaced with a very small quotient [Pop10]. For instance, in the case $\mu_p \subseteq k$, it suffices to work with the $\mathbb{Z}/p\mathbb{Z}$-metabelian quotient of G_K. It is this latter variant that we are going to generalise.

1.2 The section conjecture for localisations of curves

The present work is concerned with intermediate versions between the full section conjecture for a proper curve and the birational section conjecture for its generic point. Let X/k be a smooth, proper, geometrically connected curve.

Definition 1.2.1. For an arbitrary set $S \subseteq X_{\mathrm{cl}}$ of closed points, define the **localisation of X at S** as the pro-(open subscheme) of X

$$X_S \coloneqq \bigcap \big\{ U \subseteq X \text{ dense open containing } S \big\}. \tag{1.2.1}$$

The intersection is taken inside X in the scheme-theoretic sense, i.e. as a fibre product of X-schemes.

The limit (1.2.1) exists by Proposition 2.4.3 below. Intuitively, X_S is obtained from X by removing all closed points except those in S. The underlying topological space $|X_S|$ of X_S is the subspace of $|X|$ consisting of the generic point η_X and the points in S. Thus, if $S = X_{\mathrm{cl}}$, then $X_S = X$; if $S = \emptyset$, then $X_S = \eta_X$. In general, X_S lies in between η_X and X.

Let $\overline{k}/k$ be an algebraic closure and $\overline{x}_0$ a geometric point of $X_S \otimes_k \overline{k}$. Let $X_S^{\mathrm{univ}} \to X_S$ be the associated universal profinite étale cover and $\widetilde{X} \to X$ the normalisation of X in the function field of X_S^{univ}. Given a k-rational point $x \in X(k)$ and a point $\tilde{x}$ in $\widetilde{X}$ over x, we have a decomposition group $\mathrm{D}_{\tilde{x}|x}$ (the stabiliser of $\tilde{x}$ under the $\pi_1(X_S, \overline{x}_0)$-action on $\widetilde{X}$) and an inertia subgroup $\mathrm{I}_{\tilde{x}|x}$ as above, forming a diagram that generalises (1.1.5):

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{I}_{\tilde{x}|x} \arrow[r] \arrow[d, hook] & \mathrm{D}_{\tilde{x}|x} \arrow[r] \arrow[d, hook] & \mathrm{Gal}(\overline{k}/k) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \pi_1(X_S \otimes_k \overline{k}, \overline{x}_0) \arrow[r] & \pi_1(X_S, \overline{x}_0) \arrow[r] & \mathrm{Gal}(\overline{k}/k) \arrow[r] & 1.
\end{tikzcd} \tag{1.2.2}$$

Again, we say that a section of the map $\pi_1(X_S, \overline{x}_0) \to \mathrm{Gal}(\overline{k}/k)$ is a *section over x* if its image is contained in a decomposition group $\mathrm{D}_{\tilde{x}|x}$ for some $\tilde{x}|x$ in $\widetilde{X}$. We prove in Proposition 3.1.7 below that sections exist over every k-rational point of X.

Definition 1.2.2. We say that the localisation X_S/k **satisfies the section conjecture** if every section $s\colon \mathrm{Gal}(\overline{k}/k) \to \pi_1(X_S, \overline{x}_0)$ lies over exactly one k-rational point of X.

One of our main results is the identification of sufficient conditions on X_S over p-adic base fields which imply the section conjecture for X_S. We show in this way that the section conjecture holds for instance whenever S is at most countable (see Statement of Main Results in Section 1.4 below).

1.3 The liftable section conjecture

This work is mostly concerned with a variant of the section conjecture which uses only small quotients of the fundamental group. In order to state it, we fix a prime number p and introduce the following notation:

Notation 1.3.1. Let Π be a profinite group. Let $\Pi = \Pi^{(0)} \trianglerighteq \Pi^{(1)} \trianglerighteq \dots$ be the $\mathbb{Z}/p\mathbb{Z}$**-derived series**:

$$\Pi^{(0)} \coloneqq \Pi, \quad \Pi^{(i+1)} = [\Pi^{(i)}, \Pi^{(i)}]\big(\Pi^{(i)}\big)^p.$$

We denote by

$$\begin{aligned} \Pi' &\coloneqq \Pi/\Pi^{(1)} = \Pi^{\mathrm{ab}} \otimes \mathbb{Z}/p\mathbb{Z}, \\ \Pi'' &\coloneqq \Pi/\Pi^{(2)} \end{aligned}$$

the maximal $\mathbb{Z}/p\mathbb{Z}$-abelian and $\mathbb{Z}/p\mathbb{Z}$-metabelian quotient of Π, respectively.

Observe that the associations $\Pi \mapsto \Pi'$ and $\Pi \mapsto \Pi''$ are functorial and that surjective homomorphisms of profinite groups remain surjective under $(-)'$ and $(-)''$.

Definition 1.3.2. Let $\Pi \twoheadrightarrow G$ be a surjective homomorphism of profinite groups. A section $s' \colon G' \to \Pi'$ is called **liftable** if there exists a section $s'' \colon G'' \to \Pi''$ such that the following diagram commutes:

$$\begin{tikzcd} \Pi'' \arrow[r, two heads] \arrow[d, two heads] & G'' \arrow[l, dashed, bend right, "s''"'] \arrow[d, two heads] \\ \Pi' \arrow[r, two heads] & G'. \arrow[l, bend right, "s'"'] \end{tikzcd}$$

Let k be a field of characteristic zero and X_S the localisation of a smooth, proper, geometrically connected curve X/k at a set of closed points $S \subseteq X_{\mathrm{cl}}$. Let $\overline{k}/k$ be an algebraic closure, let $G_k \coloneqq \mathrm{Gal}(\overline{k}/k)$ be the absolute Galois group of k and let $\pi_1(X_S)$ be the fundamental group of X_S with respect to a geometric base point on $X_S \otimes_k \overline{k}$, so that we have a surjective homomorphism $\pi_1(X_S) \to G_k$. We can again define what it means for a liftable section $s' \colon G_k' \to \pi_1(X_S)'$ to lie over a k-rational point of X (Definition 3.1.3), and liftable sections do exist over every k-rational point (see Proposition 3.3.2 below).

Definition 1.3.3. We say that X_S/k satisfies the **liftable section conjecture** if every liftable section $s' \colon G_k' \to \pi_1(X_S)'$ lies over a unique k-rational point of X.

1.4 Statement of main results

1.4.1 The liftable section conjecture for good localisations

Our starting point is Pop's proof of the birational liftable section conjecture over finite extensions $k/\mathbb{Q}_p$ containing the p-th roots of unity [Pop10, Theorem A]. The aim of the present work is to generalise this result and its proof to localisations of curves, thereby making a step from the birational case towards the case of open or proper curves, where the section conjecture is still unknown. Our main result is the identification of conditions on the localisation of a curve which ensure that the liftable section conjecture holds. To this end, we introduce in Definition 5.1.1 below the notion of a **good localisation**, defined by four conditions roughly saying that there is a large supply of invertible functions on X_S. Our main theorem reads as follows:

Theorem A. *Let k be a finite extension of $\mathbb{Q}_p$ with $\mu_p \subseteq k$. Let X/k be a smooth, proper, geometrically connected curve and $S \subseteq X_{\mathrm{cl}}$ a set of closed points. If X_S is a good localisation, then X_S/k satisfies the liftable section conjecture.*

The proof of Theorem A is the content of Chapter 5 and constitutes the technical heart of this work.

As a demonstration of the usefulness of the theorem, we verify the conditions for a good localisation in some cases, thereby obtaining concrete examples of localisations of curves where the liftable section conjecture holds:

Theorem B. *Let k be a finite extension of $\mathbb{Q}_p$ with $\mu_p \subseteq k$, let X/k be a smooth, proper, geometrically connected curve and $S \subseteq X_{\mathrm{cl}}$ a set of closed points. Assume that one of the following holds:*

(a) $S \subseteq X_{\mathrm{cl}}$ is at most countable; or

(b) X is defined over a subfield $k_0 \subseteq k$ and $S \subseteq X_{\mathrm{cl}}$ contains only finitely many algebraic points over k_0.

Then X_S satisfies the liftable section conjecture.

Remark 1.4.1. In the case $S = \emptyset$, Theorem B (a) specialises to Pop's birational result which was the starting point of our investigation [Pop10, Theorem A], that every liftable section $s'\colon G'_k \to G'_K$ lies over a unique k-rational point $x \in X(k)$.

Theorem B (a) for at most countable sets S is proved in §7.3. The main ingredient is a new approximation theorem with invertibility conditions for general valuations (Theorem 7.3.1). It is used to verify the conditions of a good localisation in Theorem 7.3.3.

Theorem B (b) for sets with only finitely many algebraic points over a subfield k_0 is proved in §7.4. See also Definition 7.4.1 for the definition of algebraic points over k_0.

As a byproduct of the proof of Theorem A, we also obtain a result on the index of a curve X/k. The **index** of X is defined as the greatest common divisor of the degrees of all closed points:

$$\mathrm{index}(X) = \gcd\{[\kappa(x) : k] \ : \ x \in X_{\mathrm{cl}}\}.$$

If X contains a k-rational point, then clearly $\mathrm{index}(X) = 1$. In light of Theorem A, the existence of a liftable section for a good localisation X_S implies the existence of a rational point and thus $\mathrm{index}(X) = 1$. But only part of the definition of a good localisation is needed to prove the following:

Theorem C (= Proposition 5.5.4, Corollary 5.5.5)**.** *Let k be a finite extension of $\mathbb{Q}_p$ with $\mu_p \subseteq k$ and let X/k be a smooth, proper, geometrically connected curve of genus g. Let $S \subseteq X_{\mathrm{cl}}$ be a set of closed points such that every geometrically connected, finite $\mathbb{Z}/p\mathbb{Z}$-elementary abelian cover $W \to X_S$ satisfies* $\mathrm{Pic}(W)/p = 0$.

(a) We have the implication:

$$\exists \textit{ liftable section } s' \colon G'_k \to \pi_1(X_S)' \ \Rightarrow \ p \nmid \mathrm{index}(X).$$

(b) If $g > 0$, we have the implication:

$$\exists \textit{ section } s \colon G_k \to \pi_1(X_S) \ \Rightarrow \ \mathrm{index}(X) = 1.$$

Part (b) of Theorem C is proved by combining part (a) with a result of Stix. The assumptions of Theorem C are satisfied in particular when the complement of S is *uniformly dense* in X (see Definition 6.3.8).

Remark 1.4.2. As explained in [Pop10, Remark (a) after Theorem B], the assumption $\mu_p \subseteq k$ in Theorem A is necessary even in the birational case: otherwise, the maximal pro-p quotient $G_k(p)$ of G_k is a free pro-p group of rank $[k : \mathbb{Q}_p] + 1$ [NSW08, Theorem 7.5.11]. This implies that there exist sections $s' \colon G'_k \to \pi_1(X_S)'$ and they are all liftable, even when $X(k)$ is empty.

1.4.2 The liftable section conjecture without p-th roots of unity

If k is a finite extension of $\mathbb{Q}_p$ which does not necessarily contain the p-th roots of unity, then, by Remark 1.4.2, we cannot expect the liftable section conjecture to hold over k. Nevertheless, quite generally, the validity of the liftable section conjecture over a field extension ℓ/k can be used to deduce a form of the liftable section conjecture over k.

To state the precise result, let k be a field of characteristic zero and X_S a localisation of a smooth, proper, geometrically connected curve over k. Let ℓ/k be a finite Galois extension. Denote by

$$(X_S \otimes_k \ell)''' \to (X_S \otimes_k \ell)'' \to (X_S \otimes_k \ell)' \to X_S \otimes_k \ell$$

the beginning of the tower that corresponds to the $\mathbb{Z}/p\mathbb{Z}$-derived series for $\pi_1(X_S \otimes \ell)$. Thus, from right to left, we have the maximal $\mathbb{Z}/p\mathbb{Z}$-elementary abelian cover of $X_S \otimes_k \ell$, the maximal $\mathbb{Z}/p\mathbb{Z}$-metabelian cover, and the maximal three-step $\mathbb{Z}/p\mathbb{Z}$-solvable cover. Denote by $\ell'''/\ell''/\ell'/\ell$ the corresponding tower of field extensions. Note that each of the three covers is Galois also over X_S, being a characteristic cover of the Galois cover $X_S \otimes_k \ell \to X_S$, and similarly, each of the field extensions is Galois over k.

Definition 1.4.3. We say that a section $s'\colon \operatorname{Gal}(\ell'/k) \to \operatorname{Gal}((X_S \otimes_k \ell)'/X_S)$ is **liftable** (respectively, **twice-liftable**) if it admits a lift s'' (respectively, s'''), forming a commutative diagram as follows:

$$\begin{tikzcd}
\operatorname{Gal}((X_S \otimes_k \ell)'''/X_S) \arrow[r, two heads] \arrow[d, two heads] & \operatorname{Gal}(\ell'''/k) \arrow[d, two heads] \arrow[l, dashed, bend right, "s'''"'] \\
\operatorname{Gal}((X_S \otimes_k \ell)''/X_S) \arrow[r, two heads] \arrow[d, two heads] & \operatorname{Gal}(\ell''/k) \arrow[d, two heads] \arrow[l, dashed, bend right, "s''"'] \\
\operatorname{Gal}((X_S \otimes_k \ell)'/X_S) \arrow[r, two heads] & \operatorname{Gal}(\ell'/k) \arrow[l, bend right, "s'"']
\end{tikzcd}$$

In the case $\ell = k$, if we set $\pi_1(X_S)' \coloneqq \operatorname{Gal}(X_S'/X_S)$ and $G_k' \coloneqq \operatorname{Gal}(k'/k)$, then this definition of a liftable section specialises to the definition of a liftable section for the projection $\pi_1(X_S)' \to G_k'$ as above. We show in Proposition 3.3.2 below that liftable and twice-liftable sections exist over every k-rational point of X. The following result assumes the validity of the liftable section conjecture for certain connected finite étale covers of X_S. All those are again localisations of curves, as is proved in Corollary 2.4.11 below.

Theorem 1.4.4 (= Corollary 3.5.3, Propositions 3.5.10 and 3.5.4)**.** *Let k be a field of characteristic zero, let X/k be a smooth, proper, geometrically connected curve and $S \subseteq X_{\mathrm{cl}}$ a set of closed points. Let ℓ/k be a finite Galois extension such that $X_S \otimes_k \ell$ satisfies the liftable section conjecture. Let $s'\colon \operatorname{Gal}(\ell'/k) \to \operatorname{Gal}((X_S \otimes_k \ell)'/X_S)$ be a liftable section. Then there exists a unique k-rational point x of X such that the restricted section*

$$s'|_{\operatorname{Gal}(\ell'/\ell)}\colon \operatorname{Gal}(\ell'/\ell) \to \operatorname{Gal}((X_S \otimes_k \ell)'/(X_S \otimes_k \ell))$$

lies over $x \otimes_k \ell$. If moreover one of the following conditions holds:

(a) the prime p does not divide the degree $[\ell : k]$; or

(b) $W \otimes_k \ell$ satisfies the liftable section conjecture for every geometrically connected, finite étale subcover $(X_S \otimes_k \ell)' \to W \to X_S$, and s' is twice-liftable;

then already s' lies over x.

If k is a finite extension of $\mathbb{Q}_p$ and ℓ/k is a finite Galois extension with $\mu_p \subseteq \ell$, we can verify the hypotheses of Theorem 1.4.4 by showing that $X_S \otimes_k \ell$ (respectively, $W \otimes_k \ell$ for every geometrically connected, finite étale subcover $(X_S \otimes_k \ell)' \to W \to X_S$) is a good localisation. In this way, we obtain the following results:

Theorem D (= Theorems 7.3.4 and 7.4.16)**.** *Let k be a finite extension of $\mathbb{Q}_p$ and ℓ/k a finite Galois extension with $\mu_p \subseteq \ell$. Let X/k be a smooth, proper, geometrically connected curve and $S \subseteq X_{\mathrm{cl}}$ a set of closed points. Assume that one of the following holds:*

(a) $S \subseteq X_{\mathrm{cl}}$ is at most countable; or

(b) X is defined over a subfield $k_0 \subseteq k$ and $S \subseteq X_{\mathrm{cl}}$ contains only finitely many algebraic points over k_0.

Then, if $s' \colon \operatorname{Gal}(\ell'/k) \to \operatorname{Gal}((X_S \otimes_k \ell)'/X_S)$ is a liftable section, there exists a unique k-rational point x of X such that $s'|_{\operatorname{Gal}(\ell'/\ell)}$ lies over $x \otimes_k \ell$. If moreover one of the following holds:

- *p does not divide $[\ell : k]$; or*
- *we are in case (a) and s' is twice-liftable; or*
- *we are in case (b), every transcendental point over k_0 is contained in S, and s' is twice-liftable;*

then s' itself lies over x.

Remark 1.4.5. If we choose $S = \emptyset$, then Theorem D (a) specialises to Pop's birational result for the function field K [Pop10, Theorem B]. However, it is claimed there that every liftable section $s' \colon \operatorname{Gal}(\ell'/k) \to \operatorname{Gal}((K\ell)'/K)$ lies over a unique k-rational point, not only the restriction of such s' to $\operatorname{Gal}(\ell'/\ell)$. The discrepancy is explained by an error in the proof in §5 of loc. cit., and while we do not have a counterexample we believe that extra conditions such as in our Theorem D are probably necessary in order to conclude that the liftable section s' itself lies over the k-rational point.

1.4.3 The full section conjecture for localisations of curves

The liftable section conjecture is interesting in that it extracts information about rational points from very small quotients of the fundamental groups. But one can still deduce the section conjecture for the full fundamental groups if the liftable version holds over all covers of a given localisation of a curve:

Theorem 1.4.6 (= Proposition 3.5.10)**.** *Let k be a field of characteristic zero, let X/k be a smooth, proper, geometrically connected curve and let $S \subseteq X_{\mathrm{cl}}$ be a set of closed points. Assume that there exists a finite Galois extension ℓ/k such that $W \otimes_k \ell$ satisfies the liftable section conjecture for every geometrically connected, finite étale cover $W \to X_S$. Then X_S satisfies the section conjecture, i.e. every section $s\colon G_k \to \pi_1(X_S)$ lies over a unique k-rational point of X.*

As an application we obtain new examples of localisations of curves satisfying the section conjecture:

Theorem E (= Theorems 7.3.5 and 7.4.17)**.** *Let k be a finite extension of $\mathbb{Q}_p$, let X/k be a smooth, proper, geometrically connected curve and let $S \subseteq X_{\mathrm{cl}}$ be a set of closed points. Assume that one of the following holds:*

(a) S is at most countable; or

(b) X is defined over a subfield $k_0 \subseteq k$ and S contains all transcendental points and only finitely many algebraic points over k_0.

Then X_S satisfies the section conjecture.

Remark 1.4.7. In the case $S = \emptyset$, Theorem E (a) specialises to the Birational Section Conjecture for curves over p-adic fields, as proved by Königsmann [Koe05], which states that every section $s\colon G_k \to G_K$ of the projection of absolute Galois groups lies over a unique k-rational point $x \in X(k)$.

1.5 Outline

We start by formally defining localisations of schemes at a subspace in Chapter 2. The definition involves taking a limit over open subschemes, which is not guaranteed to exist in the category of schemes. For this reason, the localisation is defined in general only as a pro-scheme, essentially using the inverse system of open subschemes as a placeholder for its limit. We prove a few properties concerning the functorial behaviour of localisations. We also explain how to define the fundamental group of a pro-scheme and show that it coincides with the fundamental group of the limit scheme if the latter exists. We then specialise to localisations of curves, which can be described very explicitly. We then explain profinite étale covers of general schemes and end the chapter with a discussion of Kummer theory for a regular, integral scheme.

In Chapter 3 we develop the language around the section conjecture for localisations of curves using general quotients of the fundamental groups. The existence of sections over each rational point is shown for a large class of quotients. This entails in particular the maximal $\mathbb{Z}/p\mathbb{Z}$-abelian or maximal $\mathbb{Z}/p\mathbb{Z}$-metabelian quotient which are relevant in the liftable section conjecture. We then consider questions of descent type, i.e. we assume some statement over a Galois extension ℓ/k and deduce a similar statement over k. In particular, we analyse how the property of a section to lie over a rational point behaves in a Galois extension by translating the question into the language of nonabelian Galois cohomology. We apply the abstract machinery in the more special context of the liftable section conjecture. It is shown that the validity of the liftable section conjecture over a finite Galois extension ℓ/k implies a version over k. This is relevant for deducing a variant of the liftable section conjecture over p-adic base fields which do not contain the p-th roots of unity, since the liftable section conjecture itself holds only over base fields containing them. We also show how the validity of the liftable section conjecture for all finite étale covers implies the section conjecture for the full fundamental group.

In Chapter 4 we give a brief review of valuation theory for the convenience of the reader. Besides discrete valuations associated to closed points on a curve, more general valuations play an important role in the proof of the liftable section conjecture for good localisations. While we can restrict ourselves to rank 1 valuations for the most part of the proof, the use of Pop's local-to-global principle for Brauer groups, in particular, makes it necessary to discuss valuations of higher rank. The relevant definitions and properties are collected and proofs are given for a few statements.

Chapter 5 finally contains the proof of the liftable section conjecture for good localisations. Good localisations are defined via four conditions which roughly express that there is a large supply of invertible functions. They are precisely what is needed in order to extend Pop's proof in the birational case to more general localisations of curves. The uniqueness statement in the liftable section conjecture is proved first, before the more difficult existence question is treated. This is the most technically involved part of this work. We refer to Section 5.2 for a summary of the proof strategy.

In Chapter 6, the conditions for a good localisation are studied in more detail. We prove general criteria which can be used to verify them. One set of criteria uses approximation statements on the function field with respect to valuations. Another criterion is formulated in terms of p-adic approximation of divisors on the ambient curve.

The criteria are used in Chapter 7 to verify the conditions in a number of cases, thereby obtaining concrete examples of localisations of curves over p-adic fields which satisfy the liftable and general section conjecture. We first explain how the proof specialises in the birational case. The easy case of a localisation at finitely many points is also treated. We then show that localisations at countable sets of points are good. The key ingredient here is a new approximation theorem with invertibility conditions for general valuations. Finally, we consider curves over p-adic fields which are defined over a subfield k_0 and show that localisations containing only finitely many algebraic points over k_0 are good.

We close by raising in Section 7.4.2 some questions for future research to continue this line of investigation.

Notation

We use the following notations and conventions:

- For a prime number p, a $\mathbb{Z}/p\mathbb{Z}$*-abelian* (or $\mathbb{Z}/p\mathbb{Z}$*-elementary abelian*) group is an abelian group which is annihilated by p. A $\mathbb{Z}/p\mathbb{Z}$*-metabelian group* is a group which is an extension of a $\mathbb{Z}/p\mathbb{Z}$-abelian group by a $\mathbb{Z}/p\mathbb{Z}$-abelian group.
- A field extension L/K is called $\mathbb{Z}/p\mathbb{Z}$*-abelian* (or $\mathbb{Z}/p\mathbb{Z}$*-elementary abelian*, especially when there is the danger of confusion with p-cyclic extensions) if it is Galois with $\mathbb{Z}/p\mathbb{Z}$-abelian Galois group. We use the analogous terminology for $\mathbb{Z}/p\mathbb{Z}$-metabelian field extensions, and similarly for Galois covers of connected schemes.
- As in 1.3.1 above, for a profinite group Π denote by $\Pi = \Pi^{(0)} \trianglerighteq \Pi^{(1)} \trianglerighteq \dots$ the $\mathbb{Z}/p\mathbb{Z}$-derived series and by
$$\begin{aligned} \Pi' &\coloneqq \Pi/\Pi^{(1)} = \Pi^{\mathrm{ab}} \otimes \mathbb{Z}/p\mathbb{Z}, \\ \Pi'' &\coloneqq \Pi/\Pi^{(2)} \end{aligned}$$
the *maximal* $\mathbb{Z}/p\mathbb{Z}$*-abelian quotient* and *maximal* $\mathbb{Z}/p\mathbb{Z}$*-metabelian quotient* of Π, respectively. Accordingly, the maximal $\mathbb{Z}/p\mathbb{Z}$-abelian extension of a field K will usually be denoted by K'/K, and the maximal $\mathbb{Z}/p\mathbb{Z}$-abelian cover of a scheme X will be denoted by $X' \to X$. Analogous notation is used for the $\mathbb{Z}/p\mathbb{Z}$-metabelian variants K''/K and $X'' \to X$.
- If X is a scheme over a field k, and ℓ/k is a field extension, the extension of scalars is denoted by
$$X \otimes_k \ell \coloneqq X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\ell).$$
- For a scheme X, undecorated cohomology groups $\mathrm{H}^n(X, -)$ denote étale cohomology. If L/K is a Galois extension of fields, $\mathrm{H}^n(L/K, -)$ denotes the group cohomology for the Galois group $\mathrm{Gal}(L/K)$. In the special case where L is a separable closure of K, we denote by $\mathrm{H}^n(K, -)$ the cohomology for the absolute Galois group of K (which agrees with the étale cohomology of $\mathrm{Spec}(K)$).
- For a scheme X, we denote by $\mathrm{Br}(X)$ the cohomological Brauer group, i.e. the étale cohomology group $\mathrm{H}^2(X, \mathbb{G}_m)$. The Brauer group of a field K is denoted by $\mathrm{Br}(K)$.

Acknowledgements

I am indebted to a large number of people who supported me over the years in one way or another. First and foremost I want to thank my advisor Jakob Stix for providing guidance, sharing his knowledge and experience and generally creating a pleasant working atmosphere. He consistently enabled and encouraged me to attend conferences and workshops, a merit he partially shares with his own advisor Florian Pop. A three-generational hiking trip in the mountains of Canada remains in particularly good memory.

I am also grateful to my colleagues for providing an environment which has made going to the office an enjoyable experience, no matter the temperatures or noise levels. The Kaffeerunde in particular has been an important part of the workday whose absence has been felt in the recent months since the pandemic has forced us all into homeworking.

Special thanks go to my former office mate Matteo Costantini for good conversations and jolly moments inside and occasionally outside the office. Jonathan Zachhuber has been another excellent discussion partner during all my time in Frankfurt. My academic life has been accompanied longest by Nithi Rungtanapirom, first as a tutor in Heidelberg, later as a friend and colleague. In Theresa Kumpitsch I have found a trusted companion in learning new mathematics as well as in post- and pre-conference travels. I want to thank also Marius Leonhardt, Ben Heuer, Adrian Zorbach, and Jaro Eichler for companionship in both work and travel. Other colleagues who have contributed to a pleasant atmosphere at the institute include Max Bieri, Şevda Kurul, Sara Lamboglia, Rosemarie Martienssen, Matthias Nickel, Markus Rennig, Stefan Rettenmayr, Felix Röhrle, David Torres-Teigell, and Riccardo Zuffetti.

I am grateful to the Deutsche Mathematiker-Vereinigung for awarding me a two-month stay at the Max Planck Institute for Mathematics in Bonn, which I had the pleasure of sharing with Jonas Stelzig. My participation at the Heidelberg Laureate Forum was another unique experience and I want to thank the Klaus Tschira Stiftung for making it possible. Thanks go also to the various organisers and participants of the Kleine AG for creating many stimulating meetings in which I have learned a lot.

Special thanks go also to my former piano teacher Brett Youens for his long-lasting support, encouragement and interest in all my mathematical endeavours.

Last but not least, I thank my parents for their unconditional support and for always reminding me to take care of my well-being.

2 Localisations of schemes

In this chapter, we introduce the notion of localisation of a scheme at an arbitrary subspace of its underlying topological space. We then consider more specifically localisations of curves and finally discuss profinite étale covers. The material contained in this chapter is kept on a quite general level. Rather than containing concrete results, it serves to develop the language which is used to formulate and later to prove statements about localisations of curves.

A localisation of a scheme is defined as an inverse limit of open subschemes. As such a limit is not guaranteed to exist in the category of schemes, the localisation can in general be defined only as a pro-object, essentially using the inverse system of open subschemes as a placeholder for its limit. This is similar in spirit to the construction of the real numbers where a Cauchy sequence of rational numbers is used as placeholder for its limit. After recalling the notions of pro-objects and pro-categories in Section 2.1, we define in Section 2.2 the "formal localisation" (as a pro-object) and the actual localisation (if the limit exists) of a scheme at a subspace. We prove a few properties concerning their functorial behaviour. In Section 2.3 we define the fundamental group of a pro-scheme and show that it coincides with the fundamental group of the limit scheme if the latter exists. In Section 2.4 we specialise from general schemes to normal, proper curves. Localisations of curves at sets of closed points are guaranteed to exist in the category of schemes and can be described very explicitly. Finally, in Section 2.5, we explain the notion of profinite étale covers of a general scheme. This is another place where the question arises whether to work with pro-schemes or limit schemes. We show that the two viewpoints are equivalent under only mild assumptions. We close the chapter by discussing Kummer theory for a regular integral scheme.

In the later chapters we work only with localisations of curves. We could have restricted the discussion of localisations to this special case. In particular, there is no need to define the fundamental group of a pro-scheme for the purposes of this work. However, we haven chosen a higher level of generality in the expectation that it may facilitate future investigations concerning the section conjecture for higher-dimensional varieties.

2.1 Pro-categories

A category I is **filtered** if every finite subcategory admits a cocone. More explicitly, I is filtered if it satisfies the following conditions:

(i) I is non-empty;

(ii) for any two objects i and j in I, there exists an object k and arrows $i \to k$ and $j \to k$;

(iii) for any pair of parallel arrows $f, g\colon i \rightrightarrows j$ in I, there exists an arrow $e\colon j \to k$ such that $e \circ f = e \circ g$.

The category I is **cofiltered** if the opposite category I^{op} is filtered.

Remark 2.1.1. The category associated to a directed poset is filtered. Conversely, every small filtered category receives a cofinal functor from a category associated to a directed poset [SGA 4-1, Exposé I, §8.1].

Let $\mathcal{C}$ be a locally small category. A **pro-object** of $\mathcal{C}$ is a functor $\varphi\colon I^{\mathrm{op}} \to \mathcal{C}$ with I a small filtered category. Pro-objects are usually denoted as $(X_i)_{i\in I}$ with $X_i = \varphi(i)$ for $i \in I$, the transition maps $\varphi(\alpha)\colon X_j \to X_i$ for $\alpha\colon i \to j$ in I being implicit. Another—more suggestive—notation is $"\varprojlim_{i\in I}" X_i$, emphasising the role of pro-objects as formal limits of objects in $\mathcal{C}$.

Any pro-object $X = "\varprojlim_{i\in I}" X_i$ defines a functor $\mathcal{C} \to \mathrm{Set}$ by the rule $A \mapsto \varinjlim_i \mathrm{Hom}(X_i, A)$, called the functor **pro-represented** by X. This functor is the limit of the representable functors $\mathrm{Hom}(X_i, -)$ in the opposite functor category $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})^{\mathrm{op}}$. Morphisms between pro-objects are defined as morphisms between their pro-represented functors in $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})^{\mathrm{op}}$. Explicitly:

$$\mathrm{Hom}("\varprojlim_{i\in I}" X_i, "\varprojlim_{j\in J}" Y_j) = \varprojlim_{j\in J} \varinjlim_{i\in I} \mathrm{Hom}_{\mathcal{C}}(X_i, Y_j).$$

In this way, the pro-objects of $\mathcal{C}$ form the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})^{\mathrm{op}}$ on the functors which are cofiltered limits of representable functors. This category is called the **pro-category** of $\mathcal{C}$ and is denoted by $\mathrm{Pro}(\mathcal{C})$.

The category $\mathcal{C}$ embeds into its pro-category via the Yoneda embedding $\mathcal{C} \to \mathrm{Pro}(\mathcal{C})$, $A \mapsto \mathrm{Hom}(A, -)$. We often identify objects in $\mathcal{C}$ with their image in $\mathrm{Pro}(\mathcal{C})$. The embedding however does not preserve cofiltered limits in general. Therefore, even when the cofiltered limit $\varprojlim_{i\in I} X_i$ exists in $\mathcal{C}$, it has to be distinguished notationally (by the quotation marks) from the pro-object $"\varprojlim_{i\in I}" X_i$.

2.2 Localisations of schemes

In the following, let X be an arbitrary scheme, denote by $|X|$ its underlying topological space, and let $S \subseteq |X|$ be an arbitrary subspace.

Definition 2.2.1. The **formal localisation of** X **at** S is the pro-(open subscheme) of X

$$"X_S" = "\varprojlim_{U\supseteq S}" U,$$

where U runs over all open subschemes of X containing S. If the limit exists in the category of schemes, it is called the **localisation of X at S** and denoted by

$$X_S := \varprojlim_{U \supseteq S} U.$$

Remarks 2.2.2.

(1) The indexing category of the pro-object "X_S" is the poset category formed by the open subschemes U of X containing S, ordered by reverse inclusion. It is directed since the open subschemes containing S are closed under finite intersections.

(2) Since X itself appears in the inverse system, the localisation "X_S" has a canonical morphism to X viewed as a pro-scheme. In this way, "X_S" is a pro-scheme over X. It is clearly also a pro-(scheme over X). Those two statements are equivalent as there is a canonical equivalence of categories $\mathrm{Pro}(\mathrm{Sch}\,/X) \simeq \mathrm{Pro}(\mathrm{Sch})/X$.

(3) If X_S exists as a scheme, the projections $X_S \to U$ with $U \supseteq S$ an open subscheme of X containing S define a natural morphism $X_S \to$ "X_S" in the category of pro-schemes.

(4) If the localisation X_S exists as a scheme, then it is a fibre product over X of the open subschemes U of X with $U \supseteq S$. Hence, we can write suggestively

$$X_S = \bigcap_{U \supseteq S} U,$$

if we agree to interpret the intersection in the appropriate scheme-theoretic sense, i.e. as a fibre product over X.

(5) If $x \rightsquigarrow y$ is a specialisation of points in X with $y \in S$, then we have $x \in U$ for every $U \supseteq S$ appearing in the inverse system above. Therefore, for the purpose of defining "X_S" or X_S, the set S can be assumed stable under generisation without loss of generality.

(6) Assume that there is a cofinal system $(U_i)_{i \in I}$ of open subschemes of X containing S such that all U_i are quasi-compact and quasi-separated and all inclusions between them are affine. Then $X_S = \varprojlim_i U_i = \varprojlim_{U \supseteq S} U$ exists in the category of schemes. By [EGA IV$_3$, Prop. (8.2.9)], its underlying topological space is

$$|X_S| = \varprojlim_{U \supseteq S} |U| = \bigcap_{U \supseteq S} |U| \subseteq |X|.$$

This is the smallest subspace of $|X|$ containing S which is stable under generisation. In particular, if S itself is already stable under generisation, then $|X_S| = S$.

Example 2.2.3. Here are some simple examples of localisations:

(a) $S = \emptyset$: $X_\emptyset = \emptyset$;

(b) $S = |X|$: $X_{|X|} = X$;

(c) $S = |U|$ with $U \subseteq X$ open: $X_{|U|} = U$;

(d) $S = \{x\}$ a single point of X: $X_{\{x\}} = \mathrm{Spec}(\mathcal{O}_{X,x})$;

(e) $S = \{\eta_X\}$ the generic point for X integral: $X_{\{\eta_X\}} = \eta_X$.

Examples (a) and (b) are of course special cases of (c), and Example (e) is a special case of (d).

2.2.1 Functoriality

Localisations of schemes have the following functorial behaviour: let $f\colon Y \to X$ be a morphism of schemes and let $T \subseteq |Y|$ and $S \subseteq |X|$ be subsets with $f(T) \subseteq S$. Then for every open subscheme U of X containing S, the inverse image $f^{-1}(U)$ is an open subscheme of Y containing T, which therefore appears in the inverse system defining "Y_T". From this we obtain a morphism of pro-schemes "Y_T" $\to$ "X_S" which fits into a commutative diagram

$$\begin{tikzcd} \text{"}Y_T\text{"} \arrow[r] \arrow[d, dashed] & Y \arrow[d, "f"] \\ \text{"}X_S\text{"} \arrow[r] & X. \end{tikzcd}$$

If the two localisations Y_T and X_S exist in the category of schemes, then the maps $Y_T \to f^{-1}(U) \to U$ for $U \supseteq S$ similarly define a morphism $Y_T \to X_S$, and we have the extended commutative diagram

$$\begin{tikzcd} Y_T \arrow[r] \arrow[d] & \text{"}Y_T\text{"} \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ X_S \arrow[r] & \text{"}X_S\text{"} \arrow[r] & X. \end{tikzcd}$$

2.2.2 Transitivity

Let X be a scheme and $S \subseteq T \subseteq |X|$ subspaces. Assume that there is a cofinal system $(U_i)_{i\in I}$ of open subschemes of X containing S such that all U_i are quasi-compact and quasi-separated and all inclusions between them are affine. Then the localisation X_S exists and its underlying space is the subspace $|X_S| = \bigcap_{U\supseteq S} |U|$ of $|X|$ (see Remark 2.2.2 (6)). The subspace T of $|X|$ can then also be viewed as a subspace of $|X_S|$.

Proposition 2.2.4. *In the given situation, the localisation $(X_S)_T$ exists if and only if X_T exists, and in this case one has $(X_S)_T = X_T$.*

Proof. Since $|X_S|$ carries the subspace topology from $|X|$, the open subschemes V of X_S containing T are precisely those of the form $V = V' \times_X X_S$ with V' an open subscheme of X containing T. With this we calculate:

$$\begin{aligned}
(X_S)_T &= \varprojlim_{X_S \supseteq V \supseteq T} V \\
&= \varprojlim_{X \supseteq V' \supseteq T} (V' \times_X X_S) \\
&= \varprojlim_{X \supseteq V' \supseteq T} \; \varprojlim_{X \supseteq U \supseteq S} (V' \cap U) \\
&= \varprojlim_{X \supseteq U \supseteq S} \; \varprojlim_{X \supseteq V' \supseteq T} (V' \cap U).
\end{aligned}$$

Here we used the principle that limits commute with limits to move the limit defining X_S outside of the fibre product with V' and the limit over V'. We also used that the fibre product of two open subschemes of X is their intersection. For any fixed $X \supseteq U \supseteq S$, the open neighbourhoods of T in X which are contained in U are cofinal, so that we can drop the limit over U altogether and find

$$(X_S)_T = \varprojlim_{X \supseteq V' \supseteq T} V' = X_T. \qquad \square$$

2.2.3 Base change along closed morphisms

Localisation of schemes commutes with base change along closed morphisms:

Lemma 2.2.5. *Let X be a scheme and $S \subseteq |X|$ a subspace. Let $f \colon Y \to X$ be a closed morphism. Assume that the localisation X_S exists. Then $Y_{f^{-1}(S)}$ exists and the canonical map $Y_{f^{-1}(S)} \to X_S \times_X Y$ is an isomorphism.*

Proof. Using the fact that limits commute with fibre products, we have

$$X_S \times_X Y = (\varprojlim_{U \supseteq S} U) \times_X Y = \varprojlim_{U \supseteq S} f^{-1}(U).$$

Thus, it suffices to show that the sets $f^{-1}(U)$ with $U \subseteq X$ open containing S are cofinal among all open $V \subseteq Y$ containing $f^{-1}(S)$. For any open $V \subseteq Y$ containing $f^{-1}(S)$, the set $U := X \setminus f(Y \setminus V)$ satisfies $S \subseteq U$ and $f^{-1}(U) \subseteq V$, and U is open in X since f is closed. $\square$

2.3 Fundamental groups of pro-schemes

We want to extend the definition of the étale fundamental group of a scheme to pro-schemes and compare the resulting notion with the fundamental group of the limit scheme if it exists.

2.3.1 Galois categories

Recall that a **Galois category with fibre functor** is a pair $(\mathcal{C}, F)$ consisting of an essentially small category $\mathcal{C}$ (i.e. the set of isomorphism classes is small) and a functor $F\colon \mathcal{C} \to \mathrm{FinSet}$ to the category of finite sets with the following properties:

(i) $\mathcal{C}$ has finite limits and colimits;

(ii) every object of $\mathcal{C}$ is a finite coproduct of connected objects;

(iii) F is exact and conservative. [1]

Here, an object X of $\mathcal{C}$ is called **connected** if it has precisely two subobjects: the initial object $\emptyset$ and X itself. **Exactness** of F means that finite limits and colimits are preserved. The functor F being **conservative** means that a morphism ϕ in $\mathcal{C}$ is an isomorphism if and only if $F(\phi)$ is an isomorphism in FinSet. A **morphism of Galois categories** $(\mathcal{C}, F) \to (\mathcal{C}', F')$ consists of a functor $\varphi\colon \mathcal{C} \to \mathcal{C}'$ and a natural transformation (necessarily an isomorphism) $\eta\colon F' \circ \varphi \simeq F$. For any profinite group Π, the category Π-FinSet of continuous permutation representations of Π on finite sets forms a Galois category with the forgetful fibre functor $\Pi\text{-FinSet} \to \mathrm{FinSet}$. The fundamental fact about Galois categories is that that any Galois category $(\mathcal{C}, F)$ is equivalent to $(\Pi\text{-FinSet}, \mathrm{forget})$ for a unique profinite group Π. The group Π can be recovered as the automorphism group $\mathrm{Aut}(F)$ of the fibre functor F, equipped with the coarsest topology making the tautological action on all discrete finite sets $F(X)$ continuous for $X \in \mathcal{C}$.

To make this more precise, let ProfGp be the category of profinite groups and GalCat the 2-category of essentially small Galois categories with fibre functor. The category $\mathrm{Fun}((\mathcal{C}, F), (\mathcal{C}', F'))$ of morphisms between two Galois categories with fibre functor (with the obvious definition of 2-morphisms) is in fact a setoid, i.e. equivalent to a category with only identity morphisms, so that GalCat is really a category rather than a 2-category. We have a functor

$$\mathrm{ProfGp} \longrightarrow \mathrm{GalCat}^{\mathrm{op}} \tag{2.3.1}$$

which is given on objects by $\Pi \mapsto (\Pi\text{-FinSet}, \mathrm{forget})$. On morphisms, every homomorphism of profinite groups $f\colon \Pi \to \Pi'$ induces a morphism

$$f^*\colon (\Pi'\text{-FinSet}, \mathrm{forget}) \to (\Pi\text{-FinSet}, \mathrm{forget})$$

which pulls back a Π'-action on a finite set to a Π-action on the same set along f.

[1] This definition of a Galois category is taken from [Stacks, Tag 0BMQ]. This list of axioms is equivalent but more concise compared to the original definition in [SGA 1, Exposé V, §4].

Theorem 2.3.1 (Main Theorem of Galois Categories)**.** *The functor* (2.3.1) *is an equivalence of categories*

$$\mathrm{ProfGp} \simeq \mathrm{GalCat}^{\mathrm{op}}$$

with quasi-inverse $(\mathcal{C}, F) \mapsto \mathrm{Aut}(F)$*.*

Proof. The essential surjectivity is [SGA 1, Exp. V, Thm. 4.1]. Fully faithfulness is proved in [SGA 1, Exp. V, Cor. 6.3]. □

2.3.2 Colimits of Galois categories

Let $(\mathcal{C}_i, F_i)_{i\in I}$ be a filtered diagram of Galois categories with fibre functor. For simplicity, we assume that the index category I is a directed poset (cf. Remark 2.1.1). Let

$$(\mathcal{C}, F) \coloneqq \operatorname*{colim}_i (\mathcal{C}_i, F_i).$$

The underlying category $\mathcal{C}$ is given by the 2-colimit $\mathcal{C} = 2\text{-}\operatorname{colim}_i \mathcal{C}_i$, whose objects are pairs (i, X_i) of an index $i \in I$ and an object $X_i \in \mathcal{C}_i$, and whose morphisms are defined by

$$\mathrm{Hom}_{\mathcal{C}}((i, X_i), (j, Y_j)) = \operatorname*{colim}_{k\geq i,j} \mathrm{Hom}_{\mathcal{C}_k}(\varphi_{ik}(X_i), \varphi_{jk}(Y_j)),$$

where $\varphi_{ik} \colon \mathcal{C}_i \to \mathcal{C}_k$ and $\varphi_{jk} \colon \mathcal{C}_j \to \mathcal{C}_k$ are the transition functors for the direct system.

Corollary 2.3.2. *The colimit* $(\mathcal{C}, F) = \operatorname{colim}_i(\mathcal{C}_i, F_i)$ *is a Galois category with fibre functor, and there is a canonical isomorphism of profinite groups* $\mathrm{Aut}(F) \cong \varprojlim_i \mathrm{Aut}(F_i)$*.*

Proof. By the Main Theorem of Galois Categories 2.3.1, the filtered diagram of Galois categories $(\mathcal{C}_i, F_i)$ is equivalent to a cofiltered diagram of profinite groups $(\Pi_i)_{i\in I}$. The limit $\Pi = \lim_i \Pi_i$ exists in the category of profinite groups. Its image $(\Pi\text{-}\mathrm{FinSet}, \mathrm{forget})$ under the equivalence of categories (2.3.1) is the colimit $(\mathcal{C}, F) = \operatorname{colim}_i(\mathcal{C}_i, F_i)$. □

2.3.3 Fundamental groups of pro-schemes

Let $X = (X_i)_{i\in I}$ be a pro-scheme. Assume that all X_i are connected. Let $\mathrm{Cov}(X_i)$ be the category of finite étale covers of X_i. Assume we are given compatible fibre functors F_i on the $\mathrm{Cov}(X_i)$ (for instance, via compatible geometric points), so that $(\mathrm{Cov}(X_i), F_i)$ becomes a direct system of Galois categories with fibre functor and we are in the situation of Section 2.3.2.

Definition 2.3.3. The **category of finite étale covers** of the pro-scheme $X = (X_i)_{i\in I}$ is defined as

$$\mathrm{Cov}(X) = 2\text{-}\operatorname*{colim}_{i\in I} \mathrm{Cov}(X_i).$$

The **profinite fundamental group** of X with respect to the fibre functors $(F_i)_{i\in I}$ is defined as $\pi_1(X,F) := \operatorname{Aut}(F)$, the automorphism group of the limit fibre functor $F = \lim F_i\colon \operatorname{Cov}(X) \to \operatorname{FinSet}$.

The associations $(X,F) \mapsto (\operatorname{Cov}(X),F)$ and hence $(X,F) \mapsto \pi_1(X,F)$, in Definition 2.3.3 are well-defined (independent of the presentation up to canonical isomorphism) and functorial. Namely, denoting by $\operatorname{ConnSch}_*$ the category of pairs (X,F) with X a connected scheme and F a fibre functor on $\operatorname{Cov}(X)$, the functor $\operatorname{ConnSch}_*^{\mathrm{op}} \to \operatorname{GalCat}$, $(X,F) \mapsto (\operatorname{Cov}(X),F)$ extends to $\operatorname{Pro}(\operatorname{ConnSch}_*)^{\mathrm{op}}$ via the 2-universal property of ind-categories [SGA 4-1, Exposé I, Prop. 8.7.3]:

$$\begin{tikzcd}
(\operatorname{ConnSch}_*)^{\mathrm{op}} \arrow[r, "\operatorname{Cov}"] \arrow[d, hook] & \operatorname{GalCat} \arrow[d, hook] \arrow[dr, "\mathrm{id}"] & \\
\operatorname{Pro}(\operatorname{ConnSch}_*)^{\mathrm{op}} \arrow[r, "\operatorname{Cov}"] & \operatorname{Ind}(\operatorname{GalCat}) \arrow[r, "\operatorname{colim}"] & \operatorname{GalCat}\,.
\end{tikzcd}$$

This uses the fact that the category GalCat admits small filtered colimits by Corollary 2.3.2. It follows also from Corollary 2.3.2 that we have a canonical isomorphism for every connected pro-scheme $(X_i)_{i\in I}$ with fibre functors $(F_i)_{i\in I}$:

$$\pi_1((X_i,F_i)_{i\in I}) = \varprojlim_i \pi_1(X_i,F_i). \tag{2.3.2}$$

2.3.4 Comparison with the fundamental group of the limit scheme

Assume now that the schemes X_i in the cofiltered diagram are quasi-compact and quasi-separated ("qcqs" for short) and that the transition maps are affine. Then the limit scheme $X_\infty := \lim_i X_i$ exists and is connected [EGA IV_3, Prop. (8.4.1) (ii)]. We want to compare the profinite fundamental group of the limit scheme X_∞ with that of the pro-scheme $"\varprojlim_i" X_i$.

For simplicity, assume that the index category I is a directed poset (cf. Remark 2.1.1).

Proposition 2.3.4. *With the assumptions above, the canonical functor is an equivalence*

$$2\text{-}\operatorname{colim} \operatorname{Cov}(X_i) \simeq \operatorname{Cov}(X_\infty). \tag{2.3.3}$$

Proof. For a scheme X, denote by $(\mathrm{Sch}/X)_{\text{fin.pres.}}$ the category of X-schemes of finite presentation. By [EGA IV_3, Thm. (8.8.2)], the canonical functor given by

$$(i, Y \to X_i) \mapsto (Y \times_{X_i} X_\infty \to X_\infty)$$

is an equivalence

$$2\text{-}\operatorname{colim}\,(\mathrm{Sch}/X_i)_{\text{fin.pres.}} \simeq (\mathrm{Sch}/X_\infty)_{\text{fin.pres.}}.$$

By restricting to the full subcategories $\mathrm{Cov}(-) \subseteq (\mathrm{Sch}/-)_{\text{fin.pres.}}$, it follows that the functor (2.3.3) is fully faithful. To show the essential surjectivity, let $Y_\infty \to X_\infty$ be a finite étale cover. Then there exist $i \in I$ and a morphism $Y_i \to X_i$ of finite presentation, such that $Y_\infty = Y_i \times_{X_i} X_\infty$ as an X_∞-scheme. It is enough to show that there exists some $j \geq i$ such that the base change $Y_i \times_{X_i} X_j \to X_j$ is finite étale. By [EGA IV$_3$, Thm. (8.10.5)], there exists some such $j_1 \geq i$ for the property "finite", and by [EGA IV$_4$, Prop. (17.7.8)] there exists some such $j_2 \geq i$ for the property "étale", so we can choose any j in the directed set I with $j \geq j_1$ and $j \geq j_2$. □

Corollary 2.3.5. *Let $(X_i)_{i\in I}$ be an inverse system of connected qcqs schemes with affine transition maps and let $X_\infty = \lim_i X_i$ be its limit. Let F_∞ be a fibre functor on $\mathrm{Cov}(X_\infty)$ and F_i the fibre functor induced by F_∞ on $\mathrm{Cov}(X_i)$ for each $i \in I$. Then the canonical map is an isomorphism of profinite groups*

$$\pi_1(X_\infty, F_\infty) \cong \varprojlim_i \pi_1(X_i, F_i).$$

Proof. The equivalence from Proposition 2.3.4 exhibits $(\mathrm{Cov}(X_\infty), F_\infty)$ as the colimit of the Galois categories with fibre functors $(\mathrm{Cov}(X_i), F_i)$. By the Main Theorem of Galois Categories 2.3.1 (and more specifically, Corollary 2.3.2), this is equivalent to the claimed isomorphism of fundamental groups. □

Remark 2.3.6. Corollary 2.3.5 can be rephrased as saying that for an inverse system $(X_i)_{i\in I}$ of connected qcqs schemes with affine transition maps, the canonical morphism $\varprojlim_i X_i \to$ "$\varprojlim_i X_i$" from the limit scheme to the pro-scheme induces an isomorphism on fundamental groups.

2.4 Localisations of curves

When considering localisations of curves, it is convenient to specify a set of *closed* points at which to localise. The result should, in addition to the specified closed points, always include the generic point. Hence, we define:

Definition 2.4.1 (Localisation of a curve at a set of closed points)**.** Let X be a one-dimensional integral scheme with generic point η_X. For a set of closed points $S \subseteq X_{\mathrm{cl}}$, define "$X_S$" (resp. X_S) as the formal localisation (resp. the localisation) of X at $S \cup \{\eta_X\}$ in the sense of Definition 2.2.1.

Remarks 2.4.2.

(1) If $S \neq \emptyset$, then the localisations at S and at $S \cup \{\eta_X\}$ are automatically equal (see Remark 2.2.2 (5)). In the case $S = \emptyset$, the localisation $X_\emptyset$ at $\emptyset$ in the sense of Definition 2.2.1 would result in the empty scheme, but with our convention of always including the generic point, we have $X_\emptyset = \{\eta_X\}$.

(2) An equivalent way of defining the localisation of a curve X at a set of closed points $S \subseteq X_{\mathrm{cl}}$ is via "X_S" = "$\varprojlim_{U \supseteq S} U$" and $X_S = \varprojlim_{U \supseteq S} U$, with U running through the *dense* open subschemes of X containing S.

(3) Assume that there is a cofinal system $(U_i)_{i \in I}$ of open subschemes of X containing S such that all U_i are quasi-compact and quasi-separated and all inclusions between them are affine. Then the localisation X_S exists and has underlying topological space $|X_S| = S \cup \{\eta_X\} \subseteq |X|$ according to Remark 2.2.2 (6).

Proposition 2.4.3. *Let k be a field and X/k a normal, connected, one-dimensional variety over k. Then for every set of closed points $S \subseteq X_{\mathrm{cl}}$, the localisation X_S exists. It is affine unless X is proper and $S = X_{\mathrm{cl}}$.*

Proof. Let $X \hookrightarrow \overline{X}$ be the normal completion of X (see e.g. [EGA II, Cor. (7.4.11)]). The localisation of X at S, if it exists, is equal to the localisation of $\overline{X}$ at the same set S by transitivity (Proposition 2.2.4). So we may replace X with $\overline{X}$ and assume that X/k is proper, or equivalently projective [Stacks, Tag 0A26]. If $S = X_{\mathrm{cl}}$, then X_S exists and is equal to X (see Example 2.2.3 (b)), which is not affine. Assume $S \subsetneq X_{\mathrm{cl}}$. It suffices to show that every open subscheme $U \subsetneq X$ strictly smaller than X is affine, for then the limit $X_S = \varprojlim_{S \subseteq U \subsetneq X} U$ exists and is affine, being an inverse limit of affine schemes. Fix $U = X \setminus \{x_1, \ldots, x_n\} \subsetneq X$. Using the Riemann–Roch Theorem [Liu02, Theorem 7.3.17], there exists a rational function f on X which is regular on U and has poles at all x_i. It defines a finite morphism $f\colon X \to \mathbb{P}^1_k$ under which U is the preimage of the affine line $\mathbb{A}^1_k$, hence U is affine. $\square$

2.4.1 Ringed space structure

As a ringed space, the localisation of a curve can be described as follows:

Proposition 2.4.4. *Let k be a field, let X/k a normal, connected, one-dimensional variety over k and $S \subseteq X_{\mathrm{cl}}$ a set of closed points. The localisation X_S is integral of dimension ≤ 1 with function field K equal to that of X. Its underlying space is $|X_S| = S \cup \{\eta_X\} \subseteq |X|$, and the ring of regular functions on an open subset $U \subseteq |X_S|$ is given by the intersection of local rings in K:*

$$\mathcal{O}_{X_S}(U) = \bigcap_{x \in U} \mathcal{O}_{X,x}.$$

Proof. The fact that the underlying space of X_S is equal to $S \cup \{\eta_X\}$ was noted in Remark 2.4.2 (3) above. This implies that X_S is irreducible and of dimension ≤ 1. For $x \in |X_S|$, we have the equality of local rings $\mathcal{O}_{X_S,x} = \mathcal{O}_{X,x}$ by the transitivity of localisations for the subspaces $\{x\} \subseteq S \cup \{\eta_X\} \subseteq |X|$. In particular, X_S is reduced, hence integral, with function field K equal to that of X. For an open subset $U \subseteq X_S$, the comparison of local rings also implies $\mathcal{O}_{X_S}(U) = \bigcap_{x \in U} \mathcal{O}_{X_S,x} = \bigcap_{x \in U} \mathcal{O}_{X,x}$. $\square$

Corollary 2.4.5. *Let X/k and $S \subseteq X_{\mathrm{cl}}$ be as in Proposition 2.4.4. The ring of regular functions on the localisation X_S is given by the intersection of local rings in K:*

$$\mathcal{O}(X_S) = \bigcap_{x \in S} \mathcal{O}_{X,x}.$$

Proof. Immediate from Proposition 2.4.4. □

2.4.2 Compactification

Let k be a field and W/k a k-scheme which arises as the localisation of a normal, proper curve X/k at a set of closed points $S \subseteq X_{\mathrm{cl}}$. The curve X/k and the set S are uniquely determined by W. We explain how they can be functorially recovered from W.

Let K be the function field of W and define $\overline{W}/k$ as the unique normal, proper curve with function field K. It can be constructed as follows (see e.g. [Sza09, §4.4] for details): the set of closed points of $\overline{W}$ is the set of discrete valuation rings in K containing k. The underlying topological space of $\overline{W}$ consists of those plus one generic point. The proper closed subsets are finite sets of closed points. The local ring $\mathcal{O}_{\overline{W},x} \subseteq K$ of a closed point $x \in \overline{W}_{\mathrm{cl}}$ is precisely the discrete valuation ring represented by x. Finally, the structure of a ringed space is given by

$$\mathcal{O}_{\overline{W}}(U) = \bigcap_{x \in U} \mathcal{O}_{\overline{W},x}$$

for every open subset $U \subseteq \overline{W}$. The resulting locally ringed space $\overline{W}$ is a normal, proper algebraic curve over k with function field K. Every codimension one point of W can be identified with a closed point of $\overline{W}$ via its associated discrete valuation ring.

Proposition 2.4.6. *In the above situation, W/k is the localisation of $\overline{W}$ at the set of codimension one points of W.*

Proof. By the theory of proper, normal curves [Sza09, Cor. 4.4.8], the identity isomorphism $K \cong K$ between the function fields of $\overline{W}$ and X extends uniquely to an isomorphism $\overline{W} \cong X$ of curves over k. Hence, W is a localisation of $\overline{W}$. By Remark 2.4.2 (3), the set of closed points of $\overline{W}$ at which the localisation yields W is determined as the set of codimension one points of W. □

The reconstruction of X and S from the localisation X_S is functorial not only with respect to isomorphisms but more generally with respect to finite dominant morphisms:

Proposition 2.4.7. *Let k be a field. Let X/k and Y/k be normal, proper curves, and $S \subseteq X_{\mathrm{cl}}$ and $T \subseteq Y_{\mathrm{cl}}$ sets of closed points. Let $f\colon Y_T \to X_S$ be a finite dominant morphism. Then f extends uniquely to a finite dominant morphism $\overline{f}\colon Y \to X$ with $\overline{f}(T) \subseteq S$:*

$$\begin{tikzcd} Y_T \arrow[r] \arrow[d, "f"] & Y \arrow[d, dashed, "\exists!\,\overline{f}"] \\ X_S \arrow[r] & X. \end{tikzcd}$$

Proof. The finite dominant morphism $f\colon Y_T \to X_S$ induces a finite extension of function fields $f^*\colon \kappa(X) \to \kappa(Y)$, which defines a finite dominant morphism $\overline{f}\colon Y \to X$. Identifying closed points of Y (resp. X) with discrete valuation rings in $\kappa(Y)$ (resp. $\kappa(X)$) containing k, the morphism $\overline{f}$ is given on closed points by $f^{*-1}(\mathcal{O}_{Y,y}) = \mathcal{O}_{X,\overline{f}(y)}$ for $y \in Y_{\mathrm{cl}}$. Since the local rings of codimension one points of Y_T (resp. X_S) agree with those of Y (resp. X), the restriction of $\overline{f}$ to Y_T agrees with f as a map of topological spaces. But a dominant morphism of integral schemes is uniquely determined by its underlying map on topological spaces together with its induced function field extension. This shows that the diagram commutes, i.e. that $\overline{f}$ extends f. □

2.4.3 Characterisation

We can characterise those k-schemes that arise as localisations of curves:

Proposition 2.4.8. *Let k be a field and W/k a normal, integral, noetherian, separated k-scheme whose function field K is finitely generated of transcendence degree 1 over k. Then W arises as the localisation of a normal, proper curve over k at a set of closed points.*

Proof. The Dimension Inequality [Mat89, Theorem 15.5] applied to any dense open affine subscheme of W shows that $\dim(W) \leq \operatorname{trdeg}(K/k) = 1$. For any codimension one point $w \in W^{(1)}$, the local ring $\mathcal{O}_{W,w}$ is a normal, noetherian domain of Krull dimension one, hence a discrete valuation ring in K containing k. Let X/k be the smooth, proper curve with function field K (whose construction was recalled above). Sending the generic point of W to the generic point of X and every codimension one point w of W to the closed point of X which represents the discrete valuation ring $\mathcal{O}_{W,w}$ defines a map on topological spaces $|W| \to |X|$. Since W is assumed separated, the map is injective, so we can identify $|W|$ with a subspace of $|X|$. Let $S \subseteq X_{\mathrm{cl}}$ be the set of codimension one points of W. Then W and the localisation X_S have the same underlying space, in other words we have a homeomorphism $|W| \cong |X_S|$. For every $w \in W^{(1)}$, the local rings in K of W and X at w coincide by construction: $\mathcal{O}_{W,w} = \mathcal{O}_{X,w}$. For every open subset $U \subseteq W$, we have $\mathcal{O}_W(U) = \cap_{x\in U}\mathcal{O}_{W,x} = \cap_{x\in U}\mathcal{O}_{X,x}$. Comparing with Proposition 2.4.4, we see that the map $W \cong X_S$ is an isomorphism of locally ringed spaces over k, i.e. an isomorphism of k-schemes. □

2.4.4 Descent

The property of being a localisation of a curve can be checked after a scalar extension:

Proposition 2.4.9. *Let ℓ/k be an extension of fields and let W be a k-scheme such that $W \otimes_k \ell$ equals the localisation of a normal, proper curve over ℓ at a set of closed points. Then the same holds for W/k.*

Proof. We show that the properties of Proposition 2.4.8 which characterise localisations of curves are inherited by W from $W \otimes_k \ell$. The projection $W \otimes_k \ell \to W$ is surjective, which implies that W is quasi-compact and irreducible. For any affine open subscheme $U \subseteq W$, the homomorphism $\mathcal{O}(U) \to \mathcal{O}(U) \otimes_k \ell$ is injective by flatness. The ring $\mathcal{O}(U) \otimes_k \ell = \mathcal{O}(U \otimes_k \ell)$ is integral, hence so is $\mathcal{O}(U)$. Thus, the scheme W is integral. The property of being noetherian is also inherited by $\mathcal{O}(U)$: any chain of ideals in $\mathcal{O}(U)$ gives rise to a chain of ideals in $\mathcal{O}(U) \otimes_k \ell$ by tensoring; the latter stabilises, and faithful flatness of $k \to \ell$ implies that the original chain stabilises in $\mathcal{O}(U)$. So W is locally noetherian, and hence noetherian by quasi-compactness. The morphism $W \to \mathrm{Spec}(k)$ is separated [EGA IV$_2$, Prop. (2.7.1)] and W is normal [EGA IV$_2$, Cor. (6.5.4)] by faithfully flat descent. If K and L denote the function fields of W and $W \otimes_k \ell$, respectively, then we have $\mathrm{trdeg}(K/k) = \mathrm{trdeg}(L/\ell) = 1$ by [EGA IV$_2$, Prop. (4.2.1)]. In conclusion, the characterisation of localisations of curves from Proposition 2.4.8 applies to W. □

2.4.5 Finite étale covers

Finite étale covers of the localisation of a curve are uniquely determined both by their restriction to the generic point and by their extension to the complete ambient curve:

Proposition 2.4.10. *Let k be a field, X/k a normal, proper curve and $S \subseteq X_{\mathrm{cl}}$ a set of closed points. Let K be the common function field of X and X_S. There are canonical equivalences between the following categories:*

(1) connected finite étale covers of X_S;

(2) $\big($finite separable extensions L/K which are unramified over $X_S\big)^{\mathrm{op}}$;

(3) finite branched covers $Y \to X$ which are unramified over X_S.

Here, an extension L/K is called **unramified over** X_S if the normalisation of X_S in L is unramified (equivalently, étale) over X_S. A **finite branched cover** $Y \to X$ is a finite morphism of integral, normal schemes such that the function field extension $\kappa(Y)/\kappa(X)$ is separable. The finite branched cover is called **unramified over** X_S if the base change $X_S \times_X Y \to X_S$ is unramified (equivalently, étale). The equivalences (1) $\simeq$ (2) and (3) $\simeq$ (2) are both given by taking function fields (from left to right) and normalisations of X_S, resp. X, in L/K (from right to left).

Proof of Proposition 2.4.10. The equivalence (1) $\simeq$ (2) holds more generally for any normal, integral, locally noetherian scheme [SGA 1, Exp. I, Cor. 10.3]. By the theory of normal, proper curves [Sza09, Cor. 4.4.8], we have an anti-equivalence between finite extensions L/K and finite dominant morphisms from normal, connected curves $Y \to X$. It remains to see that $Y \to X$ being unramified over X_S is equivalent to $\kappa(Y)/K$ being separable and unramified over X_S. The fibre product $X_S \times_X Y$ is a localisation of Y by the compatibility of localisation with base change along closed morphisms (Lemma 2.2.5), hence $X_S \times_X Y$ is in particular normal and integral with function field $\kappa(Y)$. It is also finite over X_S, thus integral over X_S. So $X_S \times_X Y$ equals the normalisation of X_S in $\kappa(Y)$. This shows that $\kappa(Y)/K$ is unramified over X_S if and only if $Y \to X$ is. In this case, $\kappa(Y)/K$ must be separable since $\mathrm{Spec}(\kappa(Y)) \to \mathrm{Spec}(K)$ is the restriction of the unramified morphism $X_S \times_X Y \to X_S$ to the generic point of X_S. □

Corollary 2.4.11. *Let k be a field, X/k a normal, proper curve and $S \subseteq X_{\mathrm{cl}}$ a set of closed points. Every connected finite étale cover of X_S is of the form $Y_{f^{-1}(S)} \to X_S$ for some finite branched cover $f\colon Y \to X$ which is unramified over X_S.*

Proof. We have the equivalence (1) $\simeq$ (3) in Proposition 2.4.10 between connected finite étale covers of X_S and finite branched covers of X which are unramified over X_S. The functor from right to left sends $Y \to X$ to the normalisation of X_S in the function field of Y. As shown in the proof of Proposition 2.4.10, the result is the fibre product $X_S \times_X Y$, which equals $Y_{f^{-1}(S)}$ by Lemma 2.2.5. □

2.5 Profinite étale covers

2.5.1 General remarks on profinite étale covers

Generally, in the theory of the étale fundamental group of a scheme, one considers only covers of *finite* degree. One reason for this restriction is that one would like to have a comparison isomorphism with the topological fundamental group in the case of a smooth complex algebraic variety. This amounts to an equivalence of categories between topological covers and algebraic (étale) covers. For finite covers, one has such an equivalence by Riemann's Existence Theorem [SGA 1, Exp. XII, Thm. 5.1], which however fails for covers of infinite degree. For example, every finite connected topological cover of $\mathbb{G}_{\mathrm{m}}(\mathbb{C}) = \mathbb{C}^\times$ is of the form $\mathbb{C}^\times \to \mathbb{C}^\times$, $z \mapsto z^n$ for some $n \in \mathbb{N}$, and hence induced by an algebraic finite étale cover of $\mathbb{G}_m$, wheras the topological universal cover $\exp\colon \mathbb{C} \to \mathbb{C}^\times$ is transcendental and has no analogue on the algebraic side. In the world of schemes, the role of the universal cover of is played instead by the universal pro-(finite étale) cover.

Assume that X is a connected scheme and F is a fibre functor on the category $\mathrm{Cov}(X)$ of finite étale covers. By a *pointed cover* of X, we mean a pair (Y, y) of a finite étale cover $Y \to X$ together with a distinguished element $y \in F(Y)$ of the fibre. The universal pro-(finite étale cover) of X is by definition the pro-object $(X^{\mathrm{univ}}, x^{\mathrm{univ}})$ of the category of pointed finite étale covers of X pro-representing the fibre functor F. This means, there are natural bijections

$$\mathrm{Hom}_{\mathrm{Pro}(\mathrm{Cov}(X))}(X^{\mathrm{univ}}, Y) \cong F(Y),$$

given by $f \mapsto f(x^{\mathrm{univ}})$, for all $Y \in \mathrm{Cov}(X)$. The pro-object X^{univ} is not a scheme but rather given by an inverse system of finite étale covers $(X_i)_{i \in I}$, and the universal point x^{univ} is really a compatible family of points $(x_i)_{i \in I}$ with $x_i \in F(X_i)$.

Consider more generally any pro-object $(Y_i)_{i \in I}$ of the category of finite étale covers of a scheme X. Since finite morphisms are affine, the inverse system $(Y_i)_{i \in I}$ corresponds to a direct system $(\mathcal{A}_i)_{i \in I}$ of quasi-coherent $\mathcal{O}_X$-algebras. The direct limit $\mathcal{A}_\infty := \varinjlim_{i \in I} \mathcal{A}_i$ corresponds via the relative Spec functor to a scheme $Y_\infty := \underline{\mathrm{Spec}}(\mathcal{A}_\infty)$ which is affine over X. The scheme Y_∞ is then the inverse limit of the Y_i:

$$Y_\infty = \varprojlim_i Y_i.$$

In this way, every pro-(finite étale cover) gives rise to an actual scheme by taking the limit. It is sometimes convenient to work with the limit scheme rather than the pro-scheme. For example, this conforms with the habit of studying infinite algebraic field extensions rather than direct systems of finite extensions.

Definition 2.5.1. Let X be a scheme. A **profinite étale cover** of X is a morphism $Y_\infty \to X$ which arises as the limit $\varprojlim_{i \in I} Y_i$ of an inverse system of finite étale covers $Y_i \to X$.

The following proposition provides justification for not always carefully distinguishing between a pro-(finite étale cover) and the associated profinite étale cover arising as its limit, at least if the base is quasi-compact and quasi-separated.

Proposition 2.5.2. *Let X be a quasi-compact and quasi-separated scheme. Then the functor $(Y_i)_{i \in I} \mapsto \varprojlim_{i \in I} Y_i$ from* $\mathrm{Pro}(\mathrm{Cov}(X))$ *to profinite étale covers of X is an equivalence.*

Proof. To define a quasi-inverse functor in the other direction, let $Z \to X$ be any profinite étale cover. Let J_Z be the category of factorisations $Z \to Z_j \to X$ with Z_j finite étale over X. This category is essentially small (i.e. the set of isomorphism classes of objects is small) and filtered, so that $(Z_j)_{j\in J_Z}$ is a pro-object in $\mathrm{Pro}(\mathrm{Cov}(X))$. We claim that the functor $Z \mapsto (Z_j)_{j\in J_Z}$ is left adjoint to the functor $\varprojlim$, i.e. we have natural bijections

$$\mathrm{Hom}_{\mathrm{Pro}(\mathrm{Cov}(X))}((Z_j)_{j\in J_Z}, (Y_i)_{i\in I}) \cong \mathrm{Hom}_X(Z, \varprojlim_{i\in I} Y_i)$$

for all $(Y_i)_{i\in I} \in \mathrm{Pro}(\mathrm{Cov}(X))$ and all profinite étale covers $Z \to X$. Since both sides commute with inverse limits in the second argument, we may assume that the pro-object $(Y_i)_{i\in I}$ consists of a single finite étale cover $Y_0 \to X$. Using the definition of morphisms in the pro-category, the claimed bijection then takes the form

$$\varinjlim_{j\in J_Z} \mathrm{Hom}_X(Z_j, Y_0) \cong \mathrm{Hom}_X(Z, Y_0).$$

This bijection exists tautologically since every X-morphism in $\mathrm{Hom}_X(Z, Y_0)$ is a factorisation $Z \to Y_0 \to X$ which appears in the inverse system $(Z_j)_{j\in J_Z}$.

For a profinite étale cover $Z \to X$, the adjunction morphism $Z \to \varprojlim_{j\in J_Z} Z_j$ is an isomorphism since Z is by definition an inverse limit of finite étale covers of X, and all of those appear in the inverse system $(Z_j)_{j\in J_Z}$. For a pro-(finite étale cover) $(Y_i)_{i\in I}$, we have the adjunction morphism $(Y_j)_{j\in J_Y} \to (Y_i)_{i\in I}$, whose source is the pro-system of all finite étale factorisations $\varprojlim_{i\in I} Y_i \to Y_j \to X$. It is an isomorphism if the Y_i are cofinal among all finite étale factorisations. This holds since X is assumed quasi-compact and quasi-separated [EGA IV_3, Prop. 8.13.1]. □

For a connected, quasi-compact and quasi-separated scheme X with a fibre functor F on $\mathrm{Cov}(X)$, Grothendieck's Galois Theory for finite étale covers extends in a straightforward way to profinite étale covers. For example, the equivalence between finite étale covers and finite $\pi_1(X,F)$-sets extends (by taking pro-categories) to an equivalence between profinite étale covers and profinite $\pi_1(X,F)$-sets, i.e. profinite sets with a continuous $\pi_1(X,F)$-action. Just like for finite étale covers, we may define a **profinite étale Galois cover** of X to be a profinite étale cover $Y \to X$ with Y connected such that $\mathrm{Aut}_X(Y)$ acts transitively on $F(Y)$. In this case, the automorphism group

$$\mathrm{Gal}(Y/X) \coloneqq \mathrm{Aut}_X(Y)^{\mathrm{op}}$$

carries a profinite topology with a neighbourhood basis of the identity consisting of the subgroups $\mathrm{Gal}(Y/Y')$ with $Y \to Y' \to X$ a finite connected subcover. In particular, it makes sense to speak of the universal profinite étale cover, the maximal abelian (profinite étale) cover and so on.

2.5.2 Profinite étale covers of integral, normal schemes

Proposition 2.5.3. *Let Z be a normal, integral, noetherian scheme with function field K. There is a canonical equivalence between the following categories:*

(1) connected profinite étale covers of Z;

(2) $\big($separable extensions L/K which are unramified over $Z\big)^{\mathrm{op}}$.

Proof. The equivalence, given by taking function fields and normalisations, respectively, holds for finite covers and finite extensions by [SGA 1, Exp. I, Cor. 10.3]. The profinite version follows with Proposition 2.5.2 by taking pro-categories, using the fact that noetherian schemes are quasi-compact and quasi-separated. □

2.5.3 Kummer theory for regular, integral schemes

Let Z be a regular, integral, noetherian scheme and let K be its function field. Let $n \in \mathbb{N}$ be a natural number which is invertible on Z, and assume $\mu_n \subseteq \mathcal{O}(Z)$. By Proposition 2.5.3, abelian covers of exponent n of Z (i.e. Galois covers with abelian, n-torsion Galois group) correspond to unramified abelian extensions of exponent n of K. Kummer theory tells us that all such extensions arise by adjoining n-th roots of subgroups of $K^\times$ containing $K^{\times n}$. We determine the subgroup corresponding to the maximal abelian cover of exponent n. For a codimension one point $z \in Z^{(1)}$, we denote by $v_z \colon K^\times \twoheadrightarrow \mathbb{Z}$ the associated discrete valuation on the function field.

Proposition 2.5.4. *In the above situation, let $Z' \to Z$ be the maximal abelian cover of exponent n and let K'_Z/K be the corresponding extension of the function field. Then K'_Z is obtained from K by adjoining n-th roots of the subgroup $\Delta_Z \subseteq K^\times$ given by*

$$\Delta_Z = \{f \in K^\times \mid v_z(f) \equiv 0 \bmod n \text{ for all } z \in Z^{(1)}\}.$$

Proof. By Kummer theory, the extension K'_Z/K is generated by elements of the form $f^{1/n}$ with $f \in K^\times$. It suffices to show that an extension $K(f^{1/n})/K$ is unramified over Z if and only if $v_z(f) \equiv 0 \bmod n$ for all codimension one points $z \in Z^{(1)}$. Given $f \in K^\times$, assume that $v_z(f) \not\equiv 0 \bmod n$ for some $z \in Z^{(1)}$. Let $w|v_z$ be a valuation on $K(f^{1/n})$ extending v_z. Then we have $w(f^{1/n}) = \frac{1}{n}v_z(f) \notin \mathbb{Z}$, so that the extension $K(f^{1/n})/K$ is ramified over z.

Conversely, assume that $v_z(f) \equiv 0 \bmod n$ for all $z \in Z^{(1)}$ and let $Y \to Z$ be the normalisation of Z in $K(f^{1/n})$. We have to show that Y is étale over Z. Since Z is assumed regular, it is in particular normal, and since n is invertible in K, the extension $K(f^{1/n})/K$ is finite separable. This implies that Y is finite over Z. We can thus apply the Zariski–Nagata purity theorem, by which it suffices to show that Y is étale over every codimension one point $z \in Z^{(1)}$. Let $t_z \in \mathfrak{m}_z$ be a local parameter at z and write $f = t_z^{v_z(f)}u$ with $u \in \mathcal{O}_{Z,z}^\times$. Since $v_z(f) \equiv 0 \bmod n$, we have $K(f^{1/n}) = K(u^{1/n})$. Replacing f with u, we may therefore assume that f is invertible at z. The $\mathcal{O}_{Z,z}$-algebra $\mathcal{O}_{Z,z}[X]/(X^n - f)$ is then standard étale since f and n are both invertible. It is in particular normal and therefore equals the normalisation of $\mathcal{O}_{Z,z}$ in $K[X]/(X^n - f)$. The field $K(f^{1/n})$ is a direct factor of $K[X]/(X^n - f)$, so the normalisation of $\mathcal{O}_{Z,z}$ in $K(f^{1/n})$ is a direct factor of $\mathcal{O}_{Z,z}[X]/(X^n - f)$ and hence étale over $\mathcal{O}_{Z,z}$. Thus, the extension $K(f^{1/n})/K$ is unramified over z. □

Remark 2.5.5. In the setting of Proposition 2.5.4, the subgroup $\Delta_Z \subseteq K^\times$ clearly contains both $K^{\times n}$ and the functions $\mathcal{O}(Z)^\times$ which are everywhere invertible. Moreover, every n-torsion element of the Picard group $\operatorname{Pic}(Z)$ gives rise to elements of Δ_Z. Namely, if $\mathcal{L}$ is a line bundle on Z such that $\mathcal{L}^{\otimes n}$ is trivial, then let $0 \neq s \in \mathcal{L}_K$ be a nonzero rational section and $t \in \Gamma(Z, \mathcal{L}^{\otimes n})$ a global section of $\mathcal{L}^{\otimes n}$ defining a trivialisation $\mathcal{O}_Z \cong \mathcal{L}^{\otimes n}$. Then the quotient $f := s^n/t$ is an element of $K^\times$ and it is easy to see by looking at stalks of $\mathcal{L}$ at codimension one points that f is contained in Δ_Z. A different choice of s changes f by an element of $K^{\times n}$; a different choice of t changes f by an element of $\mathcal{O}(Z)^\times$.

Since the Picard group of Z coincides with the Weil divisor class group, the construction can also be described in terms of Weil divisors. Namely, given a divisor D on Z such that nD is principal, let $f \in K^\times$ be a rational function with $\operatorname{div}(f) = nD$. Then clearly, $v_z(f) \equiv 0 \bmod n$ for all $z \in Z^{(1)}$, so that $f \in \Delta_Z$. The function f with $\operatorname{div}(f) = nD$ is determined up to an invertible function in $\mathcal{O}(Z)^\times$. If D is replaced with a linearly equivalent divisor, say $D' = D + \operatorname{div}(g)$, then f is changed by $g^n \in K^{\times n}$.

We have thus a homomorphism $\operatorname{Pic}(Z)[n] \to K^\times/K^{\times n}\mathcal{O}(Z)^\times$, and the preimage in $K^\times$ of its image is precisely the group Δ_Z which corresponds by Kummer theory to the maximal abelian cover of exponent n of Z.

Remark 2.5.6. The group Δ_Z appears in the short exact sequence

$$0 \longrightarrow \mathcal{O}(Z)^\times/\mathcal{O}(Z)^{\times n} \longrightarrow \Delta_Z/K^{\times n} \longrightarrow \operatorname{Pic}(Z)[n] \longrightarrow 0,$$

with the map to $\operatorname{Pic}(Z)[n]$ given by $f \mapsto \frac{1}{n}\operatorname{div}(f)$. The exactness is easily verified. The sequence can also be obtained from the Kummer sequence on the étale site of Z. This requires the calculation of

$$\mathrm{H}^1(Z, \mu_n) \cong \Delta_Z/K^{\times n}.$$

Here is a sketch of how this isomorphism can be derived:

1. the Leray spectral sequence for the sheaf μ_n under $j\colon \operatorname{Spec}(K) \to Z$ identifies $\mathrm{H}^1(Z, \mu_n)$ with the kernel of a map $\mathrm{H}^1(K, \mu_n) \to \mathrm{H}^0(Z, \mathrm{R}^1 j_* \mu_n)$;
2. the Kummer sequence on $\operatorname{Spec}(K)$ and the vanishing of $\mathrm{R}^1 j_* \mathbb{G}_m = 0$ yield an isomorphism $\mathrm{R}^1 j_* \mu_n \cong (j_*\mathbb{G}_m)/n$;
3. the exact sequence for the sheaf of divisors $\underline{\mathrm{Div}}_Z = (j_*\mathbb{G}_m)/\mathbb{G}_m$ on Z yields an isomorphism $(j_*\mathbb{G}_m)/n \cong \underline{\mathrm{Div}}_Z/n$;
4. combining the above, $\mathrm{H}^1(Z, \mu_n)$ is identified with the kernel of the divisor map $K^\times/K^{\times n} \to \operatorname{Div}(Z)/n$, which equals $\Delta_Z/K^{\times n}$ by definition.

3 Sections and points

The starting point of the section conjecture for curves is the observation that rational points induce sections on fundamental groups. We sketched this procedure in the introduction for the fundamental group of a proper curve and in the birational variant for the absolute Galois group of its function field. Here, we want to generalise this to localisations of curves (which lie in between the generic point and the full curve), and to quotients of the fundamental groups. The relation between a rational point x and a section s induced by it is that the image of s is contained in a decomposition group of x, in which case we say that *s lies over x*. In Section 3.1, the terminology is introduced for arbitrary quotients of the fundamental group of the localisation of a curve. We then show for the class of pro-$\mathcal{C}$ quotients that sections do exist over every rational point. This entails in particular the case of liftable sections for the maximal $\mathbb{Z}/p\mathbb{Z}$-abelian quotient. In Section 3.2 we observe the functorial behaviour of sections lying over points. Afterwards, we address various questions of Galois descent in the section conjecture. That is, we are assuming the validity of some statement over a field extension ℓ/k and deduce a similar statement over k. The first such question in Section 3.3 concerns the existence of sections over rational points. We show that over each rational point there exist liftable and twice-liftable sections with respect to ℓ/k in the sense of Definition 1.4.3, i.e. on the level of the Galois groups $\mathrm{Gal}(\ell'/k)$ and $\mathrm{Gal}(\ell''/k)$. In Section 3.4 we define abstractly a Galois action on conjugacy classes of sections and discuss its interpretation in terms of nonabelian group cohomology. Finally, in Section 3.5, we apply the abstract machinery to prove descent statements in the context of the section conjecture. Specifically, we prove Theorem 1.4.4 about deducing a version of the liftable section conjecture over k from its validity over a finite Galois extension ℓ/k, and Theorem 1.4.6 about deducing the section conjecture for the full fundamental groups from the liftable variant.

3.1 Sections induced by rational points

Let k be a field of characteristic zero and let X/k be a smooth, proper, geometrically connected curve. Let $S \subseteq X_{\mathrm{cl}}$ be an arbitrary set of closed points and X_S the localisation of X at S (Definition 2.4.1). Let $\widetilde{X_S} \to X_S$ be a profinite étale Galois cover. Let $\tilde{k}$ be the field of constants of $\widetilde{X_S}$, i.e. the relative algebraic closure of k in the function field of $\widetilde{X_S}$. Then $\widetilde{X_S}$ is geometrically connected over $\tilde{k}$ and we have the following short exact sequence whose terms are quotients of the fundamental exact sequence (1.1.1) for X_S/k:

$$1 \longrightarrow \mathrm{Gal}(\widetilde{X_S}/X_S \otimes_k \tilde{k}) \longrightarrow \mathrm{Gal}(\widetilde{X_S}/X_S) \longrightarrow \mathrm{Gal}(\tilde{k}/k) \longrightarrow 1. \qquad (3.1.1)$$

This is the general setting in this Chapter 3. We want to explain how k-rational points of X can give rise to sections of (3.1.1) in order to state variants of the section conjecture which involve quotients of the full fundamental groups.

3.1.1 Sections over points

Assume first that $x \in X_{\mathrm{cl}}$ is a closed point, not necessarily k-rational. Let $\widetilde{X} \to X$ be the normalisation of X in the function field of $\widetilde{X_S}$. It can also be described as the inverse limit of the finite branched covers of X which are obtained as the compactifications of all connected finite étale subcover of $\widetilde{X_S} \to X_S$. Choose a point $\tilde{x}$ in $\widetilde{X}$ over x. Via the functoriality of the association $(\widetilde{X_S} \to X_S) \mapsto (\widetilde{X} \to X)$, the right Galois action of $\mathrm{Gal}(\widetilde{X_S}/X_S)$ on $\widetilde{X_S}$ extends to an action on $\widetilde{X}$ by X-automorphisms.

Definition 3.1.1. The **decomposition group** $\mathrm{D}_{\tilde{x}|x} \subseteq \mathrm{Gal}(\widetilde{X_S}/X_S)$ of $\tilde{x}|x$ is defined as the stabiliser of $\tilde{x}$ under the action of $\mathrm{Gal}(\widetilde{X_S}/X_S)$ on the set of closed points of $\widetilde{X}$. The normal subgroup $\mathrm{I}_{\tilde{x}|x} \subseteq \mathrm{D}_{\tilde{x}|x}$ of elements acting trivially on the residue field $\kappa(\tilde{x})$ is the **inertia group** of $\tilde{x}|x$.

Remarks 3.1.2.

(1) The decomposition group $\mathrm{D}_{\tilde{x}|x}$ and inertia group $\mathrm{I}_{\tilde{x}|x}$ are closed subgroups of the profinite group $\mathrm{Gal}(\widetilde{X_S}/X_S)$.

(2) Since we defined $\mathrm{Gal}(\widetilde{X_S}/X_S) = \mathrm{Aut}(\widetilde{X_S}/X_S)^{\mathrm{op}}$, Galois groups of coverings of schemes act from the right. We write the action of $\mathrm{Gal}(\widetilde{X_S}/X_S)$ on closed points of $\widetilde{X}$ as $(\tilde{x}, \gamma) \mapsto \tilde{x}^{\gamma}$. This is translated into a left action via the rule $\gamma(\tilde{x}) \coloneqq \tilde{x}^{\gamma^{-1}}$.

(3) Decomposition groups and inertia groups behave as follows under conjugation: for any $\gamma \in \mathrm{Gal}(\widetilde{X_S}/X_S)$ we have $\gamma\,\mathrm{D}_{\tilde{x}|x}\,\gamma^{-1} = \mathrm{D}_{\gamma(\tilde{x})|x}$ and $\gamma\,\mathrm{I}_{\tilde{x}|x}\,\gamma^{-1} = \mathrm{I}_{\gamma(\tilde{x})|x}$.

We have the following diagram with exact rows:

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{I}_{\tilde{x}|x} \arrow[r] \arrow[d, hook] & \mathrm{D}_{\tilde{x}|x} \arrow[r] \arrow[d, hook] & \mathrm{Gal}(\kappa(\tilde{x})/\kappa(x)) \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \mathrm{Gal}(\widetilde{X_S}/X_S \otimes_k \tilde{k}) \arrow[r] & \mathrm{Gal}(\widetilde{X_S}/X_S) \arrow[r] & \mathrm{Gal}(\tilde{k}/k) \arrow[r] & 1. \end{tikzcd} \qquad (3.1.2)$$

The vertical map on the right is induced by the inclusions in the field diagram

$$\begin{tikzcd}
 & \kappa(\tilde{x}) \arrow[dd, dash] \\
\tilde{k} \arrow[ur, dash] \arrow[dd, dash] & \\
 & \kappa(x) \\
k \arrow[ur, dash] &
\end{tikzcd}$$

Definition 3.1.3. A section $s\colon \operatorname{Gal}(\tilde{k}/k) \to \operatorname{Gal}(\widetilde{X_S}/X_S)$ of (3.1.1) is called **section over** x if its image is contained in a decomposition group $\mathrm{D}_{\tilde{x}|x}$ for some $\tilde{x}$ over x.

Remarks 3.1.4. (1) If $\kappa(x) = k$ and $\kappa(\tilde{x}) = \tilde{k}$, then the right vertical map in (3.1.2) is an isomorphism and sections over x are the same as sections of the top row.

(2) Sections can be conjugated by elements of the kernel. More precisely: for any section $s\colon \operatorname{Gal}(\tilde{k}/k) \to \operatorname{Gal}(\widetilde{X_S}/X_S)$ and any $\gamma \in \operatorname{Gal}(\widetilde{X_S}/X_S \otimes_k \tilde{k})$, we have a conjugate section $\gamma(-)\gamma^{-1} \circ s$. If s is a section over x, say $\operatorname{im}(s) \subseteq \mathrm{D}_{\tilde{x}|x}$, then $\gamma(-)\gamma^{-1} \circ s$ has image contained in the conjugate decomposition group $\gamma\, \mathrm{D}_{\tilde{x}|x}\, \gamma^{-1} = \mathrm{D}_{\gamma(\tilde{x})|x}$. In particular, the property of a section s to lie over x depends only its $\operatorname{Gal}(\widetilde{X_S}/X_S \otimes_k \tilde{k})$-conjugacy class.

Lemma 3.1.5. *Let $x \in X(k)$ be a k-rational point. Then* $\operatorname{Gal}(\widetilde{X_S}/X_S \otimes_k \tilde{k})$ *acts transitively on the points of $\widetilde{X}$ over x.*

Proof. The projection $X \otimes_k \tilde{k} \to X$ is closed since $\operatorname{Spec}(\tilde{k}) \to \operatorname{Spec}(k)$ is an integral and hence universally closed morphism. Denote by $S \otimes \tilde{k}$ the preimage of S under the projection $X \otimes_k \tilde{k} \to X$. The base change $X_S \otimes_k \tilde{k}$ equals the localisation $(X \otimes_k \tilde{k})_{S \otimes_k \tilde{k}}$ of $X \otimes_k \tilde{k}$ by Lemma 2.2.5. Since x is k-rational, there exists a unique point $x \otimes \tilde{k}$ in $X \otimes_k \tilde{k}$ over x. After renaming $X \otimes_k \tilde{k}$ as X and $S \otimes_k \tilde{k}$ as S and $x \otimes \tilde{k}$ as x, it suffices to show that $\operatorname{Gal}(\widetilde{X_S}/X_S)$ acts transitively on the fibres of $\widetilde{X} \to X$ over closed points x of X.

Denote by $\widetilde{K}/K$ the function field extension of $\widetilde{X_S} \to X_S$, or also of $\widetilde{X} \to X$. It is a Galois extension with group $G \coloneqq \operatorname{Gal}(\widetilde{X_S}/X_S)$. Assume first that G is finite. Let $\tilde{x}$ and $\tilde{y}$ be two points of $\widetilde{X}$ over x and assume for contradiction that $\tilde{x} \neq \sigma(\tilde{y})$ for all $\sigma \in \operatorname{Gal}(\widetilde{X_S}/X_S)$. Then, using approximation, there exists a rational function $\tilde{f} \in \widetilde{K}$ such that

$$\tilde{f}(\tilde{x}) = 0 \quad \text{and} \quad \tilde{f}(\sigma(y)) = 1 \text{ for all } \sigma \in G.$$

Set $f \coloneqq \operatorname{Nm}_{\widetilde{K}/K}(\tilde{f}) \in K$, the norm of $\tilde{f}$ in the extension $\widetilde{K}/K$. Then we have

$$f(x) = \prod_{\sigma \in G} \tilde{f}(\sigma(\tilde{x})) = 0,$$

but also

$$f(x) = \prod_{\sigma \in G} \tilde{f}(\sigma(\tilde{y})) = 1,$$

a contradiction! Assume now that G is not necessarily finite. Write $\widetilde{X_S} \to X_S$ as a limit of its finite Galois subcovers $W_i \to X_S$. Then $G = \varprojlim_i \mathrm{Gal}(W_i/X_S)$. Let $X_i \to X$ be the normalisation of X in the function field of W_i, so that $\widetilde{X} = \varprojlim_i X_i$. Given $\tilde{x} = (x_i)$ and $\tilde{y} = (y_i)$ over x, the set

$$\{\sigma \in G : \tilde{x} = \sigma(\tilde{y})\} = \varprojlim_i \{\sigma_i \in \mathrm{Gal}(W_i/X_S) : x_i = \sigma_i(y_i)\}$$

is nonempty as an inverse limit of nonempty finite sets. $\square$

Corollary 3.1.6. *Let $x \in X(k)$ be a k-rational point. If there exists some section $\mathrm{Gal}(\tilde{k}/k) \to \mathrm{Gal}(\widetilde{X_S}/X_S)$ over x, then for any $\tilde{x}$ in $\widetilde{X}$ over x there exists a section with image contained in $\mathrm{D}_{\tilde{x}|x}$.*

Proof. Let $s\colon \mathrm{Gal}(\tilde{k}/k) \to \mathrm{Gal}(\widetilde{X_S}/X_S)$ be a section with $\mathrm{im}(s) \subseteq \mathrm{D}_{\tilde{y}|x}$ for some $\tilde{y}$ in $\widetilde{X}$ over x. By Lemma 3.1.5, there exists $\gamma \in \mathrm{Gal}(\widetilde{X_S}/X_S \otimes_k \tilde{k})$ such that $\tilde{x} = \gamma(\tilde{y})$. Then the conjugate section $s' \coloneqq \gamma(-)\gamma^{-1} \circ s$ satisfies

$$\mathrm{im}(s') \subseteq \gamma\, \mathrm{D}_{\tilde{y}|x}\, \gamma^{-1} = \mathrm{D}_{\gamma(\tilde{y})|x} = \mathrm{D}_{\tilde{x}|x}\,. \qquad \square$$

3.1.2 The universal profinite étale cover

As a special case of the general setting, choose for $\widetilde{X_S} \to X_S$ a universal profinite étale cover $X_S^{\mathrm{univ}} \to X_S$. The Galois group $\mathrm{Gal}(X_S^{\mathrm{univ}}/X_S)$ is canonically isomorphic to the profinite fundamental group $\pi_1(X_S)$, formed with respect to the fibre functor pro-represented by X_S^{univ}:

$$\mathrm{Hom}_{X_S}(X_S^{\mathrm{univ}}, -)\colon \mathrm{Cov}(X_S) \longrightarrow \mathrm{FinSet}\,.$$

The field of constants of X_S^{univ} is an algebraic closure $\overline{k}$ of k, and the residue field of any closed point of X_S^{univ} is equal to $\overline{k}$. With $G_k \coloneqq \mathrm{Gal}(\overline{k}/k)$, the fundamental exact sequence (3.1.1) specialises to

$$1 \longrightarrow \pi_1(X_S \otimes_k \overline{k}) \longrightarrow \pi_1(X_S) \longrightarrow G_k \longrightarrow 1.$$

Proposition 3.1.7. *Sections of $\pi_1(X_S) \to G_k$ exist over every k-rational point of X.*

Proof. Let $\widetilde{X} \to X$ be the normalisation of X in the function field of X_S^{univ}. Let $x \in X(k)$ and $\tilde{x}|x$ in $\widetilde{X}$. The profinite étale subcover $\mathrm{X}_{S,x}^{\mathrm{h}} \to X_S$ of X_S^{univ} corresponding to the closed subgroup $\mathrm{D}_{\tilde{x}|x} \subseteq \pi_1(X_S)$ is a henselisation of X_S at x:

$$X_{S,x}^{\mathrm{h}} \coloneqq \mathrm{Spec}(\mathcal{O}_{X,x}^{\mathrm{h}}) \times_X X_S.$$

Thus, if K_x^{h} denotes the fraction field of the henselian local ring $\mathcal{O}_{X,x}^{\mathrm{h}}$, then

$$X_{S,x}^{\mathrm{h}} = \begin{cases} \mathrm{Spec}(\mathcal{O}_{X,x}^{\mathrm{h}}), & \text{if } x \in S, \\ \mathrm{Spec}(K_x^{\mathrm{h}}), & \text{if } x \notin S. \end{cases}$$

Assume that $x \in S$. Denote by x^{h} the image of $\tilde{x}$ in $X_{S,x}^{\mathrm{h}}$. Then we have $\kappa(x^{\mathrm{h}}) = \kappa(x) = k$, and a section $s\colon G_k \to \pi_1(X_{S,x}^{\mathrm{h}}) = \mathrm{D}_{\tilde{x}|x}$ is obtained by functoriality of Galois groups from the diagram

$$\begin{tikzcd} X_S^{\mathrm{univ}} \arrow[r] \arrow[d] & \mathrm{Spec}(\overline{k}) \arrow[d] \arrow[l, bend right, dashed, "\tilde{x}"'] \\ X_{S,x}^{\mathrm{h}} \arrow[r] & \mathrm{Spec}(k). \arrow[l, bend right, dashed, "x^{\mathrm{h}}"'] \end{tikzcd}$$

Assume now that $x \notin S$. By Deligne's theory of tangential base points [Del89, §15], there is an equivalence of categories between finite étale covers of $X_{S,x}^{\mathrm{h}}$ and finite étale covers of $\mathrm{T}_{X,x}^{\circ}$, the Zariski tangent space of X at x with the origin removed, viewed as a scheme over k. This implies that the group $\pi_1(X_{S,x}^{\mathrm{h}}) = \mathrm{D}_{\tilde{x}|x}$ is isomorphic, as a profinite group over G_k, to $\pi_1(\mathrm{T}_{X,x}^{\circ})$. If we choose $v \in \mathrm{T}_{X,x}^{\circ}(k)$ (a nonzero tangent vector of X at x), and $\tilde{v} \in (\mathrm{T}_{X,x}^{\circ})^{\mathrm{univ}}$ over v, then a section $s\colon G_k \to \pi_1(\mathrm{T}_{X,x}^{\circ}) \cong \mathrm{D}_{\tilde{x}|x}$ is obtained via the diagram

$$\begin{tikzcd} (\mathrm{T}_{X,x}^{\circ})^{\mathrm{univ}} \arrow[r] \arrow[d] & \mathrm{Spec}(\overline{k}) \arrow[d] \arrow[l, bend right, dashed, "\tilde{v}"'] \\ \mathrm{T}_{X,x}^{\circ} \arrow[r] & \mathrm{Spec}(k). \arrow[l, bend right, dashed, "v"'] \end{tikzcd}$$

In either case, sections over x exist. $\square$

3.1.3 Maximal pro-$\mathcal{C}$ covers

As another special case of the general setting, we can choose for $\widetilde{X_S} \to X_S$ a maximal $\mathbb{Z}/p\mathbb{Z}$-abelian or $\mathbb{Z}/p\mathbb{Z}$-metabelian cover. More generally, assume that $\mathcal{C}$ is a class of finite groups which is closed under isomorphisms, subgroups, quotients and finite products. An inverse limit of groups in $\mathcal{C}$ is called a **pro-$\mathcal{C}$ group**. Every profinite group π has a **maximal pro-$\mathcal{C}$ quotient** given by

$$\pi^{\mathcal{C}} = \varprojlim_N \pi/N,$$

where N runs through the open normal subgroups $N \trianglelefteq \pi$ for which π/N is in $\mathcal{C}$. Every homomorphism of π into a pro-$\mathcal{C}$ group factors uniquely through $\pi \to \pi^{\mathcal{C}}$. Given a connected, quasi-compact and quasi-separated scheme Y

with geometric point y, the **maximal pro-$\mathcal{C}$ cover** $(Y^{\mathcal{C}}, y^{\mathcal{C}}) \to (Y, y)$ is the pointed profinite étale Galois cover corresponding to the closed normal subgroup $\ker\big(\pi_1(Y,y) \to \pi_1(Y,y)^{\mathcal{C}}\big)$ of $\pi_1(Y,y)$. Thus, the Galois group of $Y^{\mathcal{C}}/Y$ is the maximal pro-$\mathcal{C}$ quotient of $\pi_1(Y,y)$:

$$\mathrm{Gal}(Y^{\mathcal{C}}/Y) = \pi_1(Y,y)^{\mathcal{C}}.$$

As a pointed cover, $(Y^{\mathcal{C}}, y^{\mathcal{C}}) \to (Y, y)$ is unique up to unique isomorphism; as a mere cover (without distinguished point in the fibre over y), it is unique up to non-canonical isomorphism.

Examples for $\mathcal{C}$ and $Y^{\mathcal{C}} \to Y$ include:

(1) $\mathcal{C} =$ all finite groups, $Y^{\mathrm{univ}} \to Y$ the universal profinite étale cover;

(2) $\mathcal{C} =$ finite abelian groups, $Y^{\mathrm{ab}} \to Y$ the maximal abelian cover;

(3) $\mathcal{C} =$ finite $\mathbb{Z}/p\mathbb{Z}$-elementary abelian groups, $Y' \to Y$ the maximal $\mathbb{Z}/p\mathbb{Z}$-abelian cover;

(4) $\mathcal{C} =$ finite $\mathbb{Z}/p\mathbb{Z}$-metabelian groups, $Y'' \to Y$ the maximal $\mathbb{Z}/p\mathbb{Z}$-metabelian cover;

(5) $\mathcal{C} = p$-groups, $Y(p) \to Y$ the maximal pro-p cover.

Now consider a localisation X_S of a curve as above and let $X_S^{\mathcal{C}} \to X_S$ be its maximal pro-$\mathcal{C}$ cover.

Proposition 3.1.8. *The field of constants of $X_S^{\mathcal{C}}$ is the maximal pro-$\mathcal{C}$ extension $k^{\mathcal{C}}/k$, and sections $\mathrm{Gal}(k^{\mathcal{C}}/k) \to \mathrm{Gal}(X_S^{\mathcal{C}}/X_S)$ exist over every k-rational point $x \in X(k)$.*

Proof. Since $X_S \otimes_k k^{\mathcal{C}} \to X_S$ is pro-$\mathcal{C}$, it is a subcover of $X_S^{\mathcal{C}}/X_S$, hence $k^{\mathcal{C}}$ embeds into the field of constants $\tilde{k}$ of $X_S^{\mathcal{C}}$. On the other hand, the surjective homomorphism

$$\mathrm{Gal}(X_S^{\mathcal{C}}/X_S) \twoheadrightarrow \mathrm{Gal}(\tilde{k}/k)$$

implies that $\tilde{k}/k$ is a pro-$\mathcal{C}$ extension since $\mathcal{C}$ is closed under quotients. This shows $\tilde{k} = k^{\mathcal{C}}$.

Let $x \in X(k)$. Let $X_S^{\mathrm{univ}} \to X_S^{\mathcal{C}} \to X_S$ be the universal profinite étale cover. By Proposition 3.1.7, there exists a section s above x, say $\mathrm{im}(s) \subseteq \mathrm{D}_{\tilde{x}|x}$ with $\tilde{x}$ a point over x in the normalisation of X in the function field of X_S^{univ}. Let $x^{\mathcal{C}}$ be the image of $\tilde{x}$ in the normalisation of X in the function field of $X_S^{\mathcal{C}}$ and consider the following diagram:

$$\begin{tikzcd}
\mathrm{D}_{\tilde{x}|x} \arrow[r, hook] \arrow[d, two heads] & \pi_1(X_S) \arrow[r, two heads] \arrow[d, two heads] & G_k \arrow[d] \arrow[ll, bend right, "s"'] \\
\mathrm{D}_{x^{\mathcal{C}}|x} \arrow[r, hook] & \mathrm{Gal}(X_S^{\mathcal{C}}/X_S) \arrow[r, two heads] & \mathrm{Gal}(k^{\mathcal{C}}/k). \arrow[ll, dashed, bend right, "s^{\mathcal{C}}"']
\end{tikzcd}$$

Since $\mathcal{C}$ is closed under subgroups, the decomposition group $\mathrm{D}_{x^{\mathcal{C}}|x}$ is a pro-$\mathcal{C}$ group. This implies that the section $s\colon G_k \to \mathrm{D}_{\tilde{x}|x}$ composed with the map $\mathrm{D}_{\tilde{x}|x} \to \mathrm{D}_{x^{\mathcal{C}}|x}$ factors through the maximal pro-$\mathcal{C}$ quotient $\mathrm{Gal}(k^{\mathcal{C}}/k)$, giving the claimed section $s^{\mathcal{C}}$ over x. □

3.2 Functoriality

3.2.1 Compatible sections

Assume we have for $i = 1, 2$ a field k_i, a geometrically connected qcqs scheme Y_i/k_i and a profinite étale Galois cover $\widetilde{Y_i} \to Y_i$ with field of constants $\tilde{k}_i$. A morphism $f\colon (\widetilde{Y_1}/Y_1/k_1) \to (\widetilde{Y_2}/Y_2/k_2)$ of these data by definition consists of three morphisms, all denoted by the same letter, forming the vertical arrows in a commutative diagram:

$$\begin{array}{ccccc} \widetilde{Y_1} & \longrightarrow & Y_1 & \longrightarrow & \mathrm{Spec}(k_1) \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f} \\ \widetilde{Y_2} & \longrightarrow & Y_2 & \longrightarrow & \mathrm{Spec}(k_2). \end{array}$$

The morphism f induces the following vertical maps f_* in a commutative diagram:

$$\begin{array}{ccc} \mathrm{Gal}(\widetilde{Y_1}/Y_1) & \twoheadrightarrow & \mathrm{Gal}(\tilde{k}_1/k_1) \\ \Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle f_*} \\ \mathrm{Gal}(\widetilde{Y_2}/Y_2) & \twoheadrightarrow & \mathrm{Gal}(\tilde{k}_2/k_2). \end{array} \tag{3.2.1}$$

Definition 3.2.1. Two sections $s_i\colon \mathrm{Gal}(\tilde{k}_i/k_i) \to \mathrm{Gal}(\widetilde{Y_i}/Y_i)$ $(i = 1, 2)$ are **compatible with respect to** f if they satisfy $f_* \circ s_1 = s_2 \circ f_*$.

Some cases of interest are the following:

(1) Assume that the morphism f is an isomorphism on the fields of constants: $\mathrm{Spec}(\tilde{k}_1) \cong \mathrm{Spec}(\tilde{k}_2)$ and $\mathrm{Spec}(k_1) \cong \mathrm{Spec}(k_2)$. In this case the induced map $f_*\colon \mathrm{Gal}(\tilde{k}_1/k_1) \to \mathrm{Gal}(\tilde{k}_2/k_2)$ is an isomorphism and for each section $s_1\colon \mathrm{Gal}(\tilde{k}_1/k_1) \to \mathrm{Gal}(\widetilde{Y_1}/Y_1)$ upstairs there is a unique compatible section $f(s_1)\colon \mathrm{Gal}(\tilde{k}_2/k_2) \to \mathrm{Gal}(\widetilde{Y_2}/Y_2)$ downstairs which is given by

$$f(s_1) := f_* \circ s_1 \circ f_*^{-1}.$$

(2) Let $\widetilde{Y_1} \to \widetilde{Y_2} \to Y$ be a tower of profinite étale Galois covers of Y. This is the special case where $Y_1 = Y_2 =: Y$ and $k_1 = k_2 =: k$. Then the diagram (3.2.1) looks as follows:

$$\begin{tikzcd} \mathrm{Gal}(\widetilde{Y_1}/Y) \arrow[r, two heads] \arrow[d, two heads] & \mathrm{Gal}(\tilde{k}_1/k) \arrow[d, two heads] \\ \mathrm{Gal}(\widetilde{Y_2}/Y) \arrow[r, two heads] & \mathrm{Gal}(\tilde{k}_2/k). \end{tikzcd}$$

Given a section $s_1\colon \mathrm{Gal}(\tilde{k}_1/k) \to \mathrm{Gal}(\widetilde{Y_1}/Y)$ upstairs, there is at most one compatible section $s_2\colon \mathrm{Gal}(\tilde{k}_2/k) \to \mathrm{Gal}(\widetilde{Y_2}/Y)$ downstairs. This is the case if and only if $s_1(\mathrm{Gal}(\tilde{k}_1/\tilde{k}_2)) \subseteq \mathrm{Gal}(\widetilde{Y}_1/\widetilde{Y}_2)$.

(3) Consider a single triple $\widetilde{Y}/Y/k$ as above and let ℓ/k be a subextension of $\tilde{k}/k$. Then $Y \otimes_k \ell \to Y$ is a subcover of $\widetilde{Y} \to Y$ and the maps

$$\begin{tikzcd} \widetilde{Y} \arrow[r] \arrow[d, equal] & Y \otimes_k \ell \arrow[r] \arrow[d] & \mathrm{Spec}(\ell) \arrow[d] \\ \widetilde{Y} \arrow[r] & Y \arrow[r] & \mathrm{Spec}(k) \end{tikzcd}$$

give rise to a diagram as follows:

$$\begin{tikzcd} \mathrm{Gal}(\widetilde{Y}/Y \otimes_k \ell) \arrow[r, two heads] \arrow[d, hook] & \mathrm{Gal}(\tilde{k}/\ell) \arrow[d, hook] \\ \mathrm{Gal}(\widetilde{Y}/Y) \arrow[r, two heads] & \mathrm{Gal}(\tilde{k}/k). \end{tikzcd} \tag{3.2.2}$$

The diagram is cartesian, hence every section $s\colon \mathrm{Gal}(\tilde{k}/k) \to \mathrm{Gal}(\widetilde{Y}/Y)$ downstairs induces a **restriction** $\mathrm{res}_{\ell/k}(s)\colon \mathrm{Gal}(\tilde{k}/\ell) \to \mathrm{Gal}(\widetilde{Y}/Y \otimes_k \ell)$ upstairs.

3.2.2 Functoriality for sections over points

Specialising the situation of the preceding paragraph §3.2.1, assume that each $Y_i = (X_i)_{S_i}$ is the localisation of a smooth, proper, geometrically connected curve X_i over a field k_i of characteristic zero at a set of closed points $S_i \subseteq (X_i)_{\mathrm{cl}}$ for $i = 1, 2$. We are given profinite Galois covers $\widetilde{(X_i)_{S_i}} \to (X_i)_{S_i}$ with fields of constants $\tilde{k}_i/k_i$. Denote by $\widetilde{X_i} \to X_i$ the normalisation of X_i in the function field of $\widetilde{(X_i)_{S_i}}$. We are also given compatible morphisms f as follows:

$$\begin{tikzcd} \widetilde{(X_1)_{S_1}} \arrow[r] \arrow[d, "f"] & (X_1)_{S_1} \arrow[r] \arrow[d, "f"] & \mathrm{Spec}(k_1) \arrow[d, "f"] \\ \widetilde{(X_2)_{S_2}} \arrow[r] & (X_2)_{S_2} \arrow[r] & \mathrm{Spec}(k_2). \end{tikzcd}$$

From this one obtains induced morphisms $X_1 \to X_2$ and $\widetilde{X_1} \to \widetilde{X_2}$ on the compactifications, which we also denote by f. If x_1 is a closed point of X_1 and

$\tilde{x}_1 \in \widetilde{X_1}$ lies over x_1, then $f(\tilde{x}_1) \in \widetilde{X_2}$ lies over $f(x_1) \in X_2$ and the map

$$f_*\colon \operatorname{Gal}(\widetilde{(X_1)_{S_1}}/(X_1)_{S_1}) \to \operatorname{Gal}(\widetilde{(X_2)_{S_2}}/(X_2)_{S_2})$$

maps the decomposition group of $\tilde{x}_1|x_1$ into that of $f(\tilde{x}_1)|f(x_1)$:

$$f_*(\mathrm{D}_{\tilde{x}_1|x_1}) \subseteq \mathrm{D}_{f(\tilde{x}_1)|f(x_1)}\,.$$

Lemma 3.2.2. *In the situation above, suppose f is an isomorphism $k_1 \cong k_2$ on the base fields. Let $s_i\colon \operatorname{Gal}(\tilde{k}_i/k_i) \to \operatorname{Gal}(\widetilde{(X_i)_{S_i}}/(X_i)_{S_i})$ $(i = 1, 2)$ be sections which are compatible with respect to f. If s_1 lies over a closed point x_1 of X_1, then s_2 lies over $f(x_1)$.*

Proof. Since f induces an isomorphism on the base fields by assumption, the map $f_*\colon \operatorname{Gal}(\tilde{k}_1/k_1) \to \operatorname{Gal}(\tilde{k}_2/k_2)$ is surjective. Together with the compatibility of the sections, this implies $\operatorname{im}(s_1) = f_*(\operatorname{im}(s_2))$. Now the claim follows from the preceding discussion. □

For the special cases considered in §3.2.1 above, we find the following:

Lemma 3.2.3.

(a) *Assume that f is an isomorphism on the fields of constants. If s_1 is a section over $x_1 \in X_1$, then $f(s_1)$ is a section over $f(x_1) \in X_2$.*

(b) *Let $\widetilde{\widetilde{X_S}} \to \widetilde{X_S} \to X_S$ be a tower of profinite étale Galois covers of X_S with fields of constants $\tilde{\tilde{k}}/\tilde{k}/k$. Let $\tilde{\tilde{s}}\colon \operatorname{Gal}(\tilde{\tilde{k}}/k) \to \operatorname{Gal}(\widetilde{\widetilde{X_S}}/X_S)$ be a section over $x \in X_{\mathrm{cl}}$. If $\tilde{\tilde{s}}$ descends to a section $\tilde{s}\colon \operatorname{Gal}(\tilde{k}/k) \to \operatorname{Gal}(\widetilde{X_S}/X_S)$, then also $\tilde{s}$ lies over x.*

(c) *Consider a single triple $\widetilde{X_S}/X_S/k$ as above and let ℓ/k be a subextension of $\tilde{k}/k$. Let x be a closed point of X, let $\tilde{x} \in \widetilde{X}$ be a point over x, and x_ℓ the image of $\tilde{x}$ in $X \otimes_k \ell$. If $s\colon \operatorname{Gal}(\tilde{k}/k) \to \operatorname{Gal}(\widetilde{X_S}/X_S)$ is a section over x with $\operatorname{im}(s) \subseteq \mathrm{D}_{\tilde{x}|x}$, then the restriction $\operatorname{res}_{\ell/k}(s)$ is a section over x_ℓ.*

Proof. Parts (a) and (b) follow from Lemma 3.2.3. Part (c) follows from the fact that the decomposition group $\mathrm{D}_{\tilde{x}|x_\ell}$ is precisely the preimage of $\mathrm{D}_{\tilde{x}|x}$ under the injective map $f_*\colon \operatorname{Gal}(\widetilde{X_S}/(X_S \otimes_k \ell)) \hookrightarrow \operatorname{Gal}(\widetilde{X_S}/X_S)$. □

3.3 Existence of liftable sections over rational points

We have seen in Proposition 3.1.8 that liftable sections $s'\colon G'_k \to \pi_1(X'_S)$ exist over every rational point. We now want to generalise this to liftable (and twice-liftable) sections in the sense of Definition 1.4.3, i.e. on the Galois groups $\operatorname{Gal}(\ell'/k)$ or $\operatorname{Gal}(\ell''/k)$ for some finite Galois extension ℓ/k.

Consider the following more general situation: let k be a field of characteristic zero and Y/k a geometrically connected, quasi-compact and quasi-separated scheme. Let $Y^{\mathrm{univ}} \to \widetilde{Y} \to Y$ be a profinite étale Galois subcover of the universal profinite étale cover and let $\overline{k}/\tilde{k}/k$ be the fields of constants. Set $\pi_1(Y) \coloneqq \mathrm{Gal}(Y^{\mathrm{univ}}/Y)$ and $G_k \coloneqq \mathrm{Gal}(\overline{k}/k)$. The following lemma addresses the question of when a section $s\colon G_k \to \pi_1(Y)$ on the level of full fundamental groups induces a section $\mathrm{Gal}(\tilde{k}/k) \to \mathrm{Gal}(\widetilde{Y}/Y)$ on the quotients. This property can be checked after a scalar extension:

Lemma 3.3.1. *Let ℓ/k be a finite extension contained in $\tilde{k}$ and denote by $G_\ell \coloneqq \mathrm{Gal}(\overline{k}/\ell)$ its absolute Galois group. Let $s\colon G_k \to \pi_1(Y)$ be a section and denote by $s_\ell \coloneqq \mathrm{res}_{\ell/k}(s)\colon G_\ell \to \pi_1(Y \otimes \ell)$ its restriction. If s_ℓ induces a section $\tilde{s}_\ell\colon \mathrm{Gal}(\tilde{k}/\ell) \to \mathrm{Gal}(\widetilde{Y}/Y \otimes_k \ell)$, then s induces a section $\tilde{s}\colon \mathrm{Gal}(\tilde{k}/k) \to \mathrm{Gal}(\widetilde{Y}/Y)$.*

$$\begin{tikzcd}
 & \pi_1(Y) \arrow[rr, two heads] \arrow[dd, two heads] & & G_k \arrow[ll, bend right, "s"'] \arrow[dd, two heads] \\
\pi_1(Y \otimes_k \ell) \arrow[ur, hook] \arrow[rr, two heads] \arrow[dd, two heads] & & G_\ell \arrow[ll, bend right, "s_\ell"'] \arrow[ur, hook] \arrow[dd, two heads] & \\
 & \mathrm{Gal}(\widetilde{Y}/Y) \arrow[rr, two heads] & & \mathrm{Gal}(\tilde{k}/k) \arrow[ll, bend right, dashed, "\tilde{s}"'] \\
\mathrm{Gal}(\widetilde{Y}/Y \otimes_k \ell) \arrow[ur, hook] \arrow[rr, two heads] & & \mathrm{Gal}(\tilde{k}/\ell) \arrow[ll, bend right, "\tilde{s}_\ell"'] \arrow[ur, hook] &
\end{tikzcd}$$

Proof. We have to show that the kernel $\mathrm{Gal}(\overline{k}/\tilde{k})$ of $G_k \to \mathrm{Gal}(\tilde{k}/k)$ is mapped to the identity in $\mathrm{Gal}(\widetilde{Y}/Y)$ by s. But $\mathrm{Gal}(\overline{k}/\tilde{k})$ is also the kernel of the map $G_\ell \to \mathrm{Gal}(\tilde{k}/\ell)$, so the claim can be checked for s_ℓ instead of s, where it holds by assumption. □

Now let k be a field of characteristic zero and X_S/k the localisation of a curve. Given a finite Galois extension ℓ/k, let

$$(X_S \otimes_k \ell)''' \to (X_S \otimes_k \ell)'' \to (X_S \otimes_k \ell)' \to X_S \otimes_k \ell$$

be, from right to left, the maximal $\mathbb{Z}/p\mathbb{Z}$-abelian cover of $X_S \otimes \ell$, the maximal $\mathbb{Z}/p\mathbb{Z}$-metabelian cover, and the maximal three-step $\mathbb{Z}/p\mathbb{Z}$-solvable cover. Being characteristic covers of the Galois cover $X_S \otimes \ell \to X_S$, they are all Galois over X_S. We denote by $\ell'''/\ell''/\ell'/\ell$ the corresponding fields of constants. Recall from Definition 1.4.3 that a section $s'\colon \mathrm{Gal}(\ell'/k) \to \mathrm{Gal}((X_S \otimes_k \ell)'/X_S)$ is called liftable (respectively, twice-liftable) if it admits a lift s'' (respectively, s''') as follows:

$$\begin{tikzcd}
\mathrm{Gal}((X_S \otimes_k \ell)'''/X_S) \arrow[r, two heads] \arrow[d, two heads] & \mathrm{Gal}(\ell'''/k) \arrow[l, bend right, dashed, "s'''"'] \arrow[d, two heads] \\
\mathrm{Gal}((X_S \otimes_k \ell)''/X_S) \arrow[r, two heads] \arrow[d, two heads] & \mathrm{Gal}(\ell''/k) \arrow[l, bend right, dashed, "s''"'] \arrow[d, two heads] \\
\mathrm{Gal}((X_S \otimes_k \ell)'/X_S) \arrow[r, two heads] & \mathrm{Gal}(\ell'/k) \arrow[l, bend right, "s'"']
\end{tikzcd}$$

Proposition 3.3.2. *Let ℓ/k be a finite Galois extension. For every k-rational point x of X, twice-liftable sections $s'\colon \mathrm{Gal}(\ell'/k) \to \mathrm{Gal}((X_S \otimes \ell)'/X_S)$ exist over x.*

Proof. Let $X_S^{\mathrm{univ}} \to (X_S \otimes \ell)'''$ be the universal profinite étale cover, let $\overline{k}$ be its field of constants and set $\pi_1(X_S) = \mathrm{Gal}(X_S^{\mathrm{univ}}/X_S)$ and $G_k = \mathrm{Gal}(\overline{k}/k)$. Let x be a k-rational point of X. By Proposition 3.1.7, there exists a section $s\colon G_k \to \pi_1(X_S)$ over x. Let $s_\ell \coloneqq \mathrm{res}_{\ell/k}(s)\colon G_\ell \to \pi_1(X_S \otimes \ell)$ be its restriction. By Proposition 3.1.8, s_ℓ induces a section

$$s_\ell'''\colon \mathrm{Gal}(\ell'''/\ell) \to \mathrm{Gal}((X_S \otimes \ell)'''/(X_S \otimes \ell))$$

on the three-step $\mathbb{Z}/p\mathbb{Z}$-solvable quotients. Lemma 3.3.1 implies that also s induces a section

$$s'''\colon \mathrm{Gal}(\ell'''/k) \to \mathrm{Gal}((X_S \otimes \ell)'''/X_S).$$

The same argument applied to the maximal $\mathbb{Z}/p\mathbb{Z}$-abelian quotients shows that s''' descends further to a section $s'\colon \mathrm{Gal}(\ell'/k) \to \mathrm{Gal}((X_S \otimes_k \ell)'/X_S)$ which is therefore twice-liftable. By Lemma 3.2.3 (b) applied to the tower $X_S^{\mathrm{univ}} \to X_S \otimes_k \ell \to X_S$, the liftable section s' lies over x. $\square$

3.4 The Galois action on conjugacy classes of sections

In order to prove descent statements for the liftable section conjecture with respect to a Galois extension ℓ/k, we need to analyse the Galois action of $\mathrm{Gal}(\ell/k)$ on conjugacy classes of liftable sections defined on $\mathrm{Gal}(\ell'/\ell)$. We define here the Galois action in an abstract setting and explain its interpretation in terms of nonabelian group cohomology, before applying the theory in the context of section conjecture in the next section.

Let G be a profinite group and let E be an extension of G by a profinite group A. Denote by $\mathrm{Sec}(E \to G)$ the set of sections $s\colon G \to E$ and by $\mathscr{S}(E \to G)$ the set of their A-conjugacy classes. Let $H \subseteq G$ be a closed normal subgroup.

We can form the pullback extension $E_H \coloneqq E \times_G H = \mathrm{pr}^{-1}(H)$, an extension of H by A:

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r] \arrow[d, equal] & E_H \arrow[r, "\mathrm{pr}"] \arrow[d] & H \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & A \arrow[r] & E \arrow[r, "\mathrm{pr}"] & G \arrow[r] & 1.
\end{tikzcd}$$

Every section $s\colon G \to E$ restricts to a section $\mathrm{res}(s)\colon H \to E_H$. This passes to A-conjugacy classes and defines restriction maps as follows:

$$\begin{tikzcd}
\mathrm{Sec}(E_H \to H) \arrow[r, two heads] & \mathscr{S}(E_H \to H) \\
\mathrm{Sec}(E \to G) \arrow[r, two heads] \arrow[u, "\mathrm{res}"] & \mathscr{S}(E \to G). \arrow[u, "\mathrm{res}"]
\end{tikzcd}$$

3.4.1 Definition of the Galois action

We are going to define an action of G/H on the set of conjugacy classes of sections $\mathscr{S}(E_H \to H)$. For an element $e \in E$ and a section $t\colon H \to E_H$, define the section $e(t) \in \mathrm{Sec}(E_H \to H)$ by

$$e(t) \coloneqq e(-)e^{-1} \circ t \circ \mathrm{pr}(e)^{-1}(-)\,\mathrm{pr}(e).$$

This defines an action of E on $\mathrm{Sec}(E_H \to H)$.

Lemma 3.4.1. *The action of E on $\mathrm{Sec}(E_H \to H)$ induces a well-defined action of G/H on $\mathscr{S}(E_H \to H)$.*

Proof. For $a \in A$ we have $\mathrm{pr}(a) = 1$, so that $a(t)$ equals the A-conjugate section $a(-)a^{-1} \circ t$. To see that the action of E on $\mathrm{Sec}(E_H \to H)$ passes to conjugacy classes of sections, let $e \in E$ and $t \in \mathrm{Sec}(E_H \to H)$ and $a \in A$. We can write $ea = a'e$ with $a' \in A$. Then we calculate

$$e(a(-)a^{-1} \circ t) = (ea)(t) = (a'e)(t) = a'(-)a'^{-1} \circ e(t),$$

which shows that $e(a(-)a^{-1} \circ t)$ is A-conjugate to $e(t)$. So E acts on the set $\mathscr{S}(E_H \to H)$. Since the normal subgroup A acts trivially, the action passes to G.

Let $h \in H$ and $t \in \mathrm{Sec}(E_H \to H)$. To calculate $h([t]) \in \mathscr{S}(E_H \to H)$, we can choose $t(h) \in E$ as a preimage of h under the projection and calculate

$$\begin{aligned}
(t(h))(t) &= t(h)(-)t(h)^{-1} \circ t \circ \mathrm{pr}(t(h))^{-1}(-)\,\mathrm{pr}(t(h)) \\
&= t(h)(-)t(h)^{-1} \circ t \circ h^{-1}(-)h \\
&= t(h)(-)t(h)^{-1} \circ t(h)^{-1}(-)t(h) \circ t \\
&= t.
\end{aligned}$$

Thus H acts trivially on $\mathscr{S}(E_H \to H)$ and the action passes to G/H. $\square$

Lemma 3.4.2. *The restriction map* res *on conjugacy classes of sections maps into the G/H-invariant subset:*

$$\mathrm{res}\colon \mathscr{S}(E \to G) \to \mathscr{S}(E_H \to H)^{G/H}.$$

Proof. Let $s\colon G \to E$ be a section and let $g \in G$. By definition, $g(\mathrm{res}([s]))$ is the class of $e(\mathrm{res}(s))$ for some lift $e \in E$ of g. We may choose $e = s(g)$, which satisfies $s(\mathrm{pr}(e)) = e$, and calculate

$$\begin{aligned} e(\mathrm{res}(s)) &= e(-)e^{-1} \circ s|_H \circ \mathrm{pr}(e)^{-1}(-)\,\mathrm{pr}(e) \\ &= e(-)e^{-1} \circ s(\mathrm{pr}(e))^{-1}(-)s(\mathrm{pr}(e)) \circ s|_H \\ &= e(-)e^{-1} \circ e^{-1}(-)e \circ s|_H \\ &= \mathrm{res}(s). \end{aligned}$$

□

3.4.2 Interpretation in terms of nonabelian group cohomology

The Galois action on conjugacy classes of sections and the restriction map can be understood in terms of nonabelian group cohomology. We give here a brief summary and refer to [Ser02, Ch. I, §5] for details.

First nonabelian group cohomology. Let G be a profinite group which acts on another profinite group A by continuous automorphisms. Denote the action by $(\sigma, a) \mapsto {}^{\sigma}a$. A continuous map $c\colon G \to A$ is called a **1-cocycle** if it satisfies

$$c(\sigma\tau) = c(\sigma)\ {}^{\sigma}c(\tau) \quad \text{for all } \sigma, \tau \in G.$$

Two 1-cocycles c_1 and c_2 are called cohomologous if there exists $b \in A$ such that

$$c_2(\sigma) = b^{-1}c_1(\sigma)\ {}^{\sigma}b \quad \text{for all } \sigma \in G.$$

This defines an equivalence relation on the set of 1-cocycles. The set of cohomology classes forms the cohomology set $\mathrm{H}^1(G, A)$. In the case where A is a finite abelian group, this coincides with the usual cohomology group defined as a right derived functor. In general, however, $\mathrm{H}^1(G, A)$ does not have the structure of a group but merely of a pointed set: a set with a distinguished element, given by the constant cocycle with value the identity element of A.

Interpretation via conjugacy classes of sections. To explain the connection between nonabelian cohomology and sections, assume that we have a split extension of profinite groups:

$$1 \to A \to E \to G \to 1.$$

Fix a continuous section $s_0\colon G \to E$. Then G acts on A by conjugation via s_0:

$${}^{\sigma}a := s_0(\sigma)as_0(\sigma)^{-1}.$$

Any other section $s\colon G \to E$ has the form $s(\sigma) = c(\sigma)s_0(\sigma)$ for some continuous map $c\colon G \to A$. The fact that s is a homomorphism translates into the cocycle property for c. The map c is called the **difference cocycle** of s and s_0. If two sections are conjugate via some element $b \in A$, then the corresponding difference cocycles are cohomologous. In this way, we have a bijection

$$\mathscr{S}(E \to G) \cong \mathrm{H}^1(G, A) \tag{3.4.1}$$

between the set $\mathscr{S}(E \to G)$ of conjugacy classes of sections and the nonabelian cohomology set $\mathrm{H}^1(G, A)$. Under this bijection, the distinguished point of $\mathrm{H}^1(G, A)$ corresponds to the fixed section s_0. The bijection is non-canonical in that both the action of G on A as well as the difference cocycle map depend on the chosen section s_0.

Nonabelian inflation and restriction. Let G be a profinite group which acts on another profinite group A by continuous automorphisms, and let $H \subseteq G$ be a closed normal subgroup. Then the quotient group G/H acts on the subgroup A^H fixed by H, so that also $\mathrm{H}^1(G/H, A^H)$ is defined. Every cocycle $c\colon G/H \to A^H$ induces a cocycle $G \twoheadrightarrow G/H \to A^H \hookrightarrow A$. This defines an inflation map

$$\inf\colon \mathrm{H}^1(G/H, A^H) \to \mathrm{H}^1(G, A)$$

which is easily checked to be injective. We also have a restriction map

$$\mathrm{res}\colon \mathrm{H}^1(G, A) \to \mathrm{H}^1(H, A),$$

given by restriction of cocycles from G to H. The group G acts on the set of 1-cocycles $c\colon H \to A$ by the rule

$$g(c)(h) := {}^{g}c(g^{-1}hg).$$

The action passes to cohomology classes, upon which the subgroup H acts trivially. This defines a well-defined action of G/H on $\mathrm{H}^1(H, A)$. One can check that the restriction map takes values in the G/H-invariant subset. As a result, one obtains an inflation-restriction sequence of pointed sets:

$$1 \longrightarrow \mathrm{H}^1(G/H, A^H) \xrightarrow{\inf} \mathrm{H}^1(G, A) \xrightarrow{\mathrm{res}} \mathrm{H}^1(H, A)^{G/H}.$$

This sequence is exact in the sense of pointed sets: at each term, the preimage of the distinguished point under the outgoing map equals the image of the incoming map. In the case where A is a finite abelian group, one recovers the classical inflation-restriction sequence which arises from the Hochschild–Serre spectral sequence.

To make the connection with the interpretation of nonabelian cohomology classes as conjugacy classes of sections, suppose that we have an extension E of G by a profinite group A, as in above, and let $E_H := E \times_G H$ be the pullback extension. Assume that the extension E of G is split and fix a section $s_0\colon G \to E$. Let G act on A by conjugation via s_0. We have bijections $\mathscr{S}(E \to G) \cong \mathrm{H}^1(G, A)$ and $\mathscr{S}(E_H \to H) \cong \mathrm{H}^1(H, A)$ given by taking difference cocycles with respect to s_0 and $\mathrm{res}(s_0)$, respectively. The action of G/H on conjugacy classes of sections $\mathscr{S}(E_H \to H)$ translates into the action on nonabelian cohomology $\mathrm{H}^1(H, G)$, and restriction of sections classes translates into restriction of cohomology classes:

$$\begin{tikzcd} \mathscr{S}(E_H \to H)^{G/H} \arrow[r, "\sim"] & \mathrm{H}^1(H, A)^{G/H} \\ \mathscr{S}(E \to G) \arrow[r, "\sim"] \arrow[u, "\mathrm{res}"] & \mathrm{H}^1(G, A). \arrow[u, "\mathrm{res}"'] \end{tikzcd}$$

3.5 Descent results for the section conjecture

We now apply the general theory from the previous section in the context of the section conjecture. In particular, we show Theorem 1.4.4 about deducing a form of the liftable section conjecture over k from the liftable section conjecture over a finite Galois extension ℓ/k, and Theorem 1.4.6 about deducing the section conjecture for the full fundamental group from the liftable version.

3.5.1 Galois action on sections lying over points

Let k be a field of characteristic zero, X_S/k the localisation of a curve and $\widetilde{X_S} \to X_S$ a profinite étale Galois cover with field of constants $\tilde{k}$. Let ℓ/k be a finite Galois extension contained in $\tilde{k}$. We have a diagram of Galois groups as follows:

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{Gal}(\widetilde{X_S}/X_S \otimes \tilde{k}) \arrow[r] \arrow[d, equal] & \mathrm{Gal}(\widetilde{X_S}/X_S \otimes \ell) \arrow[r, "\mathrm{pr}"'] \arrow[d] & \mathrm{Gal}(\tilde{k}/\ell) \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \mathrm{Gal}(\widetilde{X_S}/X_S \otimes \tilde{k}) \arrow[r] & \mathrm{Gal}(\widetilde{X_S}/X_S) \arrow[r, "\mathrm{pr}"'] & \mathrm{Gal}(\tilde{k}/k) \arrow[r] & 1 \end{tikzcd}$$

The extension in the top row arises from the second row via pullback. We are thus in the situation of Section 3.4. Denote by $\mathscr{S}(\widetilde{X_S}/X_S)$ the set of $\mathrm{Gal}(\widetilde{X_S}/X_S \otimes \tilde{k})$-conjugacy classes of sections $\mathrm{Gal}(\tilde{k}/k) \to \mathrm{Gal}(\widetilde{X_S}/X_S)$, and similarly for the top row. Then we have a Galois action of $\mathrm{Gal}(\ell/k)$ on $\mathscr{S}(\widetilde{X_S}/X_S \otimes \ell)$. Recall that the property of a section s_ℓ to lie over a point of $X \otimes \ell$ depends only on its conjugacy class (Remark 3.1.4 (2)). The Galois action on section classes is compatible with the Galois action on points of $X \otimes \ell$:

Lemma 3.5.1. *Let $s_\ell\colon \mathrm{Gal}(\tilde{k}/\ell) \to \mathrm{Gal}(\widetilde{X_S}/X_S \otimes \ell)$ be a section over an ℓ-rational point $x_\ell \in X(\ell)$. Then, for every $\sigma \in \mathrm{Gal}(\ell/k)$, the class $\sigma([s_\ell])$ lies over $\sigma(x_\ell)$.*

Proof. Let $\sigma \in \mathrm{Gal}(\ell/k)$ and choose a lift $\tilde{\sigma}$ in $\mathrm{Gal}(\widetilde{X_S}/X_S) = \mathrm{Aut}(\widetilde{X_S}/X_S)^{\mathrm{op}}$. We have a commutative diagram as follows:

$$\begin{array}{ccccc}
\widetilde{X_S} & \longrightarrow & X_S \otimes \ell & \longrightarrow & \mathrm{Spec}(\ell) \\
\big\downarrow{\scriptstyle \tilde{\sigma}^{\mathrm{op}}} & & \big\downarrow{\scriptstyle \sigma^{\mathrm{op}}} & & \big\downarrow{\scriptstyle \sigma^{\mathrm{op}}} \\
\widetilde{X_S} & \longrightarrow & X_S \otimes \ell & \longrightarrow & \mathrm{Spec}(\ell)
\end{array}$$

The automorphisms of Galois groups induced by functoriality are given by conjugations:

$$\begin{array}{ccc}
\mathrm{Gal}(\widetilde{X_S}/X_S \otimes \ell) & \xrightarrow{\mathrm{pr}} & \mathrm{Gal}(\tilde{k}/\ell) \\
\big\downarrow{\scriptstyle \tilde{\sigma}^{-1}(-)\tilde{\sigma}} & & \big\downarrow{\scriptstyle \mathrm{pr}(\tilde{\sigma})^{-1}(-)\,\mathrm{pr}(\tilde{\sigma})} \\
\mathrm{Gal}(\widetilde{X_S}/X_S \otimes \ell) & \xrightarrow{\mathrm{pr}} & \mathrm{Gal}(\tilde{k}/\ell).
\end{array}$$

Using this diagram, the section $s_\ell\colon \mathrm{Gal}(\tilde{k}/\ell) \to \mathrm{Gal}(\widetilde{Y}/Y \otimes \ell)$ gives rise to another section $(\tilde{\sigma}^{\mathrm{op}})_*(s_\ell)$ given by

$$(\tilde{\sigma}^{\mathrm{op}})_*(s_\ell) = \tilde{\sigma}^{-1}(-)\tilde{\sigma} \circ s_\ell \circ \mathrm{pr}(\tilde{\sigma})(-)\,\mathrm{pr}(\tilde{\sigma})^{-1}.$$

Comparing with the definition of the Galois action on conjugacy classes of sections, we find that $(\tilde{\sigma}^{\mathrm{op}})_*(s_\ell)$ represents the class $\sigma^{-1}([s_\ell]) \in \mathscr{S}(\widetilde{X_S}/X_S \otimes \ell)$. By Lemma 3.2.3 (a), $(\tilde{\sigma}^{\mathrm{op}})_*(s_\ell)$ is a section over $\sigma^{\mathrm{op}}(x_\ell) = \sigma^{-1}(x_\ell)$. □

Lemma 3.5.2. *Assume that $s_\ell\colon \mathrm{Gal}(\tilde{k}/\ell) \to \mathrm{Gal}(\widetilde{X_S}/X_S \otimes \ell)$ is a section which lies over a unique ℓ-rational point $x_\ell \in X(\ell)$. If s_ℓ arises by restriction from a section $s\colon \mathrm{Gal}(\tilde{k}/k) \to \mathrm{Gal}(\widetilde{X_S}/X_S)$, then x_ℓ is k-rational.*

Proof. If s_ℓ is the restriction of a section $s\colon \mathrm{Gal}(\tilde{k}/k) \to \mathrm{Gal}(\widetilde{X_S}/X_S)$, then the conjugacy class of s_ℓ is $\mathrm{Gal}(\ell/k)$-invariant by Lemma 3.4.2. For every $\sigma \in \mathrm{Gal}(\ell/k)$, the class $\sigma(s_\ell)$ lies over $\sigma(x_\ell)$ by Lemma 3.5.1. Hence, by the Galois invariance, s_ℓ itself lies over $\sigma(x_\ell)$. We assumed that x_ℓ is the unique point over which s_ℓ lies, so we conclude $\sigma(x_\ell) = x_\ell$ for all $\sigma \in \mathrm{Gal}(\ell/k)$ and hence, x_ℓ is k-rational. □

We can now prove that the liftable section conjecture over a field extension ℓ/k implies a weak form of the liftable section conjecture over k.

Corollary 3.5.3 (= First part of Theorem 1.4.4)**.** *Assume that the base change $X_S \otimes \ell$ satisfies the liftable section conjecture. Then, for every liftable section $s'\colon \mathrm{Gal}(\ell'/k) \to \mathrm{Gal}((X_S \otimes \ell)'/X_S)$, there exists a unique k-rational point x of X such that the restricted section $\mathrm{res}_{\ell/k}(s')$ lies over $x \otimes \ell$.*

Proof. Let s' be a liftable section with lift $s'' \colon \operatorname{Gal}(\ell''/k) \to \operatorname{Gal}((X_S \otimes \ell)''/X_S)$. Then the restriction $\operatorname{res}_{\ell/k}(s') \colon \operatorname{Gal}(\ell'/\ell) \to \operatorname{Gal}((X_S \otimes \ell)'/(X_S \otimes \ell))$ is liftable with lift $\operatorname{res}_{\ell/k}(s'')$. Since $X_S \otimes \ell$ satisfies the liftable section conjecture by assumption, $\operatorname{res}_{\ell/k}(s')$ lies over a unique ℓ-rational point of X. This point is then already k-rational by Lemma 3.5.2. The uniqueness of x follows from the uniqueness statement in the liftable section conjecture for $X_S \otimes \ell$. □

3.5.2 Twice-liftable sections

Looking at Corollary 3.5.3, it is natural to ask under which conditions the liftable section s' itself rather than only its restriction $\operatorname{res}_{\ell/k}(s')$ is guaranteed to lie over a k-rational point. We prove two different criteria which ensure this: the first assumes a stronger liftability condition on s' and the validity of the liftable section conjecture not just for $X_S \otimes \ell$ but also for certain covers; the second criterion makes use of Galois cohomology calculations and is satisfied whenever the prime p does not divide the degree $[\ell : k]$.

Proposition 3.5.4 (= Theorem 1.4.4 (b))**.** *Assume that $W \otimes \ell$ satisfies the liftable section conjecture for every geometrically connected, finite étale subcover $(X_S \otimes \ell)' \to W \to X_S$. Let $s' \colon \operatorname{Gal}(\ell'/k) \to \operatorname{Gal}((X_S \otimes \ell)'/X_S)$ be a twice-liftable section. Then there exists a unique k-rational point x of X such that s' lies over x*

Proof. Choose a lift $s'' \colon \operatorname{Gal}(\ell''/k) \to \operatorname{Gal}((X_S \otimes \ell)''/X_S)$ of the twice-liftable section s' such that s'' is itself liftable. Let

$$(X_S \otimes \ell)' \to W[s'] \to X_S$$

be the connected, profinite étale subcover of $(X_S \otimes \ell)'$ that corresponds to the image $\operatorname{im}(s') \subseteq \operatorname{Gal}((X_S \otimes \ell)'/X_S)$. Write $W[s'] = \varprojlim_i W_i$ as a limit of connected finite étale subcovers. Then the groups $\operatorname{Gal}((X_S \otimes \ell)'/W_i)$ are open subgroups of $\operatorname{Gal}((X_S \otimes \ell)'/X_S)$ with intersection equal to $\operatorname{im}(s')$. In particular, each W_i is geometrically connected since $\operatorname{Gal}((X_S \otimes \ell)'/W_i)$ surjects onto $\operatorname{Gal}(\ell'/k)$. The field of constants ℓ' of $(X_S \otimes \ell)'$ contains ℓ, therefore the cover $(X_S \otimes \ell)' \to W_i$ factors through $W_i \otimes \ell$. As a consequence, $(X_S \otimes \ell)'' \to X_S$ factors through $(W_i \otimes \ell)'$ and $(X_S \otimes \ell)''' \to X_S$ factors through $(W_i \otimes \ell)''$:

$$\begin{array}{ccc}
(X_S \otimes \ell)''' & \longrightarrow & (W_i \otimes \ell)'' \\
\big\downarrow & & \big\downarrow \\
(X_S \otimes \ell)'' & \longrightarrow & (W_i \otimes \ell)' \\
\big\downarrow & & \big\downarrow \\
(X_S \otimes \ell)' & \longrightarrow & W_i \otimes \ell
\end{array}$$

As s' maps into $\mathrm{Gal}((X_S\otimes\ell)'/W_i)$, the lift s'' of s' maps into $\mathrm{Gal}((X_S\otimes\ell)''/W_i)$ and any lift s''' of s'' maps into $\mathrm{Gal}((X_S\otimes\ell)'''/W_i)$. Hence, s'' induces a liftable section $s_i'\colon \mathrm{Gal}(\ell'/k) \to \mathrm{Gal}((W_i\otimes\ell)'/W_i)$ via the surjective homomorphism

$$\mathrm{Gal}((X_S\otimes\ell)''/W_i) \twoheadrightarrow \mathrm{Gal}((W_i\otimes\ell)'/W_i).$$

By assumption, $W_i\otimes\ell$ satisfies the liftable section conjecture. Thus, by Corollary 3.5.3, there exists a unique k-rational point x_i of the compactification $\overline{W_i}$ of W_i such that $\mathrm{res}_{\ell/k}(s_i')$ lies over $x_i\otimes\ell$. By the compatibility of the sections s_i' and the uniqueness of the points x_i, the x_i form a compatible system and thus define a k-rational point $x[s'] = \varprojlim_i x_i$ of $\overline{W[s']}$. Choose a closed point x_ℓ' over $x[s']$ in the compactification of $(X_S\otimes\ell)'$. Since ℓ' is the field of constants of $(X_S\otimes\ell)'$, the point x_ℓ' is ℓ'-rational. The decomposition group $\mathrm{D}_{x_\ell'|x[s']}$ in $\mathrm{Gal}((X_S\otimes\ell)'/W[s'])$ surjects onto $\mathrm{Gal}(\ell'/k)$. But, by definition, $\mathrm{Gal}((X_S\otimes\ell)'/W[s'])$ equals the image $\mathrm{im}(s')$, so we have the equality $\mathrm{im}(s') = \mathrm{D}_{x_\ell'|x[s']}$. If x denotes the image of $x[s']$ in X, then we have $\mathrm{im}(s') \subseteq \mathrm{D}_{x_\ell'|x}$, so that s' is a section over x. The uniqueness of x follows from the uniqueness in the liftable section conjecture for $X_S\otimes\ell$. $\square$

3.5.3 Descent for the property of lying over a point

As before, let k be a field of characteristic zero, X_S/k the localisation of a curve, and $\widetilde{X_S}\to X_S$ a profinite étale Galois cover with field of constants $\tilde{k}$. For a finite Galois extension ℓ/k contained in $\tilde{k}$ we want to analyse under which conditions a section $s\colon \mathrm{Gal}(\tilde{k}/k)\to\mathrm{Gal}(\widetilde{X_S}/X_S)$ lies over a k-rational point provided that its restriction $\mathrm{res}_{\ell/k}(s)$ does so.

Let $\widetilde{X}\to X$ be the normalisation of X in the function field of $\widetilde{X_S}$. Let x be a k-rational point of X and $\tilde{x}$ a $\tilde{k}$-rational point of $\widetilde{X}$ over x. Denote by $i\colon \mathrm{D}_{\tilde{x}|x}\hookrightarrow\mathrm{Gal}(\widetilde{X_S}/X_S)$ the inclusion of the decomposition group. We have the following diagram with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{I}_{\tilde{x}|x} \arrow[r] \arrow[d, hook] & \mathrm{D}_{\tilde{x}|x} \arrow[r] \arrow[d, hook, "i"] & \mathrm{Gal}(\tilde{k}/k) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathrm{Gal}(\widetilde{X_S}/X_S\otimes\tilde{k}) \arrow[r] & \mathrm{Gal}(\widetilde{X_S}/X_S) \arrow[r] & \mathrm{Gal}(\tilde{k}/k) \arrow[r] & 1.
\end{tikzcd}$$

Write $\mathscr{S}(\mathrm{D}_{\tilde{x}|x})$ for the set of $\mathrm{I}_{\tilde{x}|x}$-conjugacy classes of sections $\mathrm{Gal}(\tilde{k}/k)\to\mathrm{D}_{\tilde{x}|x}$ and $\mathscr{S}(\widetilde{X_S}/X_S)$ for the set of $\mathrm{Gal}(\widetilde{X_S}/X_S\otimes\tilde{k})$-conjugacy classes of sections $\mathrm{Gal}(\tilde{k}/k)\to\mathrm{Gal}(\widetilde{X_S}/X_S)$. Every section of the top row induces a section of the bottom row via the inclusion i. This defines a map on conjugacy classes of sections

$$i_*\colon \mathscr{S}(\mathrm{D}_{\tilde{x}|x})\to\mathscr{S}(\widetilde{X_S}/X_S). \tag{3.5.1}$$

Lemma 3.5.5. *A section $s\colon \mathrm{Gal}(\tilde{k}/k)\to\mathrm{Gal}(\widetilde{X_S}/X_S)$ lies over x if and only if its class is contained in the image of the map* (3.5.1).

Proof. If the class of s is contained in the image of i_*, then s is conjugate to a section with image contained in $\mathrm{D}_{\tilde{x}|x}$. Then s itself lies over x by Remark 3.1.4 (2). Conversely, assume that s lies over x, say $\operatorname{im}(s) \subseteq \mathrm{D}_{\tilde{y}|x}$ for some point $\tilde{y}$ of $\widetilde{X_S}$ over x. By Lemma 3.1.5, there exists some $\delta \in \operatorname{Gal}(\widetilde{X_S}/X_S \otimes \tilde{k})$ with $\delta(\tilde{y}) = \tilde{x}$. Then the conjugate section $\delta(-)\delta^{-1} \circ s$ has image contained in $\delta\, \mathrm{D}_{\tilde{y}|x}\, \delta^{-1} = \mathrm{D}_{\delta(\tilde{y})|x} = \mathrm{D}_{\tilde{x}|x}$, so the class of s is the image of the class of $\delta(-)\delta^{-1} \circ s$ under i_*. □

Now let ℓ/k be a finite Galois extension contained in $\tilde{k}$. Let x_ℓ be the image of $\tilde{x}$ in $X \otimes \ell$. Note that $x_\ell = x \otimes_k \ell$ because x is k-rational. In Section 3.4 we defined an action of $\operatorname{Gal}(\ell/k)$ on $\mathscr{S}(\widetilde{X_S}/X_S \otimes \ell)$ and on $\mathscr{S}(\mathrm{D}_{\tilde{x}|x_\ell})$. Given $\sigma \in \operatorname{Gal}(\ell/k)$ and $[s_\ell] \in \mathscr{S}(\widetilde{X_S}/X_S \otimes \ell)$, by definition $\sigma([s_\ell])$ is the class of

$$\gamma(s_\ell) \coloneqq \gamma(-)\gamma^{-1} \circ s_\ell \circ \operatorname{pr}(\gamma)^{-1}(-)\operatorname{pr}(\gamma)$$

for some lift $\gamma \in \operatorname{Gal}(\widetilde{X_S}/X_S)$ of σ. The action on $\mathscr{S}(\mathrm{D}_{\tilde{x}|x_\ell})$ is given by the same rule with γ chosen in $\mathrm{D}_{\tilde{x}|x}$. We have a commutative diagram with restriction maps as follows:

$$\begin{array}{ccc} \mathscr{S}(\mathrm{D}_{\tilde{x}|x_\ell})^{\operatorname{Gal}(\ell/k)} & \xrightarrow{i_*} & \mathscr{S}(\widetilde{X_S}/X_S \otimes \ell)^{\operatorname{Gal}(\ell/k)} \\ {\scriptstyle \operatorname{res}_{\ell/k}}\uparrow & & {\scriptstyle \operatorname{res}_{\ell/k}}\uparrow \\ \mathscr{S}(\mathrm{D}_{\tilde{x}|x}) & \xrightarrow{i_*} & \mathscr{S}(\widetilde{X_S}/X_S) \end{array} \tag{3.5.2}$$

Lemma 3.5.6. *Assume the following:*

(1) the map $i_*\colon \mathscr{S}(\mathrm{D}_{\tilde{x}|x_\ell}) \to \mathscr{S}(\widetilde{X_S}/X_S \otimes \ell)$ is injective;

(2) the left vertical map $\operatorname{res}_{\ell/k}\colon \mathscr{S}(\mathrm{D}_{\tilde{x}|x}) \to \mathscr{S}(\mathrm{D}_{\tilde{x}|x_\ell})^{\operatorname{Gal}(\ell/k)}$ is surjective;

(3) the right vertical map $\operatorname{res}_{\ell/k}\colon \mathscr{S}(\widetilde{X_S}/X_S) \to \mathscr{S}(\widetilde{X_S}/X_S \otimes \ell)^{\operatorname{Gal}(\ell/k)}$ is injective.

Then for every section $s\colon \operatorname{Gal}(\tilde{k}/k) \to \operatorname{Gal}(\widetilde{X_S}/X_S)$ such that $\operatorname{res}_{\ell/k}(s)$ lies over x_ℓ, already s lies over x.

Proof. If $\operatorname{res}_{\ell/k}(s)$ lies over x_ℓ, there exists some section class $[t_\ell] \in \mathscr{S}(\mathrm{D}_{\tilde{x}|x_\ell})$ with $[\operatorname{res}_{\ell/k}(s)] = i_*([t_\ell])$ by Lemma 3.5.5. The $\operatorname{Gal}(\ell/k)$-invariance of $[\operatorname{res}_{\ell/k}(s)]$ implies the $\operatorname{Gal}(\ell/k)$-invariance of $[t_\ell]$ by assumption (1). Assumption (2) yields the existence of a section class $[t] \in \mathscr{S}(\mathrm{D}_{\tilde{x}|x})$ with $[t_\ell] = \operatorname{res}_{\ell/k}([t])$. The injectivity assumption (3) implies $[s] = i_*([t])$, so that already s lies over x by Lemma 3.5.5. □

We turn to the question of whether the assumptions of Lemma 3.5.6 are satisfied in our case of interest, where $\widetilde{X_S}$ equals $(X_S \otimes \ell)'$, the maximal $\mathbb{Z}/p\mathbb{Z}$-abelian cover of $X_S \otimes \ell$ for some prime p, considered as a profinite étale Galois cover of X_S. In this case, we denote the point $\tilde{x}$ over x_ℓ in the compactification of $(X_S \otimes \ell)'$ by x'_ℓ.

Lemma 3.5.7. *For $\widetilde{X_S} = (X_S \otimes \ell)'$, assumption (1) in Lemma 3.5.6 is satisfied.*

Proof. The claim says that the map $i_*\colon \mathscr{S}(\mathrm{D}_{x'_\ell|x_\ell}) \to \mathscr{S}((X_S \otimes \ell)'/(X_S \otimes \ell))$ is injective. The group $\mathrm{Gal}((X_S \otimes \ell)'/(X_S \otimes \ell))$ is abelian, therefore conjugacy classes of sections are the same as sections. The map on sections $i_*\colon \mathrm{Sec}(\mathrm{D}_{x'_\ell|x_\ell}) \to \mathrm{Sec}((X_S \otimes \ell)'/(X_S \otimes \ell))$ induced by the inclusion of the decomposition group is clearly injective. □

Let $\widetilde{X_S} = (X_S \otimes \ell)'$ and choose a section $s_0\colon \mathrm{Gal}(\ell'/k) \to \mathrm{D}_{x'_\ell|x}$, which exists by Proposition 3.3.2. As explained in Section 3.4.2, conjugacy classes of sections can be identified with nonabelian cohomology classes by taking difference cocycles with respect to s_0. Set

$$A \coloneqq \mathrm{Gal}((X_S \otimes \ell)'/(X_S \otimes \ell')) \quad \text{and} \quad I \coloneqq \mathrm{I}_{x'_\ell|x_\ell} = \mathrm{I}_{x'_\ell|x}\,.$$

Then diagram (3.5.2) can be written as follows:

$$\begin{array}{ccc} \mathrm{H}^1(\ell'/\ell, I)^{\mathrm{Gal}(\ell/k)} & \xrightarrow{i_*} & \mathrm{H}^1(\ell'/\ell, A)^{\mathrm{Gal}(\ell/k)} \\ {\scriptstyle \mathrm{res}_{\ell/k}}\uparrow & & {\scriptstyle \mathrm{res}_{\ell/k}}\uparrow \\ \mathrm{H}^1(\ell'/k, I) & \xrightarrow{\quad i_* \quad} & \mathrm{H}^1(\ell'/k, A) \end{array} \tag{3.5.3}$$

Observe that the cohomology sets which appear are in fact cohomology groups since A and I are abelian.

Lemma 3.5.8. *If p does not divide the degree $[\ell : k]$, then both restriction maps in diagram* (3.5.2) *are bijective.*

Proof. The restriction maps in diagram (3.5.3) are part of an inflation-restriction sequence which extends into degree 2 via a transgression map since the groups A and I are abelian:

$$1 \to \mathrm{H}^1(\ell/k, M) \xrightarrow{\mathrm{inf}} \mathrm{H}^1(\ell'/k, M) \xrightarrow{\mathrm{res}} \mathrm{H}^1(\ell'/\ell, M)^{\mathrm{Gal}(\ell/k)} \xrightarrow{\mathrm{tg}} \mathrm{H}^2(\ell/k, M).$$

Here, M stands for either A or I. Note that we have $M^{\mathrm{Gal}(\ell'/\ell)} = M$ since the group $\mathrm{Gal}((X_S \otimes \ell)'/(X_S \otimes \ell))$ is abelian. By [Ser79, Ch. VIII, §2, Cor. 1], the groups $\mathrm{H}^i(\ell/k, M)$ are annihilated by the order of $\mathrm{Gal}(\ell/k)$ for $i \geq 1$. But multiplication by $\#\,\mathrm{Gal}(\ell/k)$ is an automorphism since M is $\mathbb{Z}/p\mathbb{Z}$-elementary abelian and the degree $[\ell : k]$ is coprime to p by assumption. So the groups $\mathrm{H}^1(\ell/k, M)$ and $\mathrm{H}^2(\ell/k, M)$ vanish and the restriction maps are bijective, as claimed. □

Corollary 3.5.9 (= Theorem 1.4.4 (a))**.** *If $X_S \otimes \ell$ satisfies the liftable section conjecture and p does not divide $[\ell : k]$, then every liftable section $s'\colon \mathrm{Gal}(\ell'/k) \to \mathrm{Gal}((X_S \otimes \ell)'/X_S)$ lies over a unique k-rational point.*

Proof. There exists a unique k-rational point x such that $\mathrm{res}_{\ell/k}(s')$ lies over $x \otimes \ell$ by Corollary 3.5.3. The assumptions of Lemma 3.5.6 are satisfied by Lemmas 3.5.7 and 3.5.8. Hence, already s' lies over x. □

3.5.4 Deducing the full section conjecture from the liftable section conjecture

With a similar proof as for Proposition 3.5.4, we can deduce the section conjecture for the full fundamental groups from the liftable section conjecture for geometrically connected covers. As usual, let k be a field of characteristic zero, X/k a smooth, proper, geometrically connected curve and $S \subseteq X_{\mathrm{cl}}$ a set of closed points. Let $X_S^{\mathrm{univ}} \to X_S$ be a universal profinite étale cover, let $\overline{k}/k$ be the corresponding field of constants and set $\pi_1(X_S) := \mathrm{Gal}(X_S^{\mathrm{univ}}/X_S)$ and $G_k := \mathrm{Gal}(\overline{k}/k)$.

Proposition 3.5.10 (= Theorem 1.4.6)**.** *Assume that there exists a finite Galois extension ℓ/k such that $W \otimes_k \ell$ satisfies the liftable section conjecture for every geometrically connected, finite étale cover $W \to X_S$. Then X_S satisfies the section conjecture, i.e. every section $s\colon G_k \to \pi_1(X_S)$ lies over a unique k-rational point of X.*

Proof. Let $s\colon G_k \to \pi_1(X_S)$ be a section and let $W[s] \to X_S$ be the profinite étale subcover of $X_S^{\mathrm{univ}} \to X_S$ corresponding to the closed subgroup $\mathrm{im}(s) \subseteq \pi_1(X_S)$. Write $W[s] = \varprojlim_i W_i$ as an inverse limit of connected, finite étale subcovers. The W_i are geometrically connected over k since their fundamental groups $\pi_1(W_i)$ contain $\mathrm{im}(s)$ and hence surject onto G_k. Denote by $s_i\colon G_k \to \pi_1(W_i)$ the section s with image restricted to $\pi_1(W_i)$. For all i, we get an induced liftable section $s_i'\colon \mathrm{Gal}(\ell'/k) \to \mathrm{Gal}((W_i \otimes_k \ell)'/W_i)$ by Proposition 3.3.2. By assumption, $W_i \otimes \ell$ satisfies the liftable section conjecture. Hence, by Corollary 3.5.3, there exists a unique k-rational point x_i of the compactification $\overline{W_i}$ of W_i such that $\mathrm{res}_{\ell/k}(s_i')$ lies over x_i. Since the s_i' are compatible with each other via the transition maps, and the x_i are unique, the points x_i form a compatible system, hence defining a k-rational point $x[s]$ of the compactification $\overline{W[s]}$ of $W[s]$. Let $\tilde{x} \in \widetilde{X}$ be a point over $x[s]$ and let x be the image of $x[s]$ in X. The decomposition group $\mathrm{D}_{\tilde{x}|x[s]} \subseteq \pi_1(W[s])$ is a subgroup surjecting onto G_k. But $\pi_1(W[s]) = \mathrm{im}(s)$ by definition, thus we have the equality $\mathrm{D}_{\tilde{x}|x[s]} = \mathrm{im}(s)$. In particular, $\mathrm{im}(s) \subseteq \mathrm{D}_{\tilde{x}|x}$, hence s lies over x.

The uniqueness of x follows from the uniqueness in the liftable section conjecture for $X_S \otimes \ell$. □

4 A brief review of valuation theory

We are going to recall briefly various notions from valuation theory. While we are working with rank 1 valuations for the most part, one crucial input that we are relying on is Pop's local-to-global principle for Brauer groups of function fields over p-adically closed fields, in which general Krull valuations appear. For this reason, we do not restrict ourselves to rank 1 valuations in this chapter. We give definitions and state properties mostly without proofs, referring to [EP05] for details.

4.1 Basic notions

Definition 4.1.1.

(a) An **ordered abelian group** $(\Gamma, \leq)$ is an abelian group Γ, written additively, together with a linear order $\leq$ which is translation-invariant, i.e. for all $\gamma, \delta, \lambda \in \Gamma$ the following implication holds:

$$\gamma \leq \delta \;\Rightarrow\; \gamma + \lambda \leq \delta + \lambda.$$

(b) A **convex subgroup** of an ordered abelian group Γ is a subgroup $\Delta \subseteq \Gamma$ such that for all $\gamma \in \Gamma$, if there exist $\delta_1, \delta_2 \in \Delta$ with $\delta_1 \leq \gamma \leq \delta_2$, then also $\gamma \in \Delta$.

Proposition 4.1.2. *Let Γ be an ordered abelian group. The set of convex subgroups of Γ is linearly ordered by inclusion.*

Proof. Let $\Delta, E \subseteq \Gamma$ be convex subgroups. Suppose $E \not\subseteq \Delta$ and choose $\varepsilon \in E \setminus \Delta$. We can assume $\varepsilon \geq 0$. We have to show $\Delta \subseteq E$. Let $\delta \in \Delta$. Again, we can assume $\delta \geq 0$. Since Γ is linearly ordered, we have $\delta \leq \varepsilon$ or $\varepsilon \leq \delta$. The latter would imply $\varepsilon \in \Delta$ since Δ is convex. So we have $0 \leq \delta \leq \varepsilon$ which implies $\delta \in E$ because E is convex. $\square$

Definition 4.1.3. Let Γ be an ordered abelian group. The **rank** $\mathrm{rk}(\Gamma)$ of Γ is defined as the length of the chain of convex subgroups of Γ.

Example 4.1.4.

(1) The trivial abelian group $\{0\}$ is the unique ordered abelian group of rank zero.

(2) The group $(\mathbb{R}, +)$ with the usual archimedean order has only $\{0\}$ and $\mathbb{R}$ as convex subgroups, so it has rank 1.

(3) The group $\mathbb{Z} \oplus \mathbb{Z}$ is an ordered abelian group with respect to the lexicographic ordering:

$$(a, b) \leq (a', b') \Leftrightarrow a < a' \text{ or } (a = a' \text{ and } b \leq b').$$

The subgroup $\{0\} \oplus \mathbb{Z} \subseteq \mathbb{Z} \oplus \mathbb{Z}$ is the only convex subgroup other than the trivial group and the full group, so $\mathbb{Z} \oplus \mathbb{Z}$ with lexicographic ordering has rank 2.

Lemma 4.1.5. *Ordered abelian groups are torsion-free.*

Proof. Let Γ be an ordered abelian group and let $\gamma \in \Gamma$ such that $n\gamma = 0$ for some $n \in \mathbb{N}$. We are claiming $\gamma = 0$. Replacing γ with $-\gamma$ if necessary we can assume $\gamma \geq 0$. Using this twice, we have

$$0 \leq \gamma = 0 + \gamma \leq \gamma + \gamma = 2\gamma,$$

and similarly $0 \leq \gamma \leq 2\gamma \leq 3\gamma \leq \dots$ by induction. But $n\gamma = 0$, so each inequality is an equality, and in particular we have $\gamma = 0$. $\square$

Fact 4.1.6 ([EP05, Prop. 2.1.1])**.** *An ordered abelian group* Γ *has rank 1 if and only if it is isomorphic to a nontrivial subgroup of* $(\mathbb{R}, +)$ *with the induced archimedean ordering.*

Definition 4.1.7. Let K be a field. A **valuation** on K consists of an ordered abelian group Γ and a surjective map

$$v \colon K \twoheadrightarrow \Gamma \cup \{\infty\}$$

satisfying the following axioms for $x, y \in K$:

(i) $v(x) = \infty \Leftrightarrow x = 0$;

(ii) $v(xy) = v(x) + v(y)$;

(iii) $v(x + y) \geq \min(v(x), v(y))$.

The group Γ is called the **value group** of the valuation v. The **rank** of a valuation is defined as the rank of its value group. The pair (K, v) is called a **valued field**.

Here, ∞ is an element formally added to Γ satisfying $\gamma < \infty$ and $\infty + \infty = \infty$ and $\gamma + \infty = \infty + \gamma = \infty$ for all $\gamma \in \Gamma$ by definition. Axiom (ii) can be rephrased as saying that the map $v|_{K^\times} \colon (K^\times, \cdot) \to (\Gamma, +)$ is a group homomorphism. A general valuation as defined above is sometimes called a **Krull valuation** to distinguish it from the more restrictive notion of a rank 1 valuation.

One frequently used consequence of Axiom (iii) is the following:

Lemma 4.1.8. *Let v be a valuation on a field K. If $x, y \in K$ are such that $v(x) \neq v(y)$, then equality holds in Definition 4.1.7 (iii):*

$$v(x+y) = \min(v(x), v(y)).$$

Proof. We may interchange x and y if necessary and assume $v(x) < v(y)$. By Axiom (iii) this implies

$$v(x+y) \geq \min(v(x), v(y)) = v(x).$$

We are claiming that this is an equality. The fact that the value group of v is torsion-free implies $v(-y) = v(y)$. Using Axiom (iii) again, we find

$$v(x) = v((x+y) - y) \geq \min(v(x+y), v(y)).$$

Since $v(x) < v(y)$, the minimum on the right hand side cannot be attained for $v(y)$, so we have $v(x) \geq v(x+y)$. The equality $v(x+y) = v(x)$ follows from this by anti-symmetry. □

Definition 4.1.9. Let v be a valuation on a field K. The **valuation ring** $\mathcal{O}_v$ and **valuation ideal** $\mathfrak{m}_v$ of v are defined as follows:

$$\mathcal{O}_v \coloneqq \{x \in K \ : \ v(x) \geq 0\},$$
$$\mathfrak{m}_v \coloneqq \{x \in K \ : \ v(x) > 0\}.$$

It follows from the definition of a valuation that $\mathcal{O}_k$ is a subring of K which is local with maximal ideal $\mathfrak{m}_v$. The **residue field** $\kappa(v)$ of v is defined as

$$\kappa(v) \coloneqq \mathcal{O}_v / \mathfrak{m}_v.$$

Given a valuation v with value group Γ_v on a field K, we have two canonical short exact sequences:

$$1 \longrightarrow \mathcal{O}_v^\times \longrightarrow K^\times \xrightarrow{v} \Gamma_v \longrightarrow 0,$$

$$1 \longrightarrow 1 + \mathfrak{m}_v \longrightarrow \mathcal{O}_v^\times \longrightarrow \kappa(v)^\times \longrightarrow 1.$$

The value group Γ_v is canonically isomorphic to the group $K^\times / \mathcal{O}_v^\times$ with ordering given by

$$a\mathcal{O}_v^\times \leq b\mathcal{O}_v^\times \ \Leftrightarrow \ b/a \in \mathcal{O}_v.$$

In this way, the value group Γ_v and the valuation map $v \colon K \to \Gamma_v \cup \{\infty\}$ are both canonically determined by the valuation ring $\mathcal{O}_v \subseteq K$ alone. Valuations with the same valuation ring are called **equivalent**. We usually do not distinguish between equivalent valuations. Thus, we will say that two valuations are equal when we really mean that their valuation rings are equal. Especially when their value groups are not a priori identified with each other, this is the only sensible way to interpret equality of valuations, so there should be no confusion.

Definition 4.1.10. Let K be a field. A **valuation ring** in K is a subring $R \subseteq K$ such that for all $x \in K^\times$ we have $x \in R$ or $x^{-1} \in R$.

The valuation rings in K are precisely those subrings of the form $\mathcal{O}_v$ for some valuation v on K. Thus, if we do not distinguish between equivalent valuations, valuations on K can be identified with valuation subrings.

Examples 4.1.11.

(1) On any field there is the **trivial valuation**, defined by

$$v_{\text{triv}}(x) = \begin{cases} 0, & \text{if } x \neq 0, \\ \infty, & \text{if } x = 0. \end{cases}$$

Its valuation ring is the full field K, and its valuation ideal is the zero ideal in K.

(2) Let p be a prime number. The **p-adic valuation** v_p on $\mathbb{Q}$ with value group $\mathbb{Z}$ is defined by

$$v_p(p^n \frac{a}{b}) = n$$

for $n \in \mathbb{Z}$ and $a, b \in \mathbb{Z}$, $b \neq 0$, with $p \nmid ab$. The valuation ring of v_p consists of the rational numbers of the form a/b with $a, b \in \mathbb{Z}$ such that $p \nmid b$. The residue field equals $\mathbb{F}_p$.

(3) Let X/k be a normal, proper curve over a field k. Let K be the function field of X. For every closed point $x \in X_{\text{cl}}$, the local ring $\mathcal{O}_{X,x}$ is a discrete valuation ring in K. Let t_x be a uniformiser at x, i.e. a generator of the maximal ideal $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$. Any nonzero rational function $f \in K^\times$ can be written in a unique way as $f = t_x^n u$ with $n \in \mathbb{Z}$ and $u \in \mathcal{O}_{X,x}^\times$. The integer n is called the **order** of f at x, and denoted by $v_x(f)$. This defines a valuation $v_x \colon K^\times \twoheadrightarrow \mathbb{Z}$. The valuation ring of v_x is precisely the local ring $\mathcal{O}_{X,x}$.

(4) Let X be a regular, integral scheme with function field K. For every codimension 1 point $x \in X^{(1)}$, the local ring $\mathcal{O}_{X,x}$ is noetherian of Krull dimension 1. By the Auslander–Buchsbaum Theorem [AB59], $\mathcal{O}_{X,x}$ is moreover a unique factorisation domain and hence normal. This implies that $\mathcal{O}_{X,x}$ is a discrete valuation ring, which therefore defines a valuation $v_x \colon K^\times \twoheadrightarrow \mathbb{Z}$.

(5) Let k be a field and let $F(X) \in k[\![X]\!]$ be a power series which is transcendental over the field of rational functions $k(X)$. Then the map $f \mapsto f(X, F(X))$ is a well-defined embedding $k(X,Y) \hookrightarrow k(\!(X)\!)$ and the X-adic valuation v_X on $k(\!(X)\!)$ pulls back to a discrete valuation on $k(X,Y)$:

$$v(f) \coloneqq v_X(f(X, F(X)).$$

One can think of $f(X, F(X))$ as the restriction of a rational function on the plane to the graph of F, which is guaranteed to not be constant infinity since F is transcendental. The restriction of f to the graph of F then has an order of vanishing at the point $(0, F(0))$.

(6) Let k be a field, let $\mathbb{Z} \oplus \mathbb{Z}$ be equipped with the lexicographic ordering and define a valuation $v\colon k(X,Y)^\times \to \mathbb{Z} \oplus \mathbb{Z}$ of rank 2 on the field of rational functions in two variables by the rule

$$v\big(X^n Y^m \frac{g}{h}\big) \coloneqq (n, m),$$

where $n, m \in \mathbb{Z}$ and $g, h \in k[X,Y]$ with $h \neq 0$ and $X, Y \nmid gh$. The valuation ring $\mathcal{O}_v$ consists of those functions which are not constant infinity on the y-axis and whose restriction to the y-axis is defined at the origin. The roles of X and Y are not symmetric. For example, $v(X/Y) > 0$ but $v(Y/X) < 0$.

Definition 4.1.12. Let Γ be an abelian group. The **rational rank** $\mathrm{rr}(\Gamma) \leq \infty$ of Γ is defined as

$$\mathrm{rr}(\Gamma) \coloneqq \dim_{\mathbb{Q}}(\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}).$$

Fact 4.1.13 ([EP05, Prop. 3.4.1])**.** *Let Γ be an ordered abelian group. Then the rational rank and rank of Γ satisfy the inequality*

$$\mathrm{rk}(\Gamma) \leq \mathrm{rr}(\Gamma).$$

4.2 Valuations in field extensions

Definition 4.2.1. An **extension of valued fields** is a field extension L/K where (K, v) and (L, w) are valued fields such that $w|_K = v$ (or in terms of valuation rings: $\mathcal{O}_w \cap K = \mathcal{O}_v$). We have an induced inclusion of value groups $\Gamma_v \hookrightarrow \Gamma_w$ and residue fields $\kappa(v) \hookrightarrow \kappa(w)$. The **ramification index** of L/K is defined as the index

$$e(L/K) \coloneqq e(w/v) \coloneqq (\Gamma_w : \Gamma_v).$$

The **inertia degree** of L/K is defined as the degree of the residue field extension

$$f(L/K) \coloneqq f(w/v) \coloneqq [\kappa(w) : \kappa(v)].$$

An application of Zorn's Lemma yields Chevalley's Extension Theorem:

Fact 4.2.2 ([EP05, Theorem 3.1.2])**.** *Let L/K be a field extension. Then every valuation on K admits an extension to L.*

The rank of a valuation is stable in algebraic extensions:

Fact 4.2.3 ([EP05, Cor. 3.2.5]). *Let $(K, v) \subseteq (L, w)$ be an extension of valued fields. If L/K is algebraic, then the valuations v and w have the same rank.*

In particular, every extension of the trivial valuation on a field K to some algebraic extension L/K is trivial.

In a finite extension of a valued field, the ramification indices and inertia degrees for the different extensions of the valuation satisfy the **Fundamental Inequality**:

Fact 4.2.4 (Fundamental Inequality [EP05, Theorem 3.3.4]). *Let (K, v) be a valued field and let L/K be a finite field extension. Then there are only finitely many extensions $w_1, \ldots, w_r$ of v to L and we have the inequality:*

$$\sum_{i=1}^{r} e(w_i/v) f(w_i/v) \leq [L : K].$$

Definition 4.2.5. Let (K, v) be a valued field and let L/K be a Galois extension. For a valuation w extending v, the **decomposition group** of $w|v$ is defined as the stabiliser

$$\mathrm{D}_{w|v} = \{\sigma \in \mathrm{Gal}(L/K) \ : \ \sigma(\mathcal{O}_w) = \mathcal{O}_w\}.$$

Every element of the decomposition group induces an automorphism of the residue field extension $\kappa(w)/\kappa(v)$. The **inertia group** of $w|v$ is defined as the kernel

$$\mathrm{I}_{w|v} = \ker\big(\mathrm{D}_{w|v} \to \mathrm{Aut}(\kappa(w)/\kappa(v))\big).$$

The fact that L/K is Galois implies that the residue field extension $\kappa(w)/\kappa(v)$ is normal (but not necessarily separable). The canonical map $\mathrm{D}_{w|v} \to \mathrm{Aut}(\kappa(w)/\kappa(v))$ is surjective, so that we have a short exact sequence

$$1 \longrightarrow \mathrm{I}_{w|v} \longrightarrow \mathrm{D}_{w|v} \longrightarrow \mathrm{Aut}(\kappa(w)/\kappa(v)) \longrightarrow 1.$$

For transcendental extensions, we have the following Dimension Inequality:

Fact 4.2.6 (Dimension Inequality [EP05, Theorem 3.4.3]). *Let $(K, v) \subseteq (L, w)$ be an extension of valued fields. Let Γ_v and Γ_w be the value groups of v and w. Then the following inequality holds:*

$$\mathrm{trdeg}(\kappa(w)/\kappa(v)) + \mathrm{rr}(\Gamma_w/\Gamma_v) \leq \mathrm{trdeg}(L/K).$$

4.3 The refinement relation

Definition 4.3.1. Let K be a field and let v and w be two valuations on K. We say that v is **finer** than w (and w is **coarser** than v) if the inclusion $\mathcal{O}_v \subseteq \mathcal{O}_w$ holds.

If v is a valuation on a field K, then every overring $\mathcal{O}_v \subseteq R \subseteq K$ is again a valuation ring in K and thus defines a coarsening of v.

Fact 4.3.2 ([EP05, Lemma 2.3.1])**.** *Let v be a valuation on a field K with value group Γ_v. The coarsenings of v are canonically in bijection with each of the following:*

(1) overrings of $\mathcal{O}_v$ in K;

(2) prime ideals of $\mathcal{O}_v$;

(3) convex subgroups of Γ_v.

The bijections are given as follows. An overring of $\mathcal{O}_v$ is a valuation ring $\mathcal{O}_w$ for some valuation w on K. The inclusion $\mathcal{O}_v \subseteq \mathcal{O}_w$ leads to an inclusion $\mathfrak{m}_w \subseteq \mathfrak{m}_v$ of maximal ideals in the reverse direction. Indeed, for $x \in K^\times$, we have the implications:

$$x \in \mathfrak{m}_w \;\Rightarrow\; x^{-1} \notin \mathcal{O}_w \;\Rightarrow\; x^{-1} \notin \mathcal{O}_v \;\Rightarrow\; x \in \mathfrak{m}_v.$$

The maximal ideal $\mathfrak{m}_w$ of the coarsening w is a prime ideal of $\mathcal{O}_v$. This defines the map (1)$\to$(2). In the other direction (2)$\to$(1), the bijection is given by forming the localisation $\mathcal{O}_v \subseteq (\mathcal{O}_v)_\mathfrak{p}$ at a prime ideal $\mathfrak{p}$ of $\mathcal{O}_v$. The map (3)$\to$(2) associates to each convex subgroup $\Delta \subseteq \Gamma_v$ the prime ideal

$$\mathfrak{p}_\Delta \coloneqq \{x \in K \;:\; v(x) > \delta \text{ for all } \delta \in \Delta\}.$$

In inverse map (2)$\to$(3) is given by $\mathfrak{p} \mapsto \Delta_\mathfrak{p}$ with

$$\Delta_\mathfrak{p} \coloneqq \{\gamma \in \Gamma_v \;:\; \gamma, -\gamma < v(x) \text{ for all } x \in \mathfrak{p}\}.$$

Observe that the bijections (1)$\cong$(2) and (2)$\cong$(3) are both inclusion-reversing. The fact that the set of convex subgroups of an ordered abelian group is linearly ordered implies that the set of coarsenings of v as well as the set of prime ideals of $\mathcal{O}_v$ are linearly ordered as well. In particular, chains of prime ideals of $\mathcal{O}_v$ of correspond to chains of convex subgroups of Γ_v. This shows that the rank of v equals the Krull dimension of $\mathcal{O}_v$.

Proposition 4.3.3. *Let w be a valuation on a field K. The refinements of w are canonically in bijection with valuations on the residue field $\kappa(w)$.*

Proof. Denote by $\pi\colon \mathcal{O}_w \twoheadrightarrow \kappa(w)$ the residue map. For any refinement v of w, corresponding to an inclusion of valuation rings $\mathcal{O}_v \subseteq \mathcal{O}_w$, the image $\pi(\mathcal{O}_v)$ is a valuation ring in $\kappa(w)$ and hence defines a valuation $\overline{v}$ on $\kappa(w)$. Conversely, given a valuation $\overline{v}$ on $\kappa(w)$, the preimage $\pi^{-1}(\mathcal{O}_{\overline{v}})$ is a valuation ring of K contained in $\mathcal{O}_w$, thus defining a refinement of w. The constructions are mutually inverse. $\square$

Proposition 4.3.4. *Let (K, w) be a valued field and let (K', w') be a valued field extension. Then every refinement v of w extends to a refinement v' of w'.*

Proof. A refinement v of w corresponds to a valuation $\overline{v}$ on the residue field $\kappa(w)$. By Chevalley's Extension Theorem, $\overline{v}$ extends to a valuation $\overline{v}'$ on $\kappa(v')$. This corresponds in turn to a refinement v' of w' and one checks easily that v' extends v. □

Algebraic extensions of valued fields have the following incomparability property:

Proposition 4.3.5 ([EP05, Lemma 3.2.8])**.** *Let (K, w) be a valued field and let K'/K be an algebraic extension. If v' and w' are two extensions of w to L such that $\mathcal{O}_{v'} \subseteq \mathcal{O}_{w'}$, then $v' = w'$.*

Proof. Consider the residue field extension $\kappa(w) \hookrightarrow \kappa(w')$. The valuations w and w', being refinements of themselves, correspond to the trivial valuations on $\kappa(w)$ and $\kappa(w')$, respectively. The refinement v' of w' corresponds to another valuation $\overline{v}'$ on $\kappa(w')$. Since v' extends w by assumption, $\overline{v}'$ extends the trivial valuation on $\kappa(w)$. But the fact that L/K is algebraic implies that $\kappa(w')/\kappa(w)$ is algebraic. The trivial valuation extends only trivially in an algebraic extension, so $\overline{v'}$ must be trivial and hence $v' = w'$. □

4.4 The topology induced by a valuation

Definition 4.4.1. Let v be a valuation on a field K with value group Γ_v. The subsets of the form

$$\mathcal{U}_\gamma(a) \coloneqq \{x \in K \ : \ v(x - a) > \gamma\}$$

with $a \in K$ and $\gamma \in \Gamma_v$ form the basis of a topology on K, called the **topology induced by** w.

Proposition 4.4.2. *Let K be a field and v a valuation on K. The topology induced by v gives K the structure of a topological field.*

Proof. The claim means that the ring operations addition and multiplication as well as the inversion map on $K^\times$ are continuous. (The latter is not automatic.) We omit the first two and show only the continuity of the inversion map. Denote by Γ_v the value group of v. Let $x \in K^\times$ and $\gamma \in \Gamma_v$. We have to show that there exists some $\delta \in \Gamma_v$ such that for all $y \in K^\times$ with $v(x - y) > \delta$ we have $v(x^{-1} - y^{-1}) > \gamma$. We can choose $\delta \coloneqq \max(v(x), \gamma + 2v(x))$: indeed, if

$v(x-y) > \delta$, then we have $v(y) = \min(v(y-x), v(x)) = v(x)$ which implies

$$\begin{aligned} v(x^{-1} - y^{-1}) &= v(x^{-1}y^{-1}(y-x)) \\ &= v(y-x) - v(x) - v(y) \\ &= v(y-x) - 2v(x) \\ &> \delta - 2v(x) \\ &\geq \gamma. \end{aligned}$$

□

Definition 4.4.3. Let K be a field. Two valuations v and w on K are **dependent** if they have a common nontrivial coarsening. Otherwise they are **independent**.

Consider the special case of a rank 1 valuation v on a field K. We have the chain of length 1 of coarsenings $\mathcal{O}_v \subseteq K$, corresponding to v itself and the trivial valuation. By the definition of rank, there are no coarsenings in between. In other words, a rank 1 valuation has no nontrivial proper coarsenings. As a consequence, two rank 1 valuations are dependent if and only if they are equal.

Fact 4.4.4 ([EP05, Theorem 2.3.4]). *Two valuations on a field K are dependent if and only if they induce the same topology on K.*

We have the following approximation theorem with respect to finitely many independent valuations:

Fact 4.4.5 (Approximation Theorem [EP05, Thm. 2.4.1]). *Let K be a field and let $v_1, \dots, v_n$ be pairwise independent valuations on K with value groups $\Gamma_1, \dots, \Gamma_n$. Then for all $f_i \in K$ and all $\gamma_i \in \Gamma_i$ $(i = 1, \dots, n)$, there exists $f \in K$ such that $v_i(f - f_i) > \gamma_i$ for $i = 1, \dots, n$.*

Definition 4.4.6. Let (K, v) be a valued field. A sequence $(x_n)_{n \in \mathbb{N}}$ in K is a **Cauchy sequence** if for all $\gamma \in \Gamma_v$ there exists an $N \in \mathbb{N}$ such that $v(x_n - x_m) > \gamma$ for all $n, m \geq N$. The sequence **converges** to $x \in K$ if for all $\gamma \in \Gamma_v$ there exists an $N \in \mathbb{N}$ such that $v(x - x_n) > \gamma$ for all $n \geq N$. If every Cauchy sequence in K converges to an element of K, then K is **complete** with respect to v.

By the usual process of taking Cauchy sequences modulo null sequences, one constructs the **completion** $(\widehat{K}, \hat{v})$ of a valued field (K, v). It has the same value group and residue field and contains K as a dense subfield.

Suppose that v is a rank 1 valuation on K. Then the value group can be embedded into $\mathbb{R}$ as an ordered group, so that v can be viewed as a map $v\colon K \to \mathbb{R} \cup \{\infty\}$. One can then define an absolute value $|\cdot|\colon K \to \mathbb{R}_{\geq 0}$ by

$$|x| \coloneqq \rho^{-v(x)} \text{ for } x \in K,$$

for some fixed real number $\rho > 1$. Axiom (iii) in the definition of a valuation translates into the ultrametric triangle inequality:

$$|x + y| \leq \max(|x|, |y|).$$

Thus, the rank 1 valuation defines a non-archimedean absolute value on K. Conversely, every non-archimedean absolute value defines a rank 1 valuation by setting $v(x) = -\log_\rho(x)$. The two constructions are well-defined and mutually inverse up to equivalence of valuations and absolute values.

The absolute value $|\cdot|$ associated to the rank 1 valuation v defines a metric on K by $d(x, y) := |x - y|$, and the notions of topology, Cauchy sequence, convergence and completion defined above coincide with the usual concepts for metric spaces.

4.5 Henselian fields

Definition 4.5.1. A valued field (K, v) is called **henselian** if the valuation v extends uniquely to every algebraic extension of K.

The fact that there exists an extension of the valuation to every algebraic extension is already guaranteed by Chevalley's Extension Theorem (Fact 4.2.2). The important property of henselian fields is the uniqueness of the extended valuation. Clearly, every algebraic extension of a henselian field is itself henselian with respect to the unique extension of the valuation.

An equivalent characterisation of henselian fields is that they satisfy Hensel's Lemma, expressing a way of lifting approximate roots of polynomials to actual roots. In the following, $f'(X)$ denotes the formal derivative of a polynomial $f(X) \in K[X]$:

Fact 4.5.2 (Hensel's Lemma [EP05, Theorem 4.1.3])**.** *A valued field (K, v) is henselian if and only if for all $f \in \mathcal{O}_v[X]$ and for all $a \in \mathcal{O}_v$ such that $v(f(a)) > 2v(f'(a))$, there exists $b \in K$ with $f(b) = 0$ and $v(a - b) > v(f'(a))$.*

Definition 4.5.3. A **henselisation** of a valued field (K, v) is a valued field extension $(K^{\mathrm{h}}, v^{\mathrm{h}})$ of (K, v) which is henselian and such that for every other henselian field extension (L, w) of (K, v) there exists a unique embedding $K^{\mathrm{h}} \hookrightarrow L$ of valued fields over K.

The henselisation of a valued field is unique up to unique isomorphism of valued field extensions as it is defined by a universal property. Every valued field has a henselisation:

Fact 4.5.4 ([EP05, §5.2])**.** *Let (K, v) be a valued field. Let K^{sep}/K be a separable closure and choose an extension v^{sep} of v to K^{sep}. Then the fixed field under the decomposition group $K^{\mathrm{h}} := (K^{\mathrm{sep}})^{\mathrm{D}_{v^{\mathrm{sep}}|v}}$ with the restricted valuation $v^{\mathrm{h}} := v^{\mathrm{sep}}|_{K^{\mathrm{h}}}$ is a henselisation of (K, v).*

For a rank 1 valuation, the henselisation can also be constructed as a subfield of its completion. This follows from the fact that complete fields with respect to a rank 1 valuation are henselian:

Proposition 4.5.5. *Let K be a field which is complete with respect to a rank 1 valuation v. Then (K, v) is henselian.*

Proof. Choose a real number $\rho > 1$ and define the absolute value $|x| := \rho^{-v(x)}$ on K. Then K is complete with respect to $|\cdot|$. For a finite extension L/K, any two norms on L compatible with $|\cdot|$ on K are equivalent by [EP05, Prop. 1.2.2]. This implies the uniqueness of the extension of v to L. As any algebraic extension of K is a union of its finite subextensions, v extends uniquely to every algebraic extension. □

The following does not hold for valuations of higher rank:

Fact 4.5.6 ([End72, Theorem (17.18)])**.** *Let v be a rank 1 valuation on a field K. Let $(\widehat{K}, \hat{v})$ be the completion and let K^{h} be the relative separable closure of K in $\widehat{K}$ with restricted valuation $v^{\mathrm{h}} := \hat{v}|_{K^{\mathrm{h}}}$. Then $(K^{\mathrm{h}}, v^{\mathrm{h}})$ is a henselisation of (K, v).*

One important consequence is the following:

Proposition 4.5.7. *Let (K, v) be a valued field and let $(K^{\mathrm{h}}, v^{\mathrm{h}})$ be a henselisation. If v has rank 1, then K is dense in K^{h} with respect to the topology induced by v^{h}.*

Proof. Since v has rank 1, the henselisation $(K^{\mathrm{h}}, v^{\mathrm{h}})$ embeds into the completion $(\widehat{K}, \hat{v})$. The field K is dense in $\widehat{K}$, hence also in K^{h}. □

The following fact is also needed later:

Proposition 4.5.8. *Let (K, v) be a henselian valued field. Then K is also henselian with respect to any coarsening w of v.*

Proof. Let K'/K be an algebraic extension. Let v' be the unique extension of v to K'. We have to show that w extends uniquely to K' as well. Let w' be any extension of w to K'. By Proposition 4.3.4, the refinement v of w extends to some refinement of w'. But v' is the only extension of v. This shows that every extension w' of w is a coarsening of v'. The coarsenings of v' are linearly ordered by refinement, so any two extensions w'_1 and w'_2 of w satisfy $\mathcal{O}_{w'_1} \subseteq \mathcal{O}_{w'_2}$ or $\mathcal{O}_{w'_2} \subseteq \mathcal{O}_{w'_1}$. By the incomparability property (Proposition 4.3.5), this implies $w'_1 = w'_2$. Hence, the extension of w to K' is unique. □

5 The liftable section conjecture for good localisations

This chapter, which is the technical heart of this work, contains the proof of the liftable section conjecture for good localisations over p-adic fields containing the p-th roots of unity (Theorem A). We start by introducing the four conditions which define good localisations. They are precisely what is necessary in order to generalise Pop's proof of the birational liftable section conjecture to localisations of curves.

5.1 Conditions for a good localisation

Let p be a prime number, let k be a field with $\mathrm{char}(k) \neq p$ such that $\mu_p \subseteq k$, and let X/k be a smooth, proper, geometrically connected curve. Denote by $K = \kappa(X)$ be the function field of X. For a closed point $x \in X_{\mathrm{cl}}$, the degree of x is defined as $\deg(x) \coloneqq [\kappa(x) : k]$. Let $S \subseteq X_{\mathrm{cl}}$ be a set of closed points. In this situation, we formulate the following conditions:

(Sep) *For all $x \neq y$ in $X(k)$, the map*

$$\mathcal{O}(X_{S \cup \{x,y\}})^{\times} \to k^{\times}/k^{\times p}$$

given by $f \mapsto f(x)/f(y)$ is nontrivial.

(Pic) *Every geometrically connected, finite $\mathbb{Z}/p\mathbb{Z}$-elementary abelian cover $W \to X_S$ satisfies*

$$\mathrm{Pic}(W)/p = 0.$$

(Rat) *For all non-rational closed points $x \in X_{\mathrm{cl}}$ for which* $\deg(x)$ *is not divisible by p, the map*

$$\mathcal{O}(X_{S \cup \{x\}})^{\times} \to \kappa(x)^{\times} / \, k^{\times} \kappa(x)^{\times p}$$

given by $f \mapsto f(x)$ is nontrivial.

The fourth condition assumes that k is a finite extension of $\mathbb{Q}_p$. For a valuation w on K, we denote by K_w^{h} the henselisation of K with respect to w.

(Fin) *For every rank one valuation w on K extending the p-adic valuation on k, the following map has finite cokernel:*

$$\mathcal{O}(X_S)^{\times} \to (K_w^{\mathrm{h}})^{\times}/(K_w^{\mathrm{h}})^{\times p}.$$

Definition 5.1.1. Let k be a finite extension of $\mathbb{Q}_p$ with $\mu_p \subseteq k$ and let X/k be a smooth, proper, geometrically connected curve. Given a set of closed points $S \subseteq X_{\mathrm{cl}}$ we say that X_S is a **good localisation** if the conditions (Sep), (Pic), (Rat), and (Fin) are satisfied.

Remark 5.1.2. Condition (Sep) expresses a way of *separating* rational points by functions; Condition (Pic) concerns the *Picard* group; Condition (Rat) distinguishes *rational* from non-rational closed points; (Fin) is a *finiteness* condition.

Remark 5.1.3. The assumption $\mu_p \subseteq k$ is not necessary in formulating or, at least in the cases we consider, verifying the conditions of a good localisation. We include the assumption nevertheless as the importance of the notion of good localisation is merely to provide sufficient conditions under which we can prove the liftable section conjecture, and for that purpose the assumption $\mu_p \subseteq k$ is crucial. In fact, the liftable section conjecture is false in general without this assumption (see Remark 1.4.2).

Regarding Condition (Rat), note that the target of the map

$$\mathcal{O}(X_{S\cup\{x,y\}})^\times \to \kappa(x)^\times/k^\times\kappa(x)^{\times p}$$

is trivial if x is a k-rational point. In order to have a chance of Condition (Rat) being satisfied, the group $\ell^\times/k^\times\ell^{\times p}$ should be nontrivial whenever $k \subsetneq \ell$ is a nontrivial finite extension with $p \nmid [\ell : k]$:

Lemma 5.1.4. *Let k be a finite extension of $\mathbb{Q}_p$. Then, for every nontrivial finite extension $k \subsetneq \ell$, the natural map*

$$k^\times/k^{\times p} \to \ell^\times/\ell^{\times p}$$

is not surjective.

Proof. The multiplicative group of a p-adic local field k has the structure

$$k^\times = \pi^{\mathbb{Z}} \times \mu_{q-1} \times (1 + \mathfrak{m}_k)$$

where π is a uniformiser, q is the cardinality of the residue field, and $1 + \mathfrak{m}_k$ is the group of principal units of k. The latter is a finitely generated $\mathbb{Z}_p$-module of rank $[k : \mathbb{Q}_p]$, with torsion part consisting of the p-power roots of unity in k. As a consequence, we have

$$\dim_{\mathbb{F}_p}(k^\times/k^{\times p}) = 1 + \delta_k + [k : \mathbb{Q}_p]$$

where we set

$$\delta_k = \begin{cases} 1, & \text{if } \mu_p \subseteq k, \\ 0, & \text{otherwise.} \end{cases}$$

For a nontrivial finite extension ℓ/k we have $[k : \mathbb{Q}_p] < [\ell : \mathbb{Q}_p]$ and $\delta_k \le \delta_\ell$. This implies

$$\dim_{\mathbb{F}_p}(k^\times/k^{\times p}) < \dim_{\mathbb{F}_p}(\ell^\times/\ell^{\times p}),$$

so that the map $k^\times/k^{\times p} \to \ell^\times/\ell^{\times p}$ cannot be surjective. □

Corollary 5.1.5. *Let k be a finite extension of $\mathbb{Q}_p$ with $\mu_p \subseteq k$ and let X_S/k be a localisation of a curve such that for every non-rational closed point $x \in X_{\mathrm{cl}}$ with $p \nmid \deg(x)$, the map*

$$\mathcal{O}(X_{S\cup\{x\}})^\times \to \kappa(x)^\times/\kappa(x)^{\times p}$$

given by $f \mapsto f(x)$ is surjective. Then X_S satisfies Condition (Rat).

Proof. The targets of the maps in question are nontrivial by Lemma 5.1.4, so if the maps are surjective, they are nontrivial. □

In Chapter 6 below, we prove some general criteria which imply the goodness of a localisation in terms of approximation of rational functions on X by invertible functions on X_S, and in terms of approximation of divisors on X by divisors with support outside of S. The conditions are verified in some cases in Chapter 7.

5.2 Outline of the proof

We give here a short summary of the proof of Theorem A. We start by treating in Section 5.3 the injectivity statement of the liftable section conjecture, i.e. we show that a liftable section lies over at most one rational point. This part works over general base fields containing the p-th roots of unity and uses only Condition (Sep) of a good localisation, which expresses a way of separating rational points by functions.

To discuss the proof of the existence statement, let us fix some notation. Let k be a finite extension of $\mathbb{Q}_p$ with $\mu_p \subseteq k$, let X/k be a smooth, proper, geometrically connected curve and let $S \subseteq X_{\mathrm{cl}}$ be a set of closed points such that X_S is a good localisation. Let $\overline{k}/k$ be an algebraic closure, $G_k \coloneqq \mathrm{Gal}(\overline{k}/k)$ the absolute Galois group and $\pi_1(X_S)$ the étale fundamental group of X_S with respect to a geometric base point on $X_S \otimes_k \overline{k}$. Denote by K the function field of X and by K'_S/K the maximal $\mathbb{Z}/p\mathbb{Z}$-elementary abelian extension which is unramified over X_S, so that we have $\mathrm{Gal}(K'_S/K) = \pi_1(X_S)'$. Assume we are given a liftable section $s'\colon G'_k \to \pi_1(X_S)'$. The image of s' is a closed subgroup $\mathrm{im}(s') \subseteq \pi_1(X_S)'$ and hence corresponds to a subextension $K \subseteq M[s'] \subseteq K'_S$. We write $M \coloneqq M[s']$ to ease the notation, keeping in mind that the definition of M depends on the liftable section s'. Recall that the Brauer group of the p-adic field k is canonically isomorphic to $\mathbb{Q}/\mathbb{Z}$ via the invariant map from local class field theory. We denote by $\alpha \in \mathrm{Br}(k)$ the class with invariant $\frac{1}{p} \pmod{\mathbb{Z}}$. The proof of the existence of a rational point $x \in X(k)$ over which s' lies is divided into the following steps:

Step 1 In Section 5.4 we show that the map $\mathrm{Br}(k)[p] \to \mathrm{Br}(M)[p]$ of p-torsion in Brauer groups is injective. This is where the liftability of the section and Condition (Pic) of a good localisation are used. The proof uses a combination of group cohomology and étale cohomology.

Step 2 By Step 1, the Brauer class $\alpha \in \mathrm{Br}(k)$ with invariant $\frac{1}{p}$ maps to a non-zero class in $\mathrm{Br}(M)$. Pop's local-to-global principle for Brauer groups of function fields over p-adic fields, which is used as a black box, implies the existence of a rank 1 valuation w on M such that α does not vanish in the henselisation M_w^{h}. The image of the section s' is contained in the decomposition group of w. The aim is to show that w is equal to the discrete valuation at a k-rational point of X.

Step 3 The valuation w from Step 2 restricts to the p-adic valuation or the trivial valuation on k. In Section 5.7 the former is ruled out by showing that the residue characteristic of w cannot be positive. Denoting by K_w^{h} the henselisation of K with respect to the restriction $w|_K$, the extension $K_w^{\mathrm{h}} \subseteq M_w^{\mathrm{h}}$ is "almost maximal $\mathbb{Z}/p\mathbb{Z}$-abelian" in the sense that the maximal $\mathbb{Z}/p\mathbb{Z}$-abelian extension of K_w^{h} has finite degree over M_w^{h}. This is a consequence of Condition (Fin) of a good localisation. A subtle analysis of $\mathbb{Z}/p\mathbb{Z}$-abelian extensions of mixed characteristic henselian fields shows that this makes M_w^{h} too large for the Brauer class α to survive if the residue characteristic is of w positive.

Step 4 We conclude from Step 3 that the valuation w is the discrete valuation associated to a closed point x of X. In Section 5.8, Condition (Rat) is used to prove that this point must be k-rational.

Theorem C, which deduces information about the index of X from a liftable section $s' \colon G_k \to \pi_1(X_S)'$, is proved in Section 5.5 as a byproduct of Step 1. The conclusion is weaker than the liftable section conjecture but it holds under fewer assumptions: instead of X_S being a good localisation, only the validity of Condition (Pic) is required.

5.3 Distinguishing points by their $\mathbb{Z}/p\mathbb{Z}$-abelian sections

The first question in any variant of the section conjecture concerns the uniqueness of the k-rational point over which a section lies. This question is generally much easier than the existence question. In Proposition 5.3.2 below, we give a proof on the level of $\mathbb{Z}/p\mathbb{Z}$-abelian fundamental groups over an arbitrary base field k containing the p-th roots of unity for a localisation X_S satisfying Condition (Sep), thereby proving in particular the uniqueness statement in Theorem A.

In this section 5.3, let k be any field with $\mathrm{char}(k) \neq p$ containing the p-th roots of unity. Let X/k be a smooth, proper, geometrically connected curve and $S \subseteq X_{\mathrm{cl}}$ a set of closed points. Let K be the common function field of X and X_S and let K'_S/K be the maximal $\mathbb{Z}/p\mathbb{Z}$-abelian extension which is unramified over X_S. Recall from Proposition 2.5.4 that K'_S is obtained from K by adjoining p-th roots of the elements of the group

$$\Delta_S := \{f \in K^\times \ : \ v_s(f) \equiv 0 \bmod p \text{ for all } s \in S\},$$

where $v_s \colon K^\times \twoheadrightarrow \mathbb{Z}$ denotes the discrete valuation associated to a point $s \in S$. Let $\pi_1(X_S)' \to G'_k$ be the canonical projection of $\mathbb{Z}/p\mathbb{Z}$-elementary abelian fundamental groups induced by the structural morphism $X_S \to \operatorname{Spec}(k)$. For a closed point $x \in X_{\mathrm{cl}}$, denote by $\mathrm{D}_x \subseteq \pi_1(X_S)'$ its associated decomposition group. Note that D_x does not depend on the choice of a point over x since the group $\pi_1(X_S)'$ is abelian, so that any conjugation indeterminacy vanishes. Denote by K_x^{h} a henselisation of K at x.

Lemma 5.3.1. *With the notation as above, let $x, y \in X_{\mathrm{cl}}$ be two closed points. Then the intersection of decomposition groups $\mathrm{D}_x \cap \mathrm{D}_y$ maps non-surjectively to G'_k under the projection $\pi_1(X_S)' \to G'_k$ if and only if there exist $f \in \Delta_S$ and $a \in k^\times \setminus k^{\times p}$ such that*

$$f \in (K_x^{\mathrm{h}})^{\times p} \quad \textit{and} \quad af \in (K_y^{\mathrm{h}})^{\times p}.$$

Proof. Let k'/k be the maximal $\mathbb{Z}/p\mathbb{Z}$-abelian extension of k, which agrees with the relative algebraic closure of k in K'_S. The projection $\pi_1(X_S)' \to G'_k$ can be identified with the restriction map of Galois groups $\operatorname{Gal}(K'_S/K) \to \operatorname{Gal}(k'/k)$ in the field diagram

$$\begin{tikzcd}[column sep=small]
 & K'_S \arrow[dd, dash] \\
k' \arrow[ur, dash] \arrow[dd, dash] & \\
 & K \\
k \arrow[ur, dash] &
\end{tikzcd}$$

The image of $\mathrm{D}_x \cap \mathrm{D}_y$ in G'_k equals the Galois group of k' over $k' \cap (K'_S)^{\mathrm{D}_x \cap \mathrm{D}_y}$. This is a proper subgroup of G'_k if and only if the inclusion $k \subseteq k' \cap (K'_S)^{\mathrm{D}_x \cap \mathrm{D}_y}$ is strict. By the Galois correspondence for K'_S/K, the fixed field $(K'_S)^{\mathrm{D}_x \cap \mathrm{D}_y}$ equals the compositum $(K'_S)^{\mathrm{D}_x} \cdot (K'_S)^{\mathrm{D}_y}$. Let $\overline{K}/K'_S$ be an algebraic closure and choose henselisations K_x^{h} and K_y^{h} in $\overline{K}$. Then we have $(K'_S)^{\mathrm{D}_x} = K'_S \cap K_x^{\mathrm{h}}$, so that the extension $(K'_S)^{\mathrm{D}_x}/K$ corresponds via Kummer theory to the group

$$\ker\big(\Delta_S \to (K_x^{\mathrm{h}})^\times/(K_x^{\mathrm{h}})^{\times p}\big) = \Delta_S \cap (K_x^{\mathrm{h}})^{\times p},$$

and similarly for y. In conclusion, the fixed field $(K'_S)^{\mathrm{D}_x \cap \mathrm{D}_y}$ corresponds to the subgroup $(\Delta_S \cap (K_x^{\mathrm{h}})^{\times p}) \cdot (\Delta_S \cap (K_y^{\mathrm{h}})^{\times p})$ of $K^\times$. Using the Kummer correspondence for k, we obtain the equivalences

$$\begin{aligned}
& \mathrm{D}_x \cap \mathrm{D}_y \text{ maps non-surjectively to } G'_k \\
\Leftrightarrow \quad & k \subsetneq k' \cap (K'_S)^{\mathrm{D}_x \cap \mathrm{D}_y} \\
\Leftrightarrow \quad & k^{\times p} \subsetneq k^\times \cap (\Delta_S \cap (K_x^{\mathrm{h}})^{\times p}) \cdot (\Delta_S \cap (K_y^{\mathrm{h}})^{\times p}).
\end{aligned}$$

If $\mathrm{D}_x \cap \mathrm{D}_y$ maps non-surjectively to G'_k, then there exist $a \in k^\times \setminus k^{\times p}$ and $\tilde{f} \in \Delta_S \cap (K_x^{\mathrm{h}})^{\times p}$ and $g \in \Delta_S \cap (K_y^{\mathrm{h}})^{\times p}$ such that $a = \tilde{f} g$, and then $f := \tilde{f}^{-1}$ and a satisfy the claim. If, conversely, there are $f \in \Delta_S$ and $a \in k^\times \setminus k^{\times p}$ such that $f \in (K_x^{\mathrm{h}})^{\times p}$ and $af \in (K_y^{\mathrm{h}})^{\times p}$, then setting $\tilde{f} := f^{-1}$ and $g := a/\tilde{f}$, we have $g \in \Delta_S$ since a and f are contained in the group Δ_S, so that a is contained in the right hand side but not the left hand side in the last equivalence above. $\square$

Proposition 5.3.2. *With the notation as above, assume that the localisation X_S satisfies Condition* (Sep). *If a $\mathbb{Z}/p\mathbb{Z}$-abelian section $s' \colon G'_k \to \pi_1(X_S)$ satisfies $\mathrm{im}(s') \subseteq \mathrm{D}_x \cap \mathrm{D}_y$ with $x, y \in X(k)$, then $x = y$.*

Proof. Assume that $x \neq y$. Then by Condition (Sep), there exists some function $h \in \mathcal{O}(X_{S \cup \{x,y\}})^\times$ such that $h(x)/h(y)$ is not a p-th power in k. Set $a := h(x)/h(y) \in k^\times \setminus k^{\times p}$ and $f := h/h(x)$. Then we have $f \in \mathcal{O}(X_S)^\times \subseteq \Delta_S$ and

$$f(x) = 1, \quad af(y) = 1.$$

Since $\mathrm{char}(k) \neq p$, the polynomial $X^p - 1$ is separable over k. This implies by Hensel's Lemma that f is a p-th power in K_x^{h} and that af is a p-th power in K_y^{h}. So $\mathrm{D}_x \cap \mathrm{D}_y$ does not surject onto G'_k by Lemma 5.3.1, and in particular, there exists no section s' with image contained in $\mathrm{D}_x \cap \mathrm{D}_y$. $\square$

Remark 5.3.3. In the proof of Proposition 5.3.2 above, we only needed an element in $\Delta_S \cap \mathcal{O}_{X,x}^\times \cap \mathcal{O}_{X,y}^\times$ such that $f(x)/f(y)$ is not a p-th power. This is slightly weaker than Condition (Sep) which asserts the existence of such a function in the subgroup

$$\mathcal{O}(X_{S \cup \{x,y\}})^\times \subseteq \Delta_S \cap \mathcal{O}_{X,x}^\times \cap \mathcal{O}_{X,y}^\times.$$

However, in our case of interest, where k is a finite extension of $\mathbb{Q}_p$ and X_S is a good localisation, the two statements are almost equivalent. More precisely, for a good localisation we have $\mathrm{Pic}(X_S)/p = 0$ from Condition (Pic). Assume that even the localisation at the possibly larger set $S' := S \cup \{x, y\}$ satisfies $\mathrm{Pic}(X_{S'})/p = 0$ (and k is a p-adic field). We show in Proposition 6.1.1 below that this implies that $\mathrm{Pic}(X_{S'})$ is finite, so that we have $\mathrm{Pic}(X_{S'})[p] = 0$. Then the exact sequence from Remark 2.5.6 shows that the natural map $\mathcal{O}(X_{S'})^\times/p \to \Delta_{S'}/p$ is an isomorphism. With the inclusions

$$\mathcal{O}(X_{S'})^\times \subseteq \Delta_S \cap \mathcal{O}_{X,x}^\times \cap \mathcal{O}_{X,y}^\times \subseteq \Delta_{S'},$$

it follows that the natural map

$$(\mathcal{O}(X_{S'})^\times)/p \to (\Delta_S \cap \mathcal{O}_{X,x}^\times \cap \mathcal{O}_{X,y}^\times)/p$$

is surjective. Thus, under the assumption $\mathrm{Pic}(X_{S'})/p = 0$, the map

$$\Delta_S \cap \mathcal{O}_{X,x}^\times \cap \mathcal{O}_{X,y}^\times \to k^\times/k^{\times p}$$

given by $f \mapsto f(x)/f(y)$ is nontrivial if and only if it is nontrivial on $\mathcal{O}(X_{S \cup \{x,y\}})^\times$, as required by Condition (Sep).

5.4 Injectivity of p-torsion in Brauer groups

We now turn to the proof of the existence part of Theorem A. For the first step in this Section 5.4, k is still allowed to be any field of characteristic $\mathrm{char}(k) \neq p$ satisfying $\mu_p \subseteq k$, and the only restriction we impose on X_S is that Condition (Pic) be satisfied, i.e. every $\mathbb{Z}/p\mathbb{Z}$-elementary abelian finite étale cover $W \to X_S$ satisfy $\mathrm{Pic}(W)/p = 0$. We assume we are given a liftable section $s' \colon G_k' \to \pi_1(X_S)'$ and use the notation introduced in Section 5.2. In particular, $M = M[s']$ denotes the subextension $K \subseteq M \subseteq K_S'$ corresponding to the image of the liftable section $\mathrm{im}(s') \subseteq \pi_1(X_S)' = \mathrm{Gal}(K_S'/K)$. Let $W[s'] \to X_S$ be the corresponding profinite étale cover.

The aim of this section 5.4 is to prove the following injectivity statement on p-torsion in Brauer groups:

Proposition 5.4.1. *With notations and assumptions as above, the following map is injective:*

$$\mathrm{Br}(k)[p] \hookrightarrow \mathrm{Br}(M)[p].$$

5.4.1 A lemma in group cohomology

The following lemma is where the liftability of the section s' is used.

Lemma 5.4.2. *Let $\Pi \twoheadrightarrow G$ be a surjective homomorphism of profinite groups. Let $s' \colon G' \to \Pi'$ be a liftable section (Definition 1.3.2) and let $\Gamma \subseteq \Pi$ be the preimage of $\mathrm{im}(s')$ under the projection $\Pi \twoheadrightarrow \Pi'$. Then the maps*

$$\mathrm{H}^2(G'', \mathbb{Z}/p\mathbb{Z}) \to \mathrm{H}^2(G, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{H}^2(\Gamma, \mathbb{Z}/p\mathbb{Z})$$

are injective on the image of $\mathrm{H}^2(G', \mathbb{Z}/p\mathbb{Z})$ under inflation.

Proof. Consider a class in $\mathrm{H}^2(G', \mathbb{Z}/p\mathbb{Z})$ which maps to zero in $\mathrm{H}^2(\Gamma, \mathbb{Z}/p\mathbb{Z})$. It is represented by a central extension E of G' by $\mathbb{Z}/p\mathbb{Z}$ which admits a splitting $\varphi \colon \Gamma \to E$ over Γ.

$$\begin{tikzcd}
& \Gamma \arrow[r, hook] \arrow[d, two heads] \arrow[ddr, "\varphi"] & \Pi \arrow[r, two heads] \arrow[d, two heads] & G \arrow[d, two heads] & \\
& \Gamma/\Pi^{(2)} \arrow[r, hook] \arrow[dr, dashed, "\overline{\varphi}"'] & \Pi'' \arrow[r, two heads] & G'' \arrow[d, two heads] & \\
1 \arrow[r] & \mathbb{Z}/p\mathbb{Z} \arrow[r] & E \arrow[r] & G' \arrow[r] & 1.
\end{tikzcd}$$

The claim is that there is a splitting already over G''. By definition, Γ contains the $\mathbb{Z}/p\mathbb{Z}$-commutator subgroup $\Pi^{(1)} = [\Pi, \Pi]\Pi^p$, which maps to zero in the $\mathbb{Z}/p\mathbb{Z}$-elementary abelian group G', hence φ restricts to a homomorphism $\varphi|_{\Pi^{(1)}} \colon \Pi^{(1)} \to \mathbb{Z}/p\mathbb{Z}$. This map similarly sends $\Pi^{(2)}$ to zero, the $\mathbb{Z}/p\mathbb{Z}$-commutator subgroup of $\Pi^{(1)}$, so that φ factors through a map $\overline{\varphi} \colon \Gamma/\Pi^{(2)} \to E$.

Since $\Gamma/\Pi^{(2)}$ is the preimage of $\mathrm{im}(s')$ under the projection $\Pi'' \twoheadrightarrow \Pi'$, any lift $s''\colon G'' \to \Pi''$ of s' lands in $\Gamma/\Pi^{(2)}$. By assumption, such a lift s'' exists, and then $\overline{\varphi} \circ s''\colon G'' \to E$ is a splitting of E over G''. ☐

We apply Lemma 5.4.2 to the surjective homomorphism $\pi_1(X_S) \twoheadrightarrow G_k$ and liftable section s'. The $\mathbb{Z}/p\mathbb{Z}$-elementary abelian cover $W[s'] \to X_S$ corresponds to the closed subgroup $\mathrm{im}(s') \subseteq \pi_1(X_S)'$ by definition, therefore the preimage of $\mathrm{im}(s')$ under the projection $\pi_1(X_S) \twoheadrightarrow \pi_1(X_S)'$ equals the fundamental group $\pi_1(W[s'])$ (using compatible choices of geometric base points). Since $\mu_p \subseteq k$ by assumption, one can replace $\mathbb{Z}/p\mathbb{Z}$ with the trivial G_k-module μ_p in the lemma. Denoting by k'/k the maximal $\mathbb{Z}/p\mathbb{Z}$-abelian extension of k in K'_S, we find that the map

$$\mathrm{H}^2(k, \mu_p) \longrightarrow \mathrm{H}^2(\pi_1(W[s']), \mu_p) \tag{$*$}$$

is injective on the image of $\mathrm{H}^2(k'/k, \mu_p)$ under inflation. The following lemma shows that the image of $\mathrm{H}^2(k'/k, \mu_p)$ under inflation is in fact the full group $\mathrm{H}^2(k, \mu_p)$, so that the map $(*)$ is injective.

Lemma 5.4.3. *Let k be a field with* $\mathrm{char}(k) \neq p$ *satisfying $\mu_p \subseteq k$. Then the inflation map*

$$\mathrm{H}^2(k'/k, \mu_p) \longrightarrow \mathrm{H}^2(k, \mu_p)$$

is surjective.

Proof. Consider the commutative diagram of cup products and inflation maps

$$\begin{array}{ccccc}
\mathrm{H}^1(k'/k, \mu_p) & \times & \mathrm{H}^1(k'/k, \mu_p) & \xrightarrow{\ \smile\ } & \mathrm{H}^2(k'/k, \mu_p^{\otimes 2}) \\
\big\downarrow{\scriptstyle \mathrm{inf}} & & \big\downarrow{\scriptstyle \mathrm{inf}} & & \big\downarrow{\scriptstyle \mathrm{inf}} \\
\mathrm{H}^1(k, \mu_p) & \times & \mathrm{H}^1(k, \mu_p) & \xrightarrow{\ \smile\ } \!\!\!\!\rightarrow & \mathrm{H}^2(k, \mu_p^{\otimes 2}).
\end{array}$$

The inflation maps on H^1 are isomorphisms since μ_p is a trivial G_k-module and the map

$$\mathrm{Hom}(G'_k, \mu_p) \to \mathrm{Hom}(G_k, \mu_p)$$

is an isomorphism by the universal property of the maximal $\mathbb{Z}/p\mathbb{Z}$-abelian quotient. The lower horizontal map is surjective by the Merkurjev–Suslin theorem [MS82]. It follows that the inflation map on the right is surjective. We have $\mu_p^{\otimes 2} \cong \mu_p$ (non-canonically) as G_k-modules since $\mu_p \subseteq k$, hence also the map

$$\mathrm{inf}\colon\ \mathrm{H}^2(k'/k, \mu_p) \twoheadrightarrow \mathrm{H}^2(k, \mu_p)$$

is surjective. ☐

5.4.2 Étale cohomology and cohomology of the fundamental group

We recall how étale cohomology of a connected scheme relates to group cohomology of the profinite fundamental group. Let Z be a connected qcqs scheme, let $\pi\colon \widetilde{Z} \to Z$ be the universal profinite étale cover and let $\pi_1(Z) = \mathrm{Gal}(\widetilde{Z}/Z)$. For any sheaf $\mathcal{F}$ of abelian groups on the étale site $Z_{\text{ét}}$, we have a Hochschild–Serre spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(\pi_1(Z), \mathrm{H}^q(\widetilde{Z}, \pi^*\mathcal{F})) \Rightarrow \mathrm{H}^{p+q}(Z, \mathcal{F}),$$

obtained as the colimit of the Hochschild–Serre spectral sequences associated to the finite Galois subcovers of $\widetilde{Z} \to Z$. The spectral sequence induces comparison maps

$$\mathrm{H}^n(\pi_1(Z), \mathcal{F}(\widetilde{Z})) \longrightarrow \mathrm{H}^n(Z, \mathcal{F}), \quad n \geq 0,$$

defined as the edge homomorphisms $\mathrm{E}_2^{n,0} \twoheadrightarrow \mathrm{E}_\infty^{n,0} \hookrightarrow \mathrm{E}^n$. Here, we have

$$\mathcal{F}(\widetilde{Z}) \coloneqq \pi^*\mathcal{F}(\widetilde{Z}) = \operatorname*{colim}_i \mathcal{F}(Z_i),$$

the colimit being taken over all connected finite étale subcovers $\widetilde{Z} \to Z_i \to Z$.

Proposition 5.4.4. *Let $\mathcal{F}$ be a locally constant torsion sheaf on $Z_{\text{ét}}$. Then the comparison map*

$$\mathrm{H}^n(\pi_1(Z), \mathcal{F}(\widetilde{Z})) \longrightarrow \mathrm{H}^n(Z, \mathcal{F})$$

is an isomorphism for $n = 0, 1$, and injective for $n = 2$.

Proof. The claim follows if we show

$$\mathrm{H}^1(\widetilde{Z}, \pi^*\mathcal{F}) = 0,$$

since then $\mathrm{E}_2^{p,1} = 0$ for all $p \geq 0$ in the Hochschild–Serre spectral sequence. As étale cohomology commutes with colimits on a qcqs scheme, we may assume that $\mathcal{F}$ is locally constant finite. Then $\mathcal{F}$ is representable by a finite étale Z-group scheme $G_\mathcal{F}$ and we can write $\mathrm{H}^1(\widetilde{Z}, \pi^*\mathcal{F}) = \mathrm{H}^1(\widetilde{Z}, G_\mathcal{F})$. Again by the commutation of étale cohomology with limits, we have

$$\mathrm{H}^1(\widetilde{Z}, G_\mathcal{F}) = \operatorname*{colim}_i \mathrm{H}^1(Z_i, G_\mathcal{F}),$$

where $Z_i \to Z$ runs through the connected finite étale subcovers of $\widetilde{Z} \to Z$. The group $\mathrm{H}^1(Z_i, G_\mathcal{F})$ classifies $G_\mathcal{F}$-torsors over Z_i. Let $T \to Z_i$ be a $G_\mathcal{F}$-torsor. Then T is finite étale over Z_i because this can be checked on a trivialising cover and $G_\mathcal{F}$ is finite étale over Z. Every torsor becomes trivial over itself, so T becomes trivial after pulling back along $Z_i \times_Z T \to Z_i$. Since this is nonempty and finite étale over Z, there exists a map $Z_j \to Z_i \times_Z T$ for some j, and hence T becomes trivial in the colimit. $\square$

Returning to the proof of Proposition 5.4.1, the comparison maps in degree 2 for $\mathrm{Spec}(k)$ and $W[s']$ and the sheaf μ_p fit into a commutative square

$$\begin{tikzcd} \mathrm{H}^2(\pi_1(W[s']), \mu_p) \arrow[r, hook] & \mathrm{H}^2(W[s'], \mu_p) \\ \mathrm{H}^2(G_k, \mu_p) \arrow[u, hook, "(*)"] \arrow[r, equal] & \mathrm{H}^2(k, \mu_p). \arrow[u] \end{tikzcd}$$

The left vertical arrow is the map $(*)$ which we have already proved to be injective. With the injectivity of the comparison map from Proposition 5.4.4, we conclude that the map in étale cohomology

$$\mathrm{H}^2(k, \mu_p) \hookrightarrow \mathrm{H}^2(W[s'], \mu_p) \tag{$**$}$$

is injective as well.

5.4.3 The Brauer group and the Kummer sequence

For any scheme Z on which the prime p is invertible, the Kummer sequence yields a short exact sequence

$$0 \longrightarrow \mathrm{Pic}(Z)/p \longrightarrow \mathrm{H}^2(Z, \mu_p) \longrightarrow \mathrm{Br}(Z)[p] \longrightarrow 0.$$

Taking $Z = \mathrm{Spec}(k)$ and $Z = W[s']$, respectively, and using $\mathrm{Pic}(k) = 0$, the resulting short exact sequences fit into a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Pic}(W[s'])/p \arrow[r] & \mathrm{H}^2(W[s'], \mu_p) \arrow[r] & \mathrm{Br}(W[s'])[p] \arrow[r] & 0 \\ 0 \arrow[r] & 0 \arrow[u] \arrow[r] & \mathrm{H}^2(k, \mu_p) \arrow[u, hook, "(**)"] \arrow[r, equal] & \mathrm{Br}(k)[p] \arrow[u] \arrow[r] & 0. \end{tikzcd}$$

The middle vertical arrow is the map $(**)$ which was previously shown to be injective.

The scheme $W[s']$ is an inverse limit of geometrically connected finite étale $\mathbb{Z}/p\mathbb{Z}$-elementary abelian covers $W_i \to X_S$. Since X_S satisfies Condition (Pic) by assumption, we have $\mathrm{Pic}(W_i)/p = 0$ for all i. From this we obtain

$$\mathrm{Pic}(W[s'])/p = \varinjlim_i \mathrm{Pic}(W_i)/p = 0$$

by the compatibility of étale cohomology with inverse limits of schemes. By the previous discussion, this implies $\mathrm{H}^2(W[s'], \mu_p) = \mathrm{Br}(W[s'])[p]$, and hence the map

$$\mathrm{Br}(k)[p] \hookrightarrow \mathrm{Br}(W[s'])[p] \tag{$***$}$$

is injective.

5.4.4 Purity for the Brauer group

Finally, we recall Grothendieck's Purity Theorem for the Brauer group:

Theorem 5.4.5 ([Gro66], Corollaire 1.10)**.** *Let Z be a regular, integral scheme with function field K. Then the following map is injective:*

$$\mathrm{Br}(Z) \hookrightarrow \mathrm{Br}(K).$$

Proof. Denote by $j\colon \mathrm{Spec}(K) \to Z$ the inclusion of the generic point, and by $i_z\colon z \to Z$ the inclusion of a codimension 1 point $z \in Z^{(1)}$. Since Z is regular, Z is locally factorial by the Auslander–Buchsbaum theorem [AB59] (i.e. all local rings are factorial), so that the groups of Cartier divisors and Weil divisors on Z are canonically isomorphic. The regularity is inherited by every étale Z-scheme, therefore even the sheaves of Cartier divisors and Weil divisors on the étale site $Z_{\text{ét}}$ are isomorphic. Hence, we have the short exact divisor sequence of étale sheaves on Z

$$0 \to \mathbb{G}_{\mathrm{m}} \to j_*\mathbb{G}_{\mathrm{m}} \to \underline{\mathrm{Div}}_Z \to 0,$$

with the Weil divisor sheaf

$$\underline{\mathrm{Div}}_Z = \bigoplus_{z \in Z^{(1)}} (i_z)_* \mathbb{Z}.$$

We calculate

$$\mathrm{H}^1(Z, \underline{\mathrm{Div}}_Z) = \bigoplus_{z \in Z^{(1)}} \mathrm{H}^1(z, \mathbb{Z}) = \bigoplus_{z \in Z^{(1)}} \mathrm{Hom}(\mathrm{Gal}_{\kappa(z)}, \mathbb{Z}) = 0,$$

where the vanishing $\mathrm{Hom}(\mathrm{Gal}_{\kappa(z)}, \mathbb{Z}) = 0$ follows from the fact that the image of any homomorphism (of topological groups) $G_k \to \mathbb{Z}$ is a compact subgroup of $\mathbb{Z}$, but the trivial group is the only such subgroup. It follows now from the long exact cohomology sequence that the following map is injective:

$$\mathrm{Br}(Z) \hookrightarrow \mathrm{H}^2(Z, j_*\mathbb{G}_{\mathrm{m}}).$$

Consider the Leray spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(Z, \mathrm{R}^q j_*\mathbb{G}_{\mathrm{m}}) \Rightarrow \mathrm{H}^{p+q}(K, \mathbb{G}_{\mathrm{m}}).$$

The stalk of $\mathrm{R}^1 j_*\mathbb{G}_{\mathrm{m}}$ at a geometric point $\overline{z}$ of Z with strictly henselian local ring $\mathcal{O}_{Z,\overline{z}}$ is given by

$$(\mathrm{R}^1 j_*\mathbb{G}_{\mathrm{m}})_{\overline{z}} = \mathrm{H}^1(\mathrm{Spec}(\mathcal{O}_{Z,\overline{z}}) \times_Z \mathrm{Spec}(K), \mathbb{G}_{\mathrm{m}}) = \mathrm{H}^1(\mathrm{Frac}(\mathcal{O}_{Z,\overline{z}}), \mathbb{G}_m) = 0$$

by Hilbert's Theorem 90, hence we have $\mathrm{R}^1 j_*\mathbb{G}_{\mathrm{m}} = 0$. This implies $\mathrm{E}_2^{2,0} = \mathrm{E}_\infty^{2,0}$ in the spectral sequence, thus the edge map is injective:

$$\mathrm{H}^2(Z, j_*\mathbb{G}_{\mathrm{m}}) \hookrightarrow \mathrm{Br}(K).$$

The map $\mathrm{Br}(Z) \to \mathrm{Br}(K)$ in question is the composite of the two maps proved to be injective. □

Write $W[s'] = \lim W_i$ as an inverse limit of connected finite étale covers of X_S and denote by M_i the function field of W_i. Then each of the maps $\mathrm{Br}(W_i) \to \mathrm{Br}(M_i)$ is injective by Theorem 5.4.5. Passing to the (co-)limit, we find that the map

$$\mathrm{Br}(W[s']) \hookrightarrow \mathrm{Br}(M)$$

is injective. Combining with the injectivity of $(***)$, the proof of Proposition 5.4.1 is finished.

5.5 Consequences for the index

Before continuing with the proof of the liftable section conjecture for good localisations, we want to explain how the injectivity result of Proposition 5.4.1 affects the index of a curve.

Let X/k be a smooth, proper, geometrically connected curve over an arbitrary field k.

Definition 5.5.1. The **index** of X/k is the greatest common divisor of the degrees $\deg(x) = [\kappa(x) : k]$ for all closed points $x \in X_{\mathrm{cl}}$:

$$\mathrm{index}(X) = \gcd\{\deg(x) \; : \; x \in X_{\mathrm{cl}}\}.$$

Definition 5.5.2. The **relative Brauer group** of X/k is defined as the kernel

$$\mathrm{Br}(X/k) := \ker\bigl(\mathrm{Br}(k) \to \mathrm{Br}(X)\bigr).$$

For every closed point $x \in X_{\mathrm{cl}}$, the restriction-corestriction map

$$\mathrm{Br}(k) \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(\kappa(x)) \xrightarrow{\mathrm{cor}_{\kappa(x)/k}} \mathrm{Br}(k)$$

equals multiplication by $\deg(x) = [\kappa(x) : k]$, which implies that the relative Brauer group $\mathrm{Br}(X/k)$ is annihilated by $\deg(x)$. It follows that $\mathrm{Br}(X/k)$ is annihilated by $\mathrm{index}(X)$. In particular, if k is a p-adic field, then under the isomorphism $\mathrm{Br}(k) \cong \mathbb{Q}/\mathbb{Z}$, the relative Brauer group $\mathrm{Br}(X/k)$ is contained in the subgroup $\frac{1}{\mathrm{index}(X)}\mathbb{Z}/\mathbb{Z}$. By a result of Roquette, this inclusion is in fact an equality:

Fact 5.5.3 ([Roq66]). *Let X/k be a smooth, projective curve over a finite extension $k/\mathbb{Q}_p$. Then, under the isomorphism $\mathrm{Br}(k) \cong \mathbb{Q}/\mathbb{Z}$, the relative Brauer group of X/k equals*

$$\mathrm{Br}(X/k) = \tfrac{1}{\mathrm{index}(X)}\mathbb{Z}/\mathbb{Z}.$$

If X contains a k-rational point, then clearly $\mathrm{index}(X) = 1$. Thus, for any set of closed points $S \subseteq X_{\mathrm{cl}}$ such that the localisation X_S satisfies the liftable section conjecture for some prime number p, we have the implication:

$$\exists \text{ liftable section } s' \colon G_k' \to \pi_1(X_S)' \;\Rightarrow\; \mathrm{index}(X) = 1.$$

But even without the full strength of the liftable section conjecture for X_S, we can draw conclusions about the index of X from the existence of liftable sections. The injectivity statement of Proposition 5.4.1 is enough to show the following:

Proposition 5.5.4 (= Theorem C (a))**.** *Let k be a finite extension of $\mathbb{Q}_p$ with $\mu_p \subseteq k$ and let X/k be a smooth, proper, geometrically connected curve. Let $S \subseteq X_{\mathrm{cl}}$ be a set of closed points such that the localisation X_S satisfies Condition (Pic). Then we have the implication:*

$$\exists \text{ liftable section } s' \colon G_k' \to \pi_1(X_S)' \ \Rightarrow\ p \nmid \operatorname{index}(X).$$

Proof. Let $s' \colon G_k' \to \pi_1(X_S)'$ be a liftable section, denote by M/K the $\mathbb{Z}/p\mathbb{Z}$-elementary abelian extension of the function field of X corresponding to the $\operatorname{im}(s') \subseteq \pi_1(X_S)'$. Proposition 5.4.1 says that the pullback map of Brauer groups along the composite morphism

$$\operatorname{Spec}(M) \to \operatorname{Spec}(K) \to X \to \operatorname{Spec}(k)$$

is injective on the p-torsion subgroup $\operatorname{Br}(k)[p]$ of the Brauer group. In particular, the map $\operatorname{Br}(k) \to \operatorname{Br}(X)$ is injective on the p-torsion subgroup, i.e. we have $\operatorname{Br}(X/k)[p] = 0$. Since k is a p-adic field, the relative Brauer group $\operatorname{Br}(X/k)$ is cyclic of order $\operatorname{index}(X)$ by Roquette's Theorem. This implies $p \nmid \operatorname{index}(X)$ as claimed. □

Combining this with a result of Stix, we obtain:

Corollary 5.5.5 (= Theorem C (b))**.** *Let k be a finite extension of $\mathbb{Q}_p$ with $\mu_p \subseteq k$ and let X/k be a smooth, proper, geometrically connected curve of genus $g > 0$. Let $S \subseteq X_{\mathrm{cl}}$ be a set of closed points such that X_S satisfies Condition (Pic). Then we have the implication:*

$$\exists \text{ section } s \colon G_k \to \pi_1(X_S) \ \Rightarrow\ \operatorname{index}(X) = 1.$$

Proof. Let $s \colon G_k \to \pi_1(X_S)$ be a section. Composing with $\pi_1(X_S) \to \pi_1(X)$ yields a section for the projection $\pi_1(X) \to G_k$. By [Sti10, Theorem 15], this implies that $\operatorname{index}(X)$ is a power of p. But s also induces a liftable section $s' \colon G_k' \to \pi_1(X_S)'$, so that we have $p \nmid \operatorname{index}(X)$ by Proposition 5.5.4, and hence $\operatorname{index}(X) = 1$. □

Remark 5.5.6. We show in Proposition 6.3.11 below that Condition (Pic) is satisfied for X_S when the complement of S is *uniformly dense* in X (Definition 6.3.8). Thus, Proposition 5.5.4 and Corollary 5.5.5 apply in particular to such localisations of curves.

5.6 A distinguished valuation

We continue with the proof of the liftable section conjecture for good localisations. From now on, k is assumed to be a finite extension of $\mathbb{Q}_p$ with $\mu_p \subseteq k$. Denote by $\mathbb{F}$ the finite residue field of k. Let X_S/k be a good localisation and assume we are given a liftable section $s'\colon G_k' \to \pi_1(X_S)'$. We use the notation introduced in Section 5.2. Thus, K_S'/K denotes the maximal $\mathbb{Z}/p\mathbb{Z}$-abelian extension which is unramified over X_S and $M = M[s']$ denotes the subextension $K \subseteq M \subseteq K_S'$ corresponding to the image of the liftable section $\mathrm{im}(s') \subseteq \pi_1(X_S)' = \mathrm{Gal}(K_S'/K)$. Moreover, $\alpha \in \mathrm{Br}(k)$ denotes the Brauer class with invariant $\frac{1}{p}$. By the injectivity statement of Proposition 5.4.1, the class α does not vanish after pulling back to $\mathrm{Br}(M)$:

$$\alpha|_M \neq 0.$$

5.6.1 The local-to-global principle for Brauer groups

For a valuation w on M, denote by M_w^{h} the henselisation.

Proposition 5.6.1. *There exists a rank* 1 *valuation w on M with $\mathcal{O}_k \subseteq \mathcal{O}_w$ such that we have*

$$\alpha|_{M_w^{\mathrm{h}}} \neq 0.$$

The key ingredient in the proof of Proposition 5.6.1 is Pop's local-to-global principle for Brauer groups of function fields over p-adically closed fields, which we use as a black box. The class of p-adically closed fields includes all finite extensions of $\mathbb{Q}_p$.

Fact 5.6.2 ([Pop88], Theorem 4.5)**.** *Let k be a p-adically closed field and M/k a field extension of transcendence degree* $\mathrm{trdeg}(M/k) = 1$. *Letting w run over the valuations of M extending the p-adic valuation v on k, the following map is injective:*

$$\mathrm{Br}(M) \hookrightarrow \prod_{w|v} \mathrm{Br}(M_w^{\mathrm{h}}).$$

Remark 5.6.3. If M is the function field of a smooth, proper, geometrically connected curve X over a p-adic field k, the above local-to-global principle is a consequence of the perfectness of Lichtenbaum's duality pairing [Lic69, Theorem 4]

$$\mathrm{Br}(X) \times \mathrm{Pic}(X) \to \mathrm{Br}(k) \cong \mathbb{Q}/\mathbb{Z}.$$

Pop's result, which is proved using model-theoretic methods, is a generalisation to arbitrary extensions of transcendence degree 1 which are not necessarily finitely generated. For a proof without the use of model theory see also [PS17, §4.1].

Proof of Proposition 5.6.1. By Proposition 5.4.1, we have $\alpha|_M \neq 0$. By Pop's local-to-global principle 5.6.2, there exists a valuation w on M extending the p-adic valuation v on k such that $\alpha|_{M_w^{\mathrm{h}}} \neq 0$. The rank of w is at least 1 since w is not trivial on k. Denote the value groups of w and v by Γ_w and $\Gamma_v \cong \mathbb{Z}$. The Dimension Inequality 4.2.6 yields

$$\operatorname{trdeg}(\kappa(w)/\mathbb{F}) + \operatorname{rr}(\Gamma_w/\Gamma_v) \leq \operatorname{trdeg}(M/k).$$

We have $\operatorname{rr}(\Gamma_v) = \operatorname{rr}(\mathbb{Z}) = 1$ and $\operatorname{trdeg}(M/k) = 1$. Using the additivity of the rational rank in short exact sequences and the fact that the rank of w is bounded by the rational rank (Fact 4.1.13), we find

$$\operatorname{rk}(\Gamma_w) \leq 2.$$

In particular, the rank of w is finite, so that there exists a unique rank 1 coarsening w_1 of w on M_w^{h}. The restriction $w_1|_k$ is a coarsening of the p-adic valuation of k, i.e. we have $\mathcal{O}_k \subseteq \mathcal{O}_{w_1}$. By Proposition 4.5.8, M_w^{h} is henselian also with respect to the coarsening w_1. By the universal property, the henselisation $M_{w_1}^{\mathrm{h}}$ embeds into M_w^{h}. Now $\alpha|_{M_w^{\mathrm{h}}} \neq 0$ implies that $\alpha|_{M_{w_1}^{\mathrm{h}}} \neq 0$. Thus, w_1 satisfies the claim. $\square$

5.6.2 The central field diagram

In the following, let w be a rank one valuation on M with $\mathcal{O}_k \subseteq \mathcal{O}_w$ such that $\alpha|_{M_w^{\mathrm{h}}} \neq 0$, which exists by Proposition 5.6.1. Extend w to a valuation $\overline{w}$ on algebraic closure $\overline{K}$ and let $L := K_w^{\mathrm{h}}$ and $\Lambda := M_w^{\mathrm{h}}$ be the henselisations of $(K, w|_K)$ and (M, w) inside $\overline{K}$ with respect to $\overline{w}$. Note that $L \subseteq \Lambda$. The compositum ML contains M and is henselian, being an algebraic extension of the henselian field Λ, so the inclusion $ML \subseteq \Lambda$ is an equality: $ML = \Lambda$. Set $L'_S := K'_S L = K'_S \Lambda$ inside $\overline{K}$, so that we have $L \subseteq \Lambda \subseteq L'_S$. Let k'/k be the maximal $\mathbb{Z}/p\mathbb{Z}$-abelian extension of k in K'_S. The fields fit into a diagram as follows:

$$\begin{tikzcd}
 & & L'_S = K'_S L \arrow[dd, dashed, dash] \\
 & K'_S \arrow[ur, dash] \arrow[dd, dashed, dash] & \\
k' \arrow[ur, dash] \arrow[dddd, dashed, dash] & & \Lambda = M_w^{\mathrm{h}} \arrow[dd, dash] \\
 & M \arrow[ur, dash] \arrow[dd, dash] & \\
 & & L = K_w^{\mathrm{h}} \\
 & K \arrow[ur, dash] & \\
k \arrow[ur, dash] & &
\end{tikzcd} \tag{5.6.1}$$

Lemma 5.6.4. *The canonical restriction maps of Galois groups for the three indicated extensions are isomorphisms:*

$$\mathrm{Gal}(L'_S/\Lambda) \xrightarrow{\sim} \mathrm{Gal}(K'_S/M) \xrightarrow{\sim} \mathrm{Gal}(k'/k).$$

Proof. The second isomorphism holds by definition of M since the canonical projection map $\pi_1(X_S)' \to G'_k$ restricts to an isomorphism $\mathrm{im}(s') \cong G'_k$. The first restriction map is injective since L'_S is defined as the compositum $K'_S L$. The surjectivity of the composite map $\mathrm{Gal}(L'_S/\Lambda) \to \mathrm{Gal}(k'/k)$ is equivalent to the linear disjointness $k' \cap \Lambda = k$. Let $k_1 := k'\Lambda$. The map of Brauer groups $\mathrm{Br}(k) \to \mathrm{Br}(k_1)$ is isomorphic to multiplication by $[k_1 : k]$ on $\mathbb{Q}/\mathbb{Z}$ [NSW08, Cor. (7.1.4)]:

$$\begin{array}{ccc}
\mathrm{Br}(k_1) & \xrightarrow[\sim]{\mathrm{inv}} & \mathbb{Q}/\mathbb{Z} \\
\uparrow & & \uparrow{\scriptstyle [k_1:k]} \\
\mathrm{Br}(k) & \xrightarrow[\sim]{\mathrm{inv}} & \mathbb{Q}/\mathbb{Z}
\end{array}$$

If k_1/k is a nontrivial subextension of k', its degree $[k_1 : k]$ is a multiple by p, so that the Brauer class in $\mathrm{Br}(k)$ with invariant $\frac{1}{p}$ vanishes in $\mathrm{Br}(k_1)$. But this class survives in Λ by construction, hence we must have $k_1 = k$. □

5.6.3 Connection with the decomposition group

We introduced in Definition 3.1.3 the notion of a section lying over a closed point. We can similarly define that a section lies over a valuation $\tilde{w}$ on K if its image is contained in a decomposition group of $\tilde{w}$. In this sense, the given liftable section $s' : G'_k \to \pi_1(X_S)'$ lies over the valuation w that we obtained from Proposition 5.6.1:

Proposition 5.6.5. *The image of the liftable section s' is contained in the decomposition group $\mathrm{D}_w \subseteq \pi_1(X_S)'$.*

Proof. The henselisation $L = K_w^{\mathrm{h}}$ is the fixed field in $\overline{K}$ of the decomposition group $\mathrm{D}_{\overline{w}|w} \subseteq \mathrm{Gal}(\overline{K}/K)$ for an extension $\overline{w}|w$. The image of $\mathrm{D}_{\overline{w}|w}$ under the surjection $\mathrm{Gal}(\overline{K}/K) \to \mathrm{Gal}(K'_S/K) = \pi_1(X_S)'$ is the decomposition group $\mathrm{D}_w = \mathrm{Gal}(K'_S/K'_S \cap L)$. Lemma 5.6.4 implies

$$K'_S \cap L \subseteq K'_S \cap \Lambda = M,$$

hence we have

$$\mathrm{D}_w \supseteq \mathrm{Gal}(K'_S/M) = \mathrm{im}(s'). \qquad \square$$

5.7 Ruling out mixed characteristics

We have obtained in Proposition 5.6.1 a rank 1 valuation w on M satisfying $\mathcal{O}_k \subseteq \mathcal{O}_w$ such that $\alpha|_\Lambda \neq 0$ and $\mathrm{im}(s') \subseteq \mathrm{D}_w$, and we want to show that its restriction to K equals the discrete valuation associated to a k-rational point of X. Since $\mathcal{O}_k \subseteq \mathcal{O}_w$, the restriction of w to k equals either the p-adic valuation or the trivial valuation. The valuations associated to closed points are precisely those that are trivial on k, so the former has to be ruled out. The key step is to show that the extension Λ/L is an "almost maximal $\mathbb{Z}/p\mathbb{Z}$-abelian" in the sense that the maximal $\mathbb{Z}/p\mathbb{Z}$-abelian extension L'/L has finite degree over Λ. If the residue field of w has characteristic p, this makes the extension Λ/L "too large" for the Brauer class $\alpha \in \mathrm{Br}(k)$ with invariant $\frac{1}{p}$ to survive in Λ.

5.7.1 An almost maximal $\mathbb{Z}/p\mathbb{Z}$-abelian extension

We start by proving the claimed finiteness of the extension L'/Λ.

Lemma 5.7.1. *Assume that w extends the p-adic valuation on k. Then the degree $[L' : L'_S]$ is finite.*

Proof. By definition, L'_S equals the compositum $K'_S L$. Recall from Proposition 2.5.4, that K'_S/K is the $\mathbb{Z}/p\mathbb{Z}$-elementary abelian Kummer extension obtained by adjoining p-th roots of the elements of the group

$$\Delta_S \coloneqq \{f \in K^\times \ : \ v_s(f) \equiv 0 \bmod p \text{ for all } s \in S\}.$$

Hence, L'_S/L is the Kummer extension corresponding to the subgroup $\Delta_S L^{\times p} \subseteq L^\times$. The Galois group $\mathrm{Gal}(L'/L'_S)$ is μ_p-dual via the Kummer pairing to the quotient $L^\times/\Delta_S L^{\times p}$, so the claimed finiteness amounts to the map $\Delta_S \to L^\times/L^{\times p}$ having finite cokernel. Since Δ_S contains $\mathcal{O}(X_S)^\times$, this follows from Condition (Fin) for a good localisation. $\square$

Remark 5.7.2. We will show in Proposition 6.1.1 below that the vanishing $\mathrm{Pic}(X_S)/p = 0$ from Condition (Pic) is equivalent to $\mathrm{Pic}(X_S)$ being finite of order not divisible by p. This implies $\mathrm{Pic}(X_S)[p] = 0$, and the exact sequence from Remark 2.5.6 shows that the inclusion $\mathcal{O}(X_S)^\times \hookrightarrow \Delta_S$ induces an isomorphism

$$\mathcal{O}(X_S)^\times/\mathcal{O}(X_S)^{\times p} \cong \Delta_S/K^{\times p}.$$

As a consequence, the map $\Delta_S \to L^\times/L^{\times p}$ having finite cokernel is equivalent to $\mathcal{O}(X_S)^\times \to L^\times/L^{\times p}$ doing so. In other words, Condition (Fin) expresses precisely the finiteness of the degree $[L' : L'_S]$.

Proposition 5.7.3. *Assume that w extends the p-adic valuation on k. Then the extension L'/Λ is finite.*

Proof. By Lemma 5.7.1, it suffices to show that L'_S/Λ is finite. By the isomorphism of Galois groups from Lemma 5.6.4, we have $[L'_S : \Lambda] = [k' : k]$. By Kummer theory (or local class field theory), the degree of the maximal $\mathbb{Z}/p\mathbb{Z}$-abelian extension k'/k is equal to the cardinality of $k^\times/k^{\times p}$, whose finiteness follows from the structure of $k^\times$. □

5.7.2 Almost maximal $\mathbb{Z}/p\mathbb{Z}$-abelian extensions of a mixed characteristic henselian field

In order to analyse the given situation, we put ourselves in the general setting where (L, w) is a henselian field of characteristic zero with residue characteristic $p > 0$ such that L contains the p-th roots of unity. Let $\mathfrak{m}_L$ be the valuation ideal and ℓ the residue field. Denote by ℓ' the residue field of the maximal $\mathbb{Z}/p\mathbb{Z}$-abelian extension L'/L with respect to the unique extension of w. We do not assume that the residue field of L is perfect; in particular, ℓ' may be strictly greater than the maximal $\mathbb{Z}/p\mathbb{Z}$-abelian extension of ℓ since it contains also the purely inseparable extension $\ell^{1/p}$.

p-th roots for the residue field

Proposition 5.7.4. *Let Λ be a subextension of L'/L such that the degree $[L' : \Lambda]$ is finite; denote by λ its residue field. Then we have $\ell^{1/p} \subseteq \lambda$.*

Proof. If ℓ is finite, then $\ell^{1/p} = \ell$ and there is nothing to prove, so we may assume that ℓ is infinite. By Kummer theory, $\Lambda = L(\Delta^{1/p})$ for a subgroup $L^{\times p} \subseteq \Delta \subseteq L^\times$. By assumption, we have

$$\#\big(L^\times/\Delta\big) = [L' : \Lambda] < \infty.$$

Let $\overline{\Delta}$ be the image of $\Delta \cap \mathcal{O}_L^\times$ under the reduction map $\mathcal{O}_L^\times \twoheadrightarrow \ell^\times$. The group $\ell^\times/\overline{\Delta}$ is a quotient of $\mathcal{O}_L^\times/(\Delta \cap \mathcal{O}_L^\times)$, which is in turn a subgroup of $L^\times/\Delta$, hence also $\ell^\times/\overline{\Delta}$ is finite. Since we are in characteristic p, the set $\ell \cap \lambda^p$ of elements of ℓ which become a p-th power in λ is a subfield of ℓ. It contains $\overline{\Delta}$ by construction. The claim now follows from the following lemma:

Lemma 5.7.5. *Let F/K be an extension of fields with F infinite and $F^\times/K^\times$ finite. Then $F = K$.*

Proof. Let $d \coloneqq [F : K] \leq \infty$. Then $\mathbb{P}_K(F) \coloneqq F^\times/K^\times$ is a $(d-1)$-dimensional projective space over K with only finitely many points. But K is infinite since $K^\times$ has finite index in the infinite group $F^\times$. So we must have $d = 1$. □

Dividing by p in the value group

Proposition 5.7.6. *Let Λ be a subextension of L'/L such that the degree $[L' : \Lambda]$ is finite. Assume that ℓ is not finite or $w(L)$ is not discrete. Then* *for every element $\gamma \in w(L)$ there exists a subextension $E \subseteq \Lambda$ of the form $E = L(t^{1/p})$ with $t \in L^\times$ such that γ becomes divisible by p in the value group of E, i.e. $\gamma \in pw(E)$.*

Lemma 5.7.7. *Let $x \in L$ such that $w(x) \leq \frac{p}{p-1} w(p)$. Then $w(x)$ becomes divisible by p in the value group of the extension $L((1+x)^{1/p})$.*

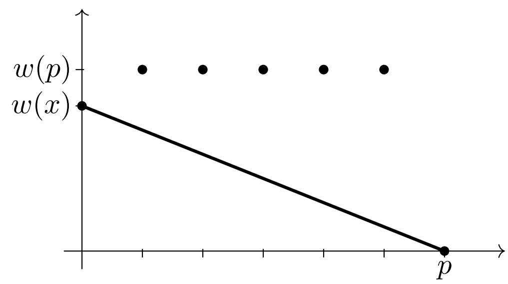
Proof. The Newton polygon of

$$f(X) := (1+X)^p - (1+x) = X^p + \sum_{i=1}^{p-1} \binom{p}{i} X^i - x,$$

is the lower convex hull of the points

$$(0, w(x)), \quad (1, w(p)), \quad \ldots \quad (p-1, w(p)), \quad (p, 0).$$

The condition $w(x) \leq \frac{p}{p-1} w(p)$ expresses that the points $(i, w(p))$ for $0 < i < p$ lie on or above the line segment connecting $(0, w(x))$ with $(p, 0)$.



The Newton polygon consists therefore only of this one line segment with slope $-w(x)/p$ and the roots of f in the splitting field $L((1+x)^{1/p})$ all have valuation $w(x)/p$. In particular, $w(x)$ becomes divisible by p in the value group of the extension. ☐

Lemma 5.7.8. *Let (L, w) be a valued field and let $x, y, z \in \mathfrak{m}_L$ be three elements satisfying $(1+x)(1+y) = 1+z$. Then we have $w(z) \geq \min(w(x), w(y))$, and equality holds if $w(x) \neq w(y)$.*

Proof. After writing $x + y + xy = z$, this follows from from analogous additive statement (Lemma 4.1.8). ☐

Proof of Proposition 5.7.6. Let $\Lambda = L(\Delta^{1/p})$ with $L^{\times p} \subseteq \Delta \subseteq L^\times$. As in the proof of Proposition 5.7.4 above, the index of Δ in $L^\times$ is finite. Observe that if $\gamma \in pw(E)$ for some subextension E of Λ, then we have $\gamma + pw(L) \subseteq pw(E)$. So given any $\gamma \in w(L)$, in order to show that γ becomes divisible by p in an extension of the form $E = L(t^{1/p}) \subseteq \Lambda$, we are free to change γ by adding an element of $pw(L)$.

Assume first that ℓ is infinite. We claim that any $\gamma > 0$ in $w(L)$ is of the form $\gamma = w(x)$ with $1 + x \in \Delta$. This is equivalent to saying that for every $t \in \mathfrak{m}_L$, there exists a unit $u \in \mathcal{O}_L^\times$ such that $1 + tu \in \Delta$. Observe that the elements $1 + tu$ with u a unit are precisely the elements not mapped to zero by the homomorphism

$$\psi\colon (1 + t\mathcal{O}_L, \cdot) \longrightarrow (\ell, +), \quad 1 + ty \mapsto y \bmod \mathfrak{m}_L.$$

Hence, we have to show that the restriction of ψ to $\Delta \cap (1 + t\mathcal{O}_L)$ is nontrivial. But $\Delta \cap (1 + t\mathcal{O}_L)$ has finite index in $1 + t\mathcal{O}_L$, so its image under ψ has finite index in ℓ and is thus infinite, in particular nontrivial, which proves the claim. Now let $\gamma \in w(L)$ be arbitrary. Set $\lambda := 1 - \zeta$ for a primitive p-th root of unity $\zeta \in L^\times$. We have $pw(\lambda) = \frac{p}{p-1} w(p)$. Since $w(L)$ embeds into $\mathbb{R}$ as an ordered group, there exists a unique integer $m \in \mathbb{Z}$ such that

$$0 < \gamma + mpw(\lambda) \leq \frac{p}{p-1} w(p).$$

As observed above, we are free to replace γ with $\gamma + mpw(\lambda)$ and may thus assume $\gamma \in (0, \frac{p}{p-1} w(p)]$. By our claim, there exists $x \in L^\times$ with $\gamma = w(x)$ and $1 + x \in \Delta$, and we are done by Lemma 5.7.7.

Assume now that $w(L)$ is nondiscrete. Then $pw(L)$ is dense in $w(L)$ and it is enough to consider only those $\gamma \in w(L)$ contained in an interval $(0, \varepsilon)$ for some suitable $\varepsilon > 0$. Since $\Delta \cap (1 + \mathfrak{m}_L)$ has finite index in $1 + \mathfrak{m}_L$, we can choose finitely many coset representatives $1 + a_i$. Then, for every $x \in \mathfrak{m}_L$ there exists an i such that $(1+x)(1+a_i) \in \Delta$. Define $z \in \mathfrak{m}_L$ by $1+z = (1+x)(1+a_i) \in \Delta$, then Lemma 5.7.8 implies $w(z) = w(x)$ if $w(x) < w(a_i)$. Therefore, if we set $\varepsilon := \min_i w(a_i)$, then every value in the interval $(0, \varepsilon)$ is of the form $w(z)$ with $1 + z \in \Delta$, and hence becomes divisible by p in an extension of the desired form by Lemma 5.7.7. $\square$

Remark 5.7.9. The hypothesis in Proposition 5.7.6 that ℓ be not finite or $w(L)$ not discrete is necessary: if $L = k$ is a finite extension of $\mathbb{Q}_p$, then k'/k is finite and hence k itself is an "almost maximal $\mathbb{Z}/p\mathbb{Z}$-abelian extension" of k, but the conclusion of Proposition 5.7.6 that every element of the value group $w(k) \cong \mathbb{Z}$ becomes divisible by p in the (trivial) extension k does not hold.

The following lemma shows that, up to taking completions, local fields are the only counterexamples:

Lemma 5.7.10. *If the residue field ℓ is finite and the value group $w(L)$ is discrete, then the completion $\widehat{L}$ of L is a finite extension of $\mathbb{Q}_p$.*

Proof. The restriction of w to $\mathbb{Q} \subseteq L$ is the p-adic valuation since $\mathrm{char}(\ell) = p$. Taking completions, we get $\mathbb{Q}_p \subseteq \widehat{L}$. With respect to the w-adic topology, $\widehat{L}$ is locally compact since $\mathcal{O}_{\widehat{L}} = \varprojlim_n \mathcal{O}_L/\mathfrak{m}_L^n$ is profinite. This implies that the dimension of $\widehat{L}$ over $\mathbb{Q}_p$ is finite by general facts about topological vector spaces [Bou87, Ch. I, §3.4, Theorem 3]. $\square$

5.7.3 The Brauer group of a henselian field in unramified extensions

Consider an arbitrary henselian field L with respect to a valuation w, and a finite unramified Galois extension E/L. Denote by e/ℓ the residue field extension, which is Galois with the same group $G \coloneqq \mathrm{Gal}(E/L) \cong \mathrm{Gal}(e/\ell)$. Write $\mathrm{Br}(E/L)$ (and similarly $\mathrm{Br}(e/\ell)$) for the **relative Brauer group** of the extension E/L:

$$\mathrm{Br}(E/L) = \ker\Big(\mathrm{Br}(L) \longrightarrow \mathrm{Br}(E)\Big).$$

The relative Brauer group $\mathrm{Br}(E/L)$ can also be described as the Galois cohomology group $\mathrm{H}^2(G, E^\times)$ by the inflation-restriction exact sequence (which starts in degree 2 by Hilbert's Theorem 90).

The aim of this subsection is to derive the following result:

Proposition 5.7.11 ([Pop88, §2])**.** *There is a natural short exact sequence:*

$$0 \longrightarrow \mathrm{Br}(e/\ell) \longrightarrow \mathrm{Br}(E/L) \longrightarrow \mathrm{Hom}(G, w(L) \otimes \mathbb{Q}/\mathbb{Z}) \longrightarrow 0.$$

Lemma 5.7.12. *The group of principal units $1 + \mathfrak{m}_E$ is G-cohomologically trivial.*

Proof. By [Ser79, Ch. IX §5, Theorem 8], we have to show that for each prime p and each p-Sylow subgroup G_p of G, the Tate cohomology group $\widehat{\mathrm{H}}^q(G_p, 1+\mathfrak{m}_E)$ vanishes for two consecutive integers q. Renaming E^{G_p} as L and G_p as G, it suffices to show that

$$\widehat{\mathrm{H}}^q(G, 1 + \mathfrak{m}_E) = 0 \quad \text{for } q = 0, 1.$$

Noting that $(1 + \mathfrak{m}_E)^G = 1 + \mathfrak{m}_L$, the claim for $q = 0$ is that the norm map

$$\mathrm{Nm}_{E/L} \colon 1 + \mathfrak{m}_E \longrightarrow 1 + \mathfrak{m}_L$$

is surjective. The fact that E/L is unramified implies that the trace maps for E/L and e/ℓ satisfy

$$\mathrm{Tr}_{E/L}(x) \bmod \mathfrak{m}_L = \mathrm{Tr}_{e/\ell}(\overline{x}) \quad \text{for } x \in \mathcal{O}_E,$$

where $\overline{x}$ is the reduction of x modulo $\mathfrak{m}_E$. As e/ℓ is separable, we can choose $\alpha \in \mathcal{O}_E$ such that $\mathrm{Tr}_{E/L}(\alpha) \not\equiv 0 \bmod \mathfrak{m}_E$. Now let $1+y \in 1+\mathfrak{m}_L$ and consider the equation

$$\mathrm{Nm}_{E/L}(1 + \alpha X) = 1 + y$$

in the variable X. The polynomial

$$\begin{aligned} P(X) &\coloneqq \mathrm{Nm}_{E/L}(1 + \alpha X) - (1 + y) \\ &= \mathrm{Nm}_{E/L}(\alpha)X^n + \ldots + \mathrm{Tr}_{E/L}(\alpha)X - y, \end{aligned}$$

where $n = [E : L]$, has coefficients in $\mathcal{O}_L$ and satisfies

$$P(0) = -y \equiv 0 \bmod \mathfrak{m}_L,$$
$$P'(0) = \mathrm{Tr}_{E/L}(\alpha) \not\equiv 0 \bmod \mathfrak{m}_L.$$

By Hensel's Lemma, P has a root $x \in \mathfrak{m}_L$, so that $\mathrm{Nm}_{E/L}(1 + \alpha x) = 1 + y$.

For $q = 1$, we have to show $\mathrm{H}^1(G, 1 + \mathfrak{m}_E) = 0$. The short exact sequence of G-modules

$$0 \longrightarrow 1 + \mathfrak{m}_E \longrightarrow \mathcal{O}_E^\times \longrightarrow e^\times \longrightarrow 0 \tag{5.7.1}$$

yields an exact sequence

$$\ldots \longrightarrow \mathcal{O}_L^\times \longrightarrow \ell^\times \longrightarrow \mathrm{H}^1(G, 1 + \mathfrak{m}_E) \longrightarrow \mathrm{H}^1(G, \mathcal{O}_E^\times) \longrightarrow \ldots$$

The first map is clearly surjective. The group $\mathrm{H}^1(G, \mathcal{O}_E^\times)$ equals the kernel of the map $\mathrm{Pic}(\mathcal{O}_L) \to \mathrm{Pic}(\mathcal{O}_E)$ by the Hochschild–Serre spectral sequence for the finite étale Galois covering $\mathrm{Spec}(\mathcal{O}_E) \to \mathrm{Spec}(\mathcal{O}_L)$ with group G. But $\mathrm{Pic}(\mathcal{O}_L) = 0$ as $\mathcal{O}_L$ is a local ring, so we have $\mathrm{H}^1(G, \mathcal{O}_E^\times) = 0$ and hence the group $\mathrm{H}^1(G, 1 + \mathfrak{m}_E)$ vanishes. □

In light of the short exact sequence (5.7.1), we obtain from Lemma 5.7.12:

Corollary 5.7.13. *For all integers $q \in \mathbb{Z}$, the natural map of Tate cohomology groups is an isomorphism:*

$$\widehat{\mathrm{H}}^q(G, \mathcal{O}_E^\times) \cong \widehat{\mathrm{H}}^q(G, e^\times).$$

In particular, $\mathrm{Br}(\mathcal{O}_E/\mathcal{O}_L) = \mathrm{Br}(e/\ell)$. □

Proposition 5.7.11 is now the special case $n = 2$ of the following proposition:

Proposition 5.7.14. *For all $n \geq 1$, there is a natural short exact sequence:*

$$0 \longrightarrow \mathrm{H}^n(G, e^\times) \longrightarrow \mathrm{H}^n(G, E^\times) \longrightarrow \mathrm{H}^{n-1}(G, w(L) \otimes \mathbb{Q}/\mathbb{Z}) \longrightarrow 0.$$

Proof. Consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_E^\times \longrightarrow E^\times \xrightarrow{w} w(E) \longrightarrow 0.$$

We claim that the valuation map w induces surjections under $\mathrm{H}^n(G, -)$ for all $n \geq 0$, so that the resulting long exact sequence splits into short exact sequences

$$0 \longrightarrow \mathrm{H}^n(G, \mathcal{O}_E^\times) \longrightarrow \mathrm{H}^n(G, E^\times) \longrightarrow \mathrm{H}^n(G, w(E)) \longrightarrow 0, \tag{5.7.2}$$

for all $n \geq 0$.

Since E/L is unramified, $L^\times$ surjects onto $w(E)$. As a totally ordered abelian group, $w(E)$ is torsion-free. Hence, every finitely generated subgroup Γ of $w(E)$ is $\mathbb{Z}$-free and admits a section s as follows:

$$
\begin{tikzcd}
 & \Gamma \arrow[dl, dashed, "s"'] \arrow[d, hook] \\
L^\times \arrow[r, two heads, "w"] & w(E).
\end{tikzcd}
$$

As a map $\Gamma \to E^\times$, the section s is G-equivariant since it takes values in the G-invariant subgroup $L^\times \subseteq E^\times$. Applying $\mathrm{H}^n(G, -)$ yields a commuting diagram

$$
\begin{tikzcd}
 & \mathrm{H}^n(G, \Gamma) \arrow[dl, dashed, "s"'] \arrow[d] \\
\mathrm{H}^n(G, E^\times) \arrow[r, "w"] & \mathrm{H}^n(G, w(E)).
\end{tikzcd}
$$

Taking the direct limit over all finitely generated subgroups Γ of $w(E)$ shows the claimed surjectivity and therefore the exactness of the sequences (5.7.2).

The proposition now follows from $w(L) = w(E)$, Corollary 5.7.13, and the isomorphism

$$\mathrm{H}^{n-1}(G, w(L) \otimes \mathbb{Q}/\mathbb{Z}) \cong \mathrm{H}^n(G, w(L)) \quad \text{for } n \geq 1$$

that arises by tensoring the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ with the torsion-free group $w(L)$. □

5.7.4 p-cyclic Brauer classes

Let K be an arbitrary field of characteristic p. A Brauer class in $\mathrm{Br}(K)$ is called **p-cyclic** if it becomes trivial over a cyclic Galois extension L/K of degree p. Such an extension is an Artin–Schreier extension $L = K(\alpha)$, obtained by adjoining a root α of a polynomial $X^p - X - a = 0$ for some $a \in K$. Interpreting $\mathrm{Br}(K)$ as equivalence classes of central simple algebras, every p-cyclic algebra which splits over L has a presentation of the form

$$[a, b) \coloneqq \left\langle x, y \;\middle|\; x^p - x = a,\ y^p = b,\ xy = yx + y \right\rangle \quad \text{for some } b \in K^\times.$$

The fact that $[a, b)$ splits over $K(\alpha)$ can be seen as follows. A central simple algebra of prime degree p (square root of the dimension) is either split or a division algebra. Over the Artin–Schreier extension $K(\alpha)$, the equation

$$\prod_{i=0}^{p-1} (x - \alpha - i) = (x - \alpha)^p - (x - \alpha) = 0$$

shows that $[a, b)$ has zero-divisors, so it must be split. Similarly, if β denotes a root of $X^p - b = 0$, we have the equation

$$(y - \beta)^p = y^p - \beta^p = 0,$$

which shows that $[a, b)$ is also split by the purely inseparable extension $K(b^{1/p})$.

Proposition 5.7.15. *Let K be a field of characteristic p. Then every p-cyclic Brauer class in* $\mathrm{Br}(K)$ *is split by* $K^{1/p}$.

Proof. We have sketched above a proof in the language of central simple algebras, but since we have so far preferred to view $\mathrm{Br}(K)$ as the cohomological Brauer group $\mathrm{H}^2(K, \mathbb{G}_m)$, we give a self-contained proof using Galois cohomology. Let L/K be a cyclic Galois extension of degree p with group G. By the 2-periodicity of the (Tate) cohomology of cyclic groups, we have an isomorphism

$$\theta_K \colon K^\times / \mathrm{Nm}_{L/K}(L^\times) \xrightarrow{\sim} \mathrm{Br}(L/K),$$

so any p-cyclic Brauer class trivialised by L is of the form $\theta_K(b)$ for some $b \in K^\times$. Let $\beta := b^{1/p}$, then the extension $L(\beta)/K(\beta)$ is also an Artin–Schreier extension with Galois group canonically isomorphic to G, and we have a commutative square of restriction maps

$$\begin{tikzcd} K^\times / \mathrm{Nm}_{L/K}(L^\times) \arrow[r, "\theta_K", "\sim"'] \arrow[d] & \mathrm{Br}(L/K) \arrow[d] \\ K(\beta)^\times / \mathrm{Nm}_{L(\beta)/K(\beta)}(L(\beta)^\times) \arrow[r, "\theta_{K(\beta)}", "\sim"'] & \mathrm{Br}(L(\beta)/K(\beta)). \end{tikzcd}$$

The element b becomes the norm of β in the degree p extension $L(\beta)/K(\beta)$, which shows that $\theta_K(b)$ becomes trivial over $K(\beta)$. □

Remark 5.7.16. The central simple algebra viewpoint and the Galois cohomology viewpoint are related as follows. Let L/K be an Artin–Schreier extension of K obtained by adjoining a root α of $X^p - X - a$. A generator σ of the Galois group G is given by $\sigma(\alpha) = \alpha + 1$. Define $\chi \colon G \cong \mathbb{Z}/p\mathbb{Z}$ by $\chi(\sigma) = 1$. Viewing χ as an element of $\mathrm{H}^1(G, \mathbb{Z}/p\mathbb{Z})$ and denoting by $\delta(\chi) \in \mathrm{H}^2(G, \mathbb{Z})$ its image along the boundary map from the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$, the periodicity isomorphism θ_K above (which depends on a through the choice of σ) is given by forming the cup product with $\delta(\chi)$:

$$\theta_K(b) = \delta(\chi) \smile b,$$

and $\theta_K(b)$ is the class representing the central simple algebra $[a, b)$ [GS17, Corollary 4.7.4].

5.7.5 Ruling out mixed characteristics

After these generalities, let us return to the proof of Theorem A. Recall that $\alpha \in \mathrm{Br}(k)$ is the Brauer class with invariant $\frac{1}{p}$, we have a rank 1 valuation w on M such that $\alpha|_{M_w^{\mathrm{h}}} \neq 0$, and we have the field diagram (5.6.1).

Proposition 5.7.17. *The residue field of the valuation w has characteristic zero.*

Proof. Assume that w has positive residue characteristic, i.e. the restriction of w to k equals the p-adic valuation. Let k_1/k be the unramified extension of degree p in a fixed algebraic closure $\overline{k}$ and let $G := \mathrm{Gal}(k_1/k)$ be its Galois group. For any extension F/k contained in $\overline{K}$, denote by F_1 the compositum $F_1 := Fk_1$. By taking the unique extension of w from L to $\overline{K}$ and restricting, F is endowed with a valuation. In this way, F/k is an extension of valued fields and the compositum F_1/F is unramified since k_1/k is. The residue field f_1 of F_1 is obtained as the compositum of the residue field f of F with the residue field $\mathbb{F}_1$ of k_1. If F is contained in Λ, then the Galois group $\mathrm{Gal}(F_1/F)$ is canonically isomorphic to G by Lemma 5.6.4.

Now take F equal to each of k, L, E, Λ in turn, for some suitable subextension E of Λ/L to be chosen later. In each case, we have the short exact sequence from Proposition 5.7.14 for the relative Brauer group in the unramified extension F_1/F. By functoriality with respect to extensions of valued fields, the sequences form a commutative diagram as follows, with obvious notations for the residue fields:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathrm{Br}(k_1/k) \arrow[r, "\sim", "\alpha\mapsto\chi_\alpha"'] \arrow[d] \arrow[ddl, dashed] & \mathrm{Hom}(G, v(k)\otimes\mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Br}(\ell_1/\ell) \arrow[r] \arrow[d, equal] & \mathrm{Br}(L_1/L) \arrow[r] \arrow[d] & \mathrm{Hom}(G, w(L)\otimes\mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d] & 0. \\
0 \arrow[r] & \mathrm{Br}(e_1/e) \arrow[r] \arrow[d, "\overset{!}{=}0"] & \mathrm{Br}(E_1/E) \arrow[r] \arrow[d] & \mathrm{Hom}(G, w(E)\otimes\mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Br}(\lambda_1/\lambda) \arrow[r] & \mathrm{Br}(\Lambda_1/\Lambda) \arrow[r] & \mathrm{Hom}(G, w(\Lambda)\otimes\mathbb{Q}/\mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

Here, the relative Brauer group $\mathrm{Br}(\mathbb{F}_1/\mathbb{F})$ for the residue field extension of k_1/k vanishes since the Brauer group of a finite field is trivial. The right vertical maps are induced by the inclusions of value groups

$$v(k) \subseteq w(L) \subseteq w(E) \subseteq w(\Lambda).$$

The Brauer class $\alpha \in \mathrm{Br}(k)$ with invariant $\frac{1}{p}$ is contained in the relative Brauer group $\mathrm{Br}(k_1/k)$ since the pullback map $\mathrm{Br}(k) \to \mathrm{Br}(k_1)$ is isomorphic to multiplication with $[k_1 : k] = p$ on $\mathbb{Q}/\mathbb{Z}$. Denoting by σ the canonical generator of G given by the Frobenius automorphism, and by π_k a uniformiser of k, the image χ_α of α in $\mathrm{Hom}(G, v(k)\otimes\mathbb{Q}/\mathbb{Z})$ is given by

$$\chi_\alpha(\sigma) = v(\pi_k) \otimes \frac{1}{p}.$$

Observe that χ_α maps to zero in $\mathrm{Hom}(G, w(E)\otimes\mathbb{Q}/\mathbb{Z})$ if $v(\pi_k)$ becomes divisible by p in $w(E)$. We claim that there exists a subextension E of Λ/L with residue field e equal to ℓ where this happens. If $v(\pi_k)$ is already divisible by p

in $w(L)$, we simply set $E := L$, so assume $v(\pi_k) \notin pw(L)$. The field L must have infinite residue field or non-discrete value group, for otherwise the completion $\widehat{L}$ would be finite over $\mathbb{Q}_p$ by Lemma 5.7.10, contradicting the fact that L contains the function field K of the curve X/k. So L satisfies the hypotheses of Proposition 5.7.6 and there exists a subextension $E = L(t^{1/p}) \subseteq \Lambda$ with $t \in L^\times$ in whose value group $v(\pi_k)$ becomes divisible by p. The extension E/L has degree at most p and ramification index divisible by p since $v(\pi_k)$ is contained in $pw(E)$ but not in $pw(L)$. By the Fundamental Inequality (Fact 4.2.4), it follows that the residue field extension is trivial: $\ell = e$.

The choice of E implies that the restriction map $\mathrm{Br}(k_1/k) \to \mathrm{Br}(E_1/E)$ actually takes values in $\mathrm{Br}(e_1/e)$. The extension e_1/e is cyclic of degree p. By Proposition 5.7.15, every class in $\mathrm{Br}(e_1/e)$ is also split by the purely inseparable extension $e^{1/p}$ $(= \ell^{1/p})$. By Proposition 5.7.4, we have $\ell^{1/p} \subseteq \lambda$. Hence, the restriction map $\mathrm{Br}(e_1/e) \to \mathrm{Br}(\lambda_1/\lambda)$ is trivial. In particular, $\alpha \in \mathrm{Br}(k_1/k)$ maps to zero in $\mathrm{Br}(\Lambda)$, contradicting the choice of the valuation w. $\square$

5.8 Rationality

We have ruled out the possibility that the valuation w has positive residue characteristic. So its restriction to k must be trivial and hence, by the valuative criterion of properness, w equals the discrete valuation associated to a unique closed point $x \in X_{\mathrm{cl}}$. It remains to see that this point x is k-rational.

Lemma 5.8.1. *The residue field extension λ/ℓ of Λ/L is trivial and $[\ell : k]$ is not divisible by p.*

Proof. Let k_1/k be any Galois degree p extension in k', for instance the unique unramified one. Set $L_1 := Lk_1$ and $\Lambda_1 := \Lambda k_1$ and let ℓ_1 and λ_1 be their residue fields. The field k, like any field, is henselian with respect to the trivial valuation $w|_k$, and the extension k_1/k as well as its translates L_1/L and Λ_1/Λ are trivially unramified with respect to w. They all have degree p since Λ and k' are linearly disjoint. Using that the exact sequence from Proposition 5.7.11 is functorial with respect to extensions of valued fields, we get a commutative diagram:

$$\begin{tikzcd}
\frac{1}{p}\mathbb{Z}/\mathbb{Z} \arrow[r, equal] \arrow[d, "{[\ell:k]}"] & \mathrm{Br}(k_1/k) \arrow[r, equal] \arrow[d] & \mathrm{Br}(k_1/k) \arrow[d] \\
\frac{1}{p}\mathbb{Z}/\mathbb{Z} \arrow[r, equal] \arrow[d, "{[\lambda:\ell]}"] & \mathrm{Br}(\ell_1/\ell) \arrow[r, hook] \arrow[d] & \mathrm{Br}(L_1/L) \arrow[d] \\
\frac{1}{p}\mathbb{Z}/\mathbb{Z} \arrow[r, equal] & \mathrm{Br}(\lambda_1/\lambda) \arrow[r, hook] & \mathrm{Br}(\Lambda_1/\Lambda).
\end{tikzcd}$$

The class $\alpha \in \mathrm{Br}(k_1/k)$ with invariant $\frac{1}{p}$ lands in the unramified Brauer group $\mathrm{Br}(\ell_1/\ell)$. As indicated in the diagram, each of the unramified relative Brauer groups is isomorphic to $\frac{1}{p}\mathbb{Z}/\mathbb{Z}$ and the vertical maps are given by multiplication by the field extension degrees. Since the generator α of $\mathrm{Br}(k_1/k)$ survives in $\mathrm{Br}(\Lambda)$ by construction, the degrees $[\ell : k]$ and $[\lambda : \ell]$ are not divisible by p. Since Λ/L is $\mathbb{Z}/p\mathbb{Z}$-elementary abelian, so is λ/ℓ, hence we must have $[\lambda : \ell] = 1$. □

Proposition 5.8.2. *The closed point $x \in X$ with valuation w is k-rational.*

Proof. The residue field ℓ of L equals the residue field $\kappa(x)$ of the closed point x. By Lemma 5.8.1, we have $p \nmid \deg(x)$. The residue field ℓ' of L' equals the maximal $\mathbb{Z}/p\mathbb{Z}$-abelian extension of ℓ, obtained by adjoining p-th roots of all elements. Consider the subextension $\ell \subseteq \ell'_S \subseteq \ell'$ given by the residue field of L'_S. As in the proof of Lemma 5.7.1, the extension L'_S/L is obtained by adjoining p-th roots of the group

$$\Delta_S \coloneqq \{f \in K^\times \ :\ v_s(f) \equiv 0 \bmod p \text{ for all } s \in S\} \subseteq K^\times.$$

Since L is henselian of residue characteristic zero, an element of ℓ admits a p-th root in ℓ'_S if and only if a lift in $\mathcal{O}_L^\times$ admits a p-th root in L'_S. The residue field ℓ'_S/ℓ is thus obtained by adjoing p-th roots of the image of

$$\Delta_S \cap \mathcal{O}_L^\times \to \ell^\times/\ell^{\times p}.$$

On the other hand, the field L'_S equals the compositum $\Lambda k'$ by the isomorphism of Galois groups $\mathrm{Gal}(L'_S/\Lambda) \cong \mathrm{Gal}(k'/k)$ from Lemma 5.6.4, which implies $\ell'_S = \lambda k'$. By Lemma 5.8.1, we have $\ell = \lambda$ and hence $\ell'_S = \ell k'$. In other words, ℓ'_S is obtained from ℓ by adjoining p-th roots of all elements of $k^\times$. Combining this with the considerations above, we find that the map $\Delta_S \cap \mathcal{O}_L^\times \to \ell^\times/\ell^{\times p}$ has image equal to $k^\times \ell^{\times p}/\ell^{\times p}$. Hence, the subgroup $\mathcal{O}(X_S)^\times \cap \mathcal{O}_L^\times \subseteq \Delta_S \cap \mathcal{O}_L^\times$ maps into $k^\times \ell^{\times p}/\ell^{\times p}$. Condition (Rat) rules this out for a non-rational point, so we must have $\kappa(x) = k$. □

We have thus shown in Proposition 5.8.2 that the valuation w belongs to a k-rational point x of X. By Proposition 5.6.5, the image of the liftable section s' is contained in the decomposition group D_x of x in $\pi_1(X_S)'$, in other words, s' is a section over x. The uniqueness of x was shown in Proposition 5.3.2. The proof the main theorem A is thus finished.

6 Criteria for good localisations

In this chapter we collect some general criteria which can be used to verify the conditions of a good localisation. Condition (Pic) is given an equivalent formulation in Proposition 6.1.1. In Section 6.2, we show that each of the four conditions of a good localisation is implied by a certain approximation property for rational functions with respect to various valuations. In Section 6.3 we show that Condition (Pic) for a localisation X_S follows also from the possibility of approximating divisors on X by divisors with support outside S. We show that this is satisfied when the complement of S is *uniformly dense* (Definition 6.3.8). Finally, in Section 6.4 we give one sufficient and one necessary condition for a localisation to have the property (Fin).

6.1 The Picard group

Let k be a field of characteristic zero and let X/k be a smooth, proper, geometrically connected curve. The **Picard scheme** $\underline{\mathrm{Pic}}_{X/k}$ is defined as the commutative group scheme over k which represents the fppf-sheafification of the presheaf $T \mapsto \mathrm{Pic}(X \times_k T)$ [Kle05]. It has a decomposition as a disjoint union

$$\underline{\mathrm{Pic}}_{X/k} = \coprod_{d \in \mathbb{Z}} \underline{\mathrm{Pic}}^d_{X/k}$$

with $\underline{\mathrm{Pic}}^d_{X/k}$ representing the subsheaf corresponding to line bundles of degree d. The degree zero part of $\underline{\mathrm{Pic}}_{X/k}$ is the identity component $\underline{\mathrm{Pic}}^\circ_{X/k}$; it is an abelian variety of dimension g, the genus of X, and each $\underline{\mathrm{Pic}}^d_{X/k}$ is an étale $\underline{\mathrm{Pic}}^\circ_{X/k}$-torsor. If the base field k it not separably closed, a k-point of $\underline{\mathrm{Pic}}_{X/k}$ does not necessarily represent an isomorphism class of line bundles in $\mathrm{Pic}(X)$ but rather a $\mathrm{Gal}(\ell/k)$-invariant element of $\mathrm{Pic}(X \otimes \ell)$ for some finite Galois extension ℓ/k. The difference between $\mathrm{Pic}(X)$ and $\underline{\mathrm{Pic}}_{X/k}(k)$ is measured by the relative Brauer group $\mathrm{Br}(X/k)$ (Definition 5.5.2) via a short exact sequence

$$0 \to \mathrm{Pic}(X) \to \underline{\mathrm{Pic}}_{X/k}(k) \to \mathrm{Br}(X/k) \to 0, \tag{6.1.1}$$

which arises from the Leray spectral sequence for the étale sheaf $\mathbb{G}_m$ under the structural morphism $X \to \mathrm{Spec}(k)$. It is the vanishing of $\mathrm{Pic}(k) = 0$ by Hilbert's Theorem 90 which ensures the injectivity of $\mathrm{Pic}(X) \to \underline{\mathrm{Pic}}_{X/k}(k)$.

Proposition 6.1.1. *Let k be a finite extension of $\mathbb{Q}_p$, let X/k be a smooth, proper, geometrically connected curve and let $S \subseteq X_{\mathrm{cl}}$ be a set of closed points. Then the following are equivalent:*

(i) $\mathrm{Pic}(X_S)/p = 0$*;*

(ii) $\mathrm{Pic}(X_S)$ *is finite of order not divisible by* p*.*

Proof. The implication (ii)⇒(i) is clear. For the forward direction, assume that we have $\mathrm{Pic}(X_S)/p = 0$. The group $\mathrm{Pic}(X_S)$ is a quotient of $\mathrm{Pic}(X)$ via restriction of line bundles from X to X_S. The group $\mathrm{Pic}(X)$ in turn is a subgroup of $\underline{\mathrm{Pic}}_{X/k}(k)$ via the exact sequence (6.1.1). The degree map defines a short exact sequence

$$0 \to \underline{\mathrm{Pic}}^\circ_{X/k}(k) \to \underline{\mathrm{Pic}}_{X/k}(k) \xrightarrow{\deg} \mathrm{period}(X)\mathbb{Z} \to 0,$$

where $\mathrm{period}(X)$ is defined as the smallest positive integer $d > 0$ for which $\underline{\mathrm{Pic}}^d_{X/k}(k) \neq \emptyset$. As a consequence, $\mathrm{Pic}(X_S)$ sits in a short exact sequence of the form

$$0 \to P \to \mathrm{Pic}(X_S) \to Q \to 0$$

where P is a subquotient of $\underline{\mathrm{Pic}}^\circ_{X/k}(k)$ and Q is a subquotient of $\mathbb{Z}$. The group scheme $\underline{\mathrm{Pic}}^\circ_{X/k}$ is an abelian variety of dimension g over k, so the group of k-points $\underline{\mathrm{Pic}}^\circ_{X/k}(k)$ has a structure of a compact p-adic analytic Lie group over k. By a theorem of Mattuck [Mat55], $\underline{\mathrm{Pic}}^\circ_{X/k}(k)$ contains an open subgroup isomorphic to $(\mathcal{O}_k^g, +)$, which has finite index by compactness. Hence, the subquotient P of $\underline{\mathrm{Pic}}^\circ_{X/k}(k)$ sits in an exact sequence of the form

$$0 \to R \to P \to F \to 0,$$

where R is a subquotient of $\mathcal{O}_k^g \cong \mathbb{Z}_p^{g\cdot[k:\mathbb{Q}_p]}$ and F is finite.

Now from $\mathrm{Pic}(X_S)/p = 0$ we get $Q/p = 0$. Since Q is a subquotient of $\mathbb{Z}$, this implies that Q is finite cyclic of order prime to p. Then also $Q[p] = 0$, which implies $P/p \cong \mathrm{Pic}(X_S)/p = 0$. With the latter exact sequence, we find $F/p = 0$, hence $F[p] = 0$ since F is finite. As a consequence, we have $R/p \cong P/p = 0$. Since R is a free $\mathbb{Z}_p$-module of finite rank, this implies $R = 0$. So $P \cong F$ is finite, and therefore $\mathrm{Pic}(X_S)$ is finite as an extension of the finite group Q by P. Finally, the order of $\mathrm{Pic}(X_S)$ is not divisible by p because of $\mathrm{Pic}(X_S)/p = 0$. □

6.2 Approximation of rational functions

Each of the four conditions of a good localisation are implied by the possibility of approximating rational functions on X with respect to various valuations by rational functions with invertibility conditions.

Definition 6.2.1. Let K be a field and $F \subseteq K$ a subset. Given valuations $v_1, \ldots, v_n$ of K with value groups $\Gamma_1, \ldots, \Gamma_n$, we say that K satisfies **approximation by F with respect to** $v_1, \ldots, v_n$ if for all $f_1, \ldots, f_n \in K$ and all $\gamma_1 \in \Gamma_1, \ldots, \gamma_n \in \Gamma_n$, there exists $f \in F$ such that $v_i(f - f_i) > \gamma_i$ for all $i = 1, \ldots, n$.

Remark 6.2.2. Definition 6.2.1 can be rephrased as follows: K satisfies approximation by F with respect to $v_1, \dots, v_n$ if and only if the image of F under the diagonal embedding $F \hookrightarrow \prod_{i=1}^n (K, v_i)$ is dense, where (K, v_i) is the topological field K with the v_i-adic topology.

Example 6.2.3. Let K be a field and let $v_1, \dots, v_n$ be finitely many pairwise independent valuations on K. Then K satisfies approximation by K itself with respect to $v_1, \dots, v_n$. This is the statement of the Approximation Theorem 4.4.5.

We can formulate the following criteria in terms of approximation of rational functions which imply the four conditions of a good localisation:

Proposition 6.2.4. *Let k be a field of $\operatorname{char}(k) \neq p$ with $\mu_p \subseteq k$, let X/k be a smooth, proper, geometrically connected curve and $S \subseteq X_{\mathrm{cl}}$ a set of closed points.*

(a) *Assume that at least one element of k is not a p-th power and that for all $x \neq y$ in $X(k)$, the function field K of X satisfies approximation by $\mathcal{O}(X_{S \setminus \{x,y\}})^\times$ with respect to v_x, v_y. Then X_S satisfies Condition (Sep).*

(b) *Assume that for every geometrically connected, $\mathbb{Z}/p\mathbb{Z}$-elementary abelian finite étale cover $Y_{f^{-1}(S)} \to X_S$ (cf. Corollary 2.4.11) and every point $y \in f^{-1}(S)$, the function field $\kappa(Y)$ satisfies approximation by $\mathcal{O}(Y_{f^{-1}(S) \setminus \{y\}})^\times$ with respect to v_y. Then X_S satisfies Condition (Pic).*

(c) *Assume that for every non-rational closed point $x \in X_{\mathrm{cl}}$ with $p \nmid \deg(x)$, the function field K satisfies approximation by $\mathcal{O}(X_{S \setminus \{x\}})^\times$ with respect to v_x. Then X_S satisfies Condition (Rat).*

(d) *Suppose that k is a finite extension of $\mathbb{Q}_p$. Assume that for every rank one valuation w on K extending the p-adic valuation on k, the function field K satisfies approximation by $\mathcal{O}(X_S)^\times$ with respect to w. Then X_S satisfies Condition (Fin).*

Proof. For (a), let $a \in k^\times$ be an element which is not a p-th power and let $x \neq y$ in $X(k)$ be different k-rational points. Using the approximation assumption, there exists $f \in \mathcal{O}(X_{S \setminus \{x,y\}})^\times$ such that $v_x(f-a) > 0$ and $v_y(f-1) > 0$. Then f is invertible at x and y with $f(x) = a$ and $f(y) = 1$, and thus $f(x)/f(y) = a$ is not a p-th power in k.

To prove (b), let $Y_{f^{-1}(S)} \to X_S$ and $y \in f^{-1}(S)$ as above be given. We show that $\operatorname{Pic}(Y_{f^{-1}(S)}) = 0$, even without quotienting by multiples of p. A prime divisor on $Y_{f^{-1}(S)}$ is given by a point $y \in f^{-1}(S)$. To show that it is a principal divisor, we need to find a rational function $f \in \kappa(Y)^\times$ on Y with $v_y(f) = 1$ and no zeroes or poles in $f^{-1}(S) \setminus \{y\}$. Let $t_y \in \kappa(Y)^\times$ be a uniformiser at y. By the approximation assumption, there exists some $f \in \mathcal{O}(Y_{f^{-1}(S) \setminus \{y\}})^\times$ such that $v_y(f - t_y) > 1$. Then it follows that we have $v_y(f) = \min\{v_y(f - t_y), v_y(t_y)\} = 1$, as desired.

To show (c), let $x \in X_{\mathrm{cl}}$ be a non-rational closed point with $p \nmid \deg(x)$. We show that the map $\mathcal{O}(X_{S\cup\{x\}})^\times \to \kappa(x)^\times$ given by $f \mapsto f(x)$ is surjective, which implies by Corollary 5.1.5 that Condition (Rat) is satisfied. For a given $a \in \kappa(x)^\times$, choose $g \in \mathcal{O}_{X,x}$ with $g(x) = a$. By the approximation assumption, there exists $f \in \mathcal{O}(X_{S\setminus\{x\}})^\times$ such that $v_x(f-g) > 0$. Then f is invertible also at x, i.e. f is an element of $\mathcal{O}(X_{S\cup\{x\}})^\times$, and satisfies $f(x) = g(x) = a$.

Finally, to show (d), let w be a rank one valuation on K extending the p-adic valuation on k such that K satisfies approximation by $\mathcal{O}(X_S)^\times$ with respect to w. We show that the map $\mathcal{O}(X_S)^\times \to (K_w^{\mathrm{h}})^\times/(K_w^{\mathrm{h}})^{\times p}$ has even trivial cokernel, i.e. is surjective. Since w has rank one, the field K is dense with respect to the w-adic topology in the henselisation K_w^{h}. By assumption, $\mathcal{O}(X_S)^\times$ is dense in K with respect to the w-adic topology, so by transitivity, $\mathcal{O}(X_S)^\times$ is dense in $(K_w^{\mathrm{h}})^\times$. It follows that the image of $\mathcal{O}(X_S)^\times$ is dense in the quotient $(K_w^{\mathrm{h}})^\times/(K_w^{\mathrm{h}})^{\times p}$. As a consequence of Hensel's Lemma (Fact 4.5.2), the group $(K_w^{\mathrm{h}})^{\times p}$ of p-th powers is open in $(K_w^{\mathrm{h}})^\times$; more precisely, if $f \in (K_w^{\mathrm{h}})^\times$ satisfies $w(f-1) > 2w(p)$, then the approximate root 1 of the polynomial $X^p - f$ implies the existence of an actual root in $(K_w^{\mathrm{h}})^\times$. Therefore, the quotient $(K_w^{\mathrm{h}})^\times/(K_w^{\mathrm{h}})^{\times p}$ is discrete and the dense image of $\mathcal{O}(X_S)^\times$ is the whole group. $\square$

6.3 Approximation of divisors

For Condition (Pic) we can formulate a criterion in terms of approximation of divisors on X by divisors with support outside S.

6.3.1 The p-adic topology on the Picard group

In order to give a precise meaning to the notion of approximation of divisors, we define a p-adic topology on the Picard group and on Hilbert schemes of points.

Definition 6.3.1. Let k be a topological field and let X/k be a k-scheme which is locally of finite type. The **analytic topology** on $X(k)$ is defined as follows: for an affine space, $\mathbb{A}^n(k) = k^n$ carries the product topology. For an affine scheme X/k of finite type, $X(k)$ carries the subspace topology from $\mathbb{A}^n(k)$ via an embedding $X \hookrightarrow \mathbb{A}^n_k$. This does not depend on the choice of embedding. In general, a subset $U \subseteq X(k)$ is open in the analytic topology if and only if its intersection with any affine Zariski-open subscheme of finite type is analytically open. If k is a p-adic field, we speak also of the **p-adic topology**.

Given a morphism $f\colon X \to Y$ between two k-schemes which are locally of finite type, the induced map $f\colon X(k) \to Y(k)$ is continuous with respect to the analytic topology. This follows from the fact that for a topological field k, every polynomial map $k^n \to k^m$ is continuous.

From now on, let k be a finite extension of $\mathbb{Q}_p$, and let X/k be a smooth, proper, geometrically connected curve. The Picard scheme of X/k is locally of finite type over k, so that we have a p-adic topology on $\underline{\mathrm{Pic}}_{X/k}(k)$. Recall that $\mathrm{Pic}(X)$ is a subgroup of $\underline{\mathrm{Pic}}_{X/k}(k)$ as a consequence of Hilbert's Theorem 90 (see Section 6.1 above).

Definition 6.3.2. The **p-adic topology** on $\mathrm{Pic}(X)$ is the subspace topology induced by the p-adic topology on $\underline{\mathrm{Pic}}_{X/k}(k)$.

Lemma 6.3.3. *For every $n \in \mathbb{N}$, the multiplication-by-n map on $\underline{\mathrm{Pic}}_{X/k}(k)$ is open with respect to the p-adic topology.*

Proof. By [Poo17, Prop. 3.5.73], it suffices to show that multiplication by n is étale on $\underline{\mathrm{Pic}}_{X/k}$ as an endomorphism of schemes. Since étaleness satisfies faithfully flat descent, this may be checked after a base change to an algebraic closure $\overline{k}/k$. On the identity component $\underline{\mathrm{Pic}}^{\circ}_{X_{\overline{k}}/\overline{k}}$, the map is étale by [Stacks, Lemma 0BFH], essentially because the multiplication-by-n map on the tangent space at the identity is invertible. Choose a point $p \in X(\overline{k})$. Then, for any $d \in \mathbb{Z}$, the degree-d component $\underline{\mathrm{Pic}}^{d}_{X_{\overline{k}}/\overline{k}}$ is the translate of $\mathrm{Pic}^{\circ}_{X_{\overline{k}}/\overline{k}}$ by the line bundle $\mathcal{O}_{X_{\overline{k}}}(dp)$, hence the multiplication-by-n map is étale there as well. $\square$

Lemma 6.3.4. *The Picard group* $\mathrm{Pic}(X)$ *is an open subgroup of* $\underline{\mathrm{Pic}}_{X/k}(k)$.

Proof. By the exact sequence (6.1.1), the quotient $\underline{\mathrm{Pic}}_{X/k}(k)/\operatorname{Pic}(X)$ is isomorphic to the relative Brauer group $\mathrm{Br}(X/k)$. As explained in Section 5.5, $\mathrm{Br}(X/k)$ is annihilated by the index of X/k. Hence, if we set $n \coloneqq \mathrm{index}(X)$, then $\mathrm{Pic}(X)$ contains $n \cdot \underline{\mathrm{Pic}}_{X/k}(k)$. By Lemma 6.3.3, the subgroup $n \cdot \underline{\mathrm{Pic}}_{X/k}(k)$ is open in $\underline{\mathrm{Pic}}_{X/k}(k)$, hence $\mathrm{Pic}(X)$, being a union of open cosets, is open as well. $\square$

Lemma 6.3.5. *The subgroup $p \cdot \mathrm{Pic}(X)$ is open in* $\mathrm{Pic}(X)$.

Proof. The group $p \cdot \mathrm{Pic}(X)$ is the image of the open subgroup $\mathrm{Pic}(X)$ of $\underline{\mathrm{Pic}}_{X/k}(k)$ under the open multiplication-by-p map. $\square$

6.3.2 The p-adic topology on the Hilbert scheme

We also want to define a p-adic topology on sets of divisors. For $d \geq 0$, the **Hilbert scheme of points** $\underline{\mathrm{Hilb}}^{d}_{X/k}$ represents the presheaf

$$T \mapsto \left\{ \begin{array}{c} D \subseteq X \times_k T \text{ closed subscheme s.t.} \\ D \to T \text{ is finite locally free of degree } d \end{array} \right\}$$

on k-schemes. Equivalently, the points in $\underline{\mathrm{Hilb}}^{d}_{X/k}(T)$ are relative effective Cartier divisors of $X \times_k T \to T$ of degree d [Stacks, Lemma 0B9D]. In particular:

$$\underline{\mathrm{Hilb}}^{d}_{X/k}(k) = \{\text{effective divisors on } X/k \text{ of degree } d\}.$$

The scheme $\underline{\mathrm{Hilb}}^d_{X/k}$ is a smooth, proper k-variety of dimension d [Stacks, Prop. 0B9I]. In particular, $\underline{\mathrm{Hilb}}^d_{X/k}$ is of finite type over k, so that we have a p-adic topology on its k-points via Definition 6.3.1 and can define:

Definition 6.3.6. For $d \geq 0$, the **p-adic topology** on the set of effective degree-d divisors on X is the p-adic topology on $\underline{\mathrm{Hilb}}^d_{X/k}(k)$.

There is a morphism of k-schemes

$$\underline{\mathrm{Hilb}}^d_{X/k} \to \underline{\mathrm{Pic}}^d_{X/k}$$

induced by mapping each relative effective Cartier divisor $D \subseteq X \times_k T$ to its associated line bundle $\mathcal{O}(D)$, functorially for k-schemes T. On k-points, the map clearly takes values in the subset $\mathrm{Pic}^d(X)$ of $\underline{\mathrm{Pic}}^d_{X/k}(k)$. Consequently, we have for all $d \geq 0$ a map

$$\underline{\mathrm{Hilb}}^d_{X/k}(k) \to \mathrm{Pic}^d(X),$$

which is continuous with respect to the p-adic topologies and which is given by $D \mapsto \mathcal{O}_X(D)$ or, in terms of linear equivalence classes of divisors, by $D \mapsto [D]$.

6.3.3 Approximation of divisors

If divisors on X can be approximated by divisors with support outside S, then we have $\mathrm{Pic}(X_S)/p = 0$:

Proposition 6.3.7. *Let k be a finite extension of $\mathbb{Q}_p$ and let $S \subseteq X_{\mathrm{cl}}$ be a set of closed points such that for all sufficiently large $d \gg 0$, the set of effective degree-d divisors on X with support outside S is p-adically dense in $\underline{\mathrm{Hilb}}^d_{X/k}(k)$. Then we have* $\mathrm{Pic}(X_S)/p = 0$.

Proof. It suffices to show that the classes of divisors on X with support outside of S are dense in $\mathrm{Pic}(X)$. Indeed, any dense subset of $\mathrm{Pic}(X)$ surjects onto $\mathrm{Pic}(X)/p$ since $p \cdot \mathrm{Pic}(X)$ is open (Lemma 6.3.5). Restriction of divisors from X to X_S defines a further surjective homomorphism $\mathrm{Pic}(X)/p \twoheadrightarrow \mathrm{Pic}(X_S)/p$. Hence, if the classes of divisors with support outside S are dense in $\mathrm{Pic}(X)$, then they surject onto $\mathrm{Pic}(X_S)/p$. But their restriction to X_S is trivial, so we have $\mathrm{Pic}(X_S)/p = 0$.

To show the claim, let D be any divisor on X and let $U \subseteq \mathrm{Pic}(X)$ be a p-adically open neighbourhood of $[D]$. We have to show that U contains the class of a divisor with support outside S. Write $D = D_+ - D_-$ with D_+ and D_- effective of sufficiently large degrees d_+ and d_-. By continuity of the group operations, there exist p-adic neighbourhoods U_+ and U_- of $[D_+]$ and $[D_-]$ such that $U_+ - U_- \subseteq U$. By the density assumption and the fact that the map $\underline{\mathrm{Hilb}}^d_{X/k}(k) \to \mathrm{Pic}(X)$ is continuous for all $d \geq 0$, there exist divisors E_+ and E_- with support outside S such that $[E_+] \in U_+$ and $[E_-] \in U_-$. Then $E := E_+ - E_-$ has support outside S and $[E]$ is contained in U. $\square$

6.3.4 Complements of uniformly dense subsets

We remain in the situation where k is a finite extension of $\mathbb{Q}_p$ and X_S/k is the localisation of a smooth, proper, geometrically connected curve at a set of closed points. Given a set of closed points $T \subseteq X_{\mathrm{cl}}$ and a field extension ℓ/k, we denote by $T(\ell) \subseteq X(\ell)$ the subset of k-morphisms $\mathrm{Spec}(\ell) \to X$ with image point contained in T.

Definition 6.3.8. A set of closed points $T \subseteq X_{\mathrm{cl}}$ is called **uniformly dense** if $T(\ell)$ is p-adically dense in $X(\ell)$ for every finite field extension ℓ/k.

Divisors on X can be approximated by divisors with support in a uniformly dense subset:

Lemma 6.3.9. *Let $T \subseteq X_{\mathrm{cl}}$ be a set of closed points which is uniformly dense. Then for all $d \geq 0$, the set of degree-d divisors on X with support in T is p-adically dense in $\underline{\mathrm{Hilb}}^d_{X/k}(k)$.*

Proof. Let $D \in \underline{\mathrm{Hilb}}^d_{X/k}(k)$ be a divisor and let $U \subseteq \underline{\mathrm{Hilb}}^d_{X/k}(k)$ be a p-adically open neighbourhood of D. We have to show that U contains a divisor with support in T. Addition of divisors $(D_1, D_2) \mapsto D_1 + D_2$ is a morphism of k-schemes

$$\underline{\mathrm{Hilb}}^{d_1}_{X/k} \times \underline{\mathrm{Hilb}}^{d_2}_{X/k} \to \underline{\mathrm{Hilb}}^{d_1+d_2}_{X/k},$$

by [Stacks, Prop. 0B9I], and therefore continuous on k-points with respect to the p-adic topology. Writing D as a sum of prime divisors, we may thus assume that D consists of a single closed point $x \in X_{\mathrm{cl}}$. Let $\ell := \kappa(x)$ be its residue field, so that we have $x \in X(\ell)$ and $d = [\ell : k]$. For any subextension $k \subseteq m \subseteq \ell$, the set $X(m)$ is p-adically closed in $X(\ell)$. This follows from m being p-adically closed in ℓ, which in turn follows from the fact that the p-adic topology on ℓ equals the product topology from k for any choice of basis, and that m is a k-linear subspace. Since there are only finitely many proper subextensions of ℓ/k, there exists an open neighbourhood V of x in $X(\ell)$ in which all points have residue field equal to ℓ.

We claim that the map $V \to \underline{\mathrm{Hilb}}^d_{X/k}(k)$ mapping an ℓ-point of X to its underlying prime divisor is continuous. Let $\overline{k}/\ell$ be an algebraic closure. It suffices to show that the composite map

$$V \to \underline{\mathrm{Hilb}}^d_{X/k}(k) \hookrightarrow \underline{\mathrm{Hilb}}^d_{X/k}(\overline{k}). \tag{$*$}$$

is continuous, since $\underline{\mathrm{Hilb}}^d_{X/k}(k)$ carries the subspace topology from $\underline{\mathrm{Hilb}}^d_{X/k}(\overline{k})$. The map $(*)$ sends $y \in V$ with residue field ℓ to the divisor $y \otimes_\ell \overline{k} = \sum_\sigma \sigma(y)$ on $X \otimes_k \overline{k}$, with σ running through the set of k-embeddings $\mathrm{Hom}_k(\ell, \overline{k})$. This follows from the isomorphism

$$\ell \otimes_k \overline{k} = \prod_\sigma \overline{k}$$

given by $a \otimes b \mapsto (\sigma(a)b)_\sigma$, which holds for any finite separable extension of fields ℓ/k. Thus, the map $(*)$ equals the sum of the d maps

$$V \hookrightarrow X(\ell) \xrightarrow{\sigma_*} X(\overline{k}) = \underline{\mathrm{Hilb}}^1_{X/k}(\overline{k})$$

induced by the k-embeddings $\sigma\colon \ell \hookrightarrow \overline{k}$. As each of these maps is continuous, the continuity of the map $(*)$ follows.

By the uniform density assumption, $T(\ell)$ is dense in $X(\ell)$. Thus, using the continuity of $(*)$, there exists a point in $T(\ell) \cap V$ mapping to U, in other words, U contains a prime divisor with support in T. □

The property of having uniformly dense complement passes to covers:

Lemma 6.3.10. *Let $S \subseteq X_{\mathrm{cl}}$ be a set of closed points whose complement is uniformly dense in X. Then for every finite branched cover $f\colon Y \to X$ which is unramified over S, the complement of $f^{-1}(S)$ is uniformly dense in Y.*

Proof. Let T be the complement of S in X_{cl}. Then $f^{-1}(T)$ is the complement of $f^{-1}(S)$ in Y_{cl} and we have to show that $f^{-1}(T)(\ell)$ is dense in $Y(\ell)$ for every finite extension ℓ/k. Since the assumptions are stable under an extension of scalars, it suffices to treat the case $\ell = k$. Note that f induces a map on k-points $f\colon Y(k) \to X(k)$ under which $f^{-1}(T)(k)$ is the preimage of $T(k) \subseteq X(k)$. So we can write $f^{-1}(T)(k) = f^{-1}(T(k))$ and have to show that $f^{-1}(T(k))$ is dense in $Y(k)$. Let $y \in Y(k)$ and let $V \subseteq Y(k)$ be a p-adically open neighbourhood. We are claiming that $f^{-1}(T(k)) \cap V \neq \emptyset$. If y is contained in $f^{-1}(T)$, there is nothing to do, so assume $y \notin f^{-1}(T)$, i.e. $y \in f^{-1}(S)$. Since f is unramified over S by assumption, the map f is étale at y. It follows that f is a local homeomorphism for the p-adic topology near y by [Poo17, Prop. 3.5.73]. In particular, $f(V)$ contains an open neighbourhood of $f(y)$ in $X(k)$. Since $T(k)$ is dense in $X(k)$ by assumption, there exists a point in $T(k) \cap f(V)$, and hence a point in $f^{-1}(T(k)) \cap V$. □

Combining the above results, we have the following criterion for Condition (Pic):

Proposition 6.3.11. *Let k be a finite extension of $\mathbb{Q}_p$, let X/k be a smooth, proper, geometrically connected curve and let $S \subseteq X_{\mathrm{cl}}$ be a set of closed points. If the complement of S is uniformly dense, then X_S satisfies Condition (Pic).*

Proof. Let $Y_{f^{-1}(S)} \to X_S$ be a geometrically connected, finite $\mathbb{Z}/p\mathbb{Z}$-elementary abelian cover, arising by restriction from a finite branched cover $f\colon Y \to X$ which is unramified over S. By Lemma 6.3.10, the complement of $f^{-1}(S)$ is uniformly dense in Y. By Lemma 6.3.9, the effective divisors with support outside $f^{-1}(S)$ are p-adically dense in $\underline{\mathrm{Hilb}}^d_{Y/k}(k)$ for all $d \geq 0$. By Proposition 6.3.7, this implies $\mathrm{Pic}(Y_{f^{-1}(S)})/p = 0$. □

6.4 Criteria for p-adic valuations

Recall from Definition 4.1.12 that the rational rank of an abelian group Γ is defined as
$$\mathrm{rr}(\Gamma) \coloneqq \dim_{\mathbb{Q}}(\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}).$$

Lemma 6.4.1. *Let Γ be a torsion-free abelian group of finite rational rank. Then for any natural number $n \geq 1$, the group $\Gamma/n\Gamma$ is finite.*

Proof. Let $r \coloneqq \dim_{\mathbb{Q}}(\Gamma \otimes_{\mathbb{Z}} \mathbb{Q})$ and choose elements $\gamma_1, \dots, \gamma_r \in \Gamma$ such that $\{\gamma_i \otimes 1\}_{i=1,\dots,r}$ is a $\mathbb{Q}$-basis of $\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$. Then the subgroup of Γ generated by the γ_i is isomorphic to $\mathbb{Z}^r$, so we may assume $\mathbb{Z}^r \subseteq \Gamma \subseteq \mathbb{Q}^r$. Define $\overline{\Gamma} \coloneqq \Gamma/\mathbb{Z}^r$ by the exact sequence
$$0 \to \mathbb{Z}^r \to \Gamma \to \overline{\Gamma} \to 0.$$
Since the functor $A \mapsto A/nA = A \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ is right exact on abelian groups, we obtain an exact sequence
$$(\mathbb{Z}/n\mathbb{Z})^r \to \Gamma/n\Gamma \to \overline{\Gamma}/n\overline{\Gamma} \to 0.$$
The first term is finite, so it suffices to show finiteness of $\overline{\Gamma}/n\overline{\Gamma}$. Note that $\overline{\Gamma}$ is a subgroup of $(\mathbb{Q}/\mathbb{Z})^r$, in particular a torsion group. We have a decomposition $\overline{\Gamma} = \bigoplus_p \overline{\Gamma}(p)$, where the p-primary part $\overline{\Gamma}(p)$ for a prime p consists of all elements which are killed by a power of p. Since multiplication by n is an isomorphism on all $\overline{\Gamma}(p)$ for which p does not divide n, it suffices to show that $\overline{\Gamma}(p)/n\overline{\Gamma}(p)$ is finite for all primes p. Write $(-)^\vee \coloneqq \mathrm{Hom}_{\mathrm{cts}}(-, \mathbb{R}/\mathbb{Z})$ for the Pontryagin dual of a locally compact Hausdorff abelian group. Recall that Pontryagin duality is a contravariant equivalence between discrete torsion abelian groups and profinite abelian groups. The finiteness of $\overline{\Gamma}(p)/n\overline{\Gamma}(p)$ is equivalent to the finiteness of its Pontryagin dual $(\overline{\Gamma}(p)/n\overline{\Gamma}(p))^\vee = \overline{\Gamma}(p)^\vee[n]$. The inclusion $\overline{\Gamma}(p) \subseteq (\mathbb{Q}_p/\mathbb{Z}_p)^r$ induces a surjection $\mathbb{Z}_p^r \twoheadrightarrow \overline{\Gamma}(p)^\vee$. By the structure theorem for finitely generated modules over principal ideal domains, there is an isomorphism $\overline{\Gamma}(p)^\vee \cong \mathbb{Z}_p^s \oplus T$ with $0 \leq s \leq r$ and T a finite direct sum of cyclic $\mathbb{Z}_p$-torsion modules. But every cyclic $\mathbb{Z}_p$-torsion module is of the form $\mathbb{Z}/p^e\mathbb{Z}$ for some $e \geq 0$, in particular finite. It follows that T and hence $\overline{\Gamma}(p)^\vee[n] = T[n]$ are finite. $\square$

In the following, for a valuation w on a field K, denote by $(K_w^{\mathrm{h}}, w^{\mathrm{h}})$ the henselisation, by $\mathcal{O}_w^{\mathrm{h}} \subseteq K_w^{\mathrm{h}}$ the corresponding valuation ring and by $\mathfrak{m}^{\mathrm{h}}$ the maximal ideal.

Proposition 6.4.2. *Let k be a finite extension of $\mathbb{Q}_p$ with $\mu_p \subseteq k$, let X/k be a smooth, proper, geometrically connected curve and $S \subseteq X_{\mathrm{cl}}$ a set of closed points. Let w be a rank one valuation on the function field K extending the p-adic valuation on k. Assume that the following two reduction maps have finite cokernel:*

(1) the map $\mathcal{O}(X_S)^\times \cap \mathcal{O}_w^\times \to \kappa(w)^\times/\kappa(w)^{\times p}$*;*

(2) the map $\mathcal{O}(X_S)^\times \cap (1+\mathfrak{m}_w) \to (1+\mathfrak{m}_w^{\mathrm{h}})/(1+\mathfrak{m}_w^{\mathrm{h}})^p$*.*

Then Condition (Fin) *is satisfied for* w*, i.e. the map* $\mathcal{O}(X_S)^\times \to (K_w^{\mathrm{h}})^\times/(K_w^{\mathrm{h}})^{\times p}$ *has finite cokernel.*

Proof. For an abelian group A, written additively or multiplicatively, we write A/p for the cokernel of the multiplication by p map. Let Γ_w be the value group of w on K, which agrees with the value group of the henselisation K_w^{h}. We have a commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{O}(X_S)^\times \cap \mathcal{O}_w^\times & \longrightarrow & \mathcal{O}(X_S)^\times & \xrightarrow{w} & w(\mathcal{O}(X_S)^\times) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathcal{O}_w^{\mathrm{h}\,\times}/p & \longrightarrow & K_w^{\mathrm{h}\,\times}/p & \xrightarrow{w^{\mathrm{h}}} & \Gamma_w/p & \longrightarrow & 0. \end{array} \tag{6.4.1}$$

The first row is clearly exact. The second row is also exact since Γ_w, being a totally ordered abelian group, is torsion-free, so that $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \Gamma_w) = 0$. The Dimension Inequality 4.2.6 yields

$$\mathrm{trdeg}(\kappa(w)/\mathbb{F}) + \mathrm{rr}(\Gamma_w/\Gamma_v) \le \mathrm{trdeg}(K/k), \tag{6.4.2}$$

where $\mathbb{F}$ denotes the residue field of k and $\Gamma_v \cong \mathbb{Z}$ denotes the value group of the p-adic valuation v on k. This implies $\mathrm{rr}(\Gamma_w) \le 2$, in particular the rational rank of Γ_w is finite. Lemma 6.4.1 now implies that the group Γ_w/p is finite, so that the middle vertical map has finite cokernel (as claimed) as soon as the left vertical map does. Denote by $\overline{\mathcal{O}(X_S)^\times \cap \mathcal{O}_w^\times}$ the image of $\mathcal{O}(X_S)^\times \cap \mathcal{O}_w^\times$ under the map $f \mapsto f(w)$, $\mathcal{O}_w^\times \to \kappa(w)^\times$. The residue field $\kappa(w)$ of (K, w) coincides with that of the henselisation $(K_w^{\mathrm{h}}, w^{\mathrm{h}})$, so we obtain a commutative diagram as follows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{O}(X_S)^\times \cap (1+\mathfrak{m}_w) & \longrightarrow & \mathcal{O}(X_S)^\times \cap \mathcal{O}_w^\times & \longrightarrow & \overline{\mathcal{O}(X_S)^\times \cap \mathcal{O}_w^\times} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & (1+\mathfrak{m}_w^{\mathrm{h}})/p & \longrightarrow & \mathcal{O}_w^{\mathrm{h}\,\times}/p & \longrightarrow & \kappa(w)^\times/p & \longrightarrow & 1. \end{array}$$

The first row is clearly exact. To see the exactness of the second row, note that $\kappa(w)$ has characteristic $p > 0$ since w extends the p-adic valuation on k. As a consequence, $\kappa(w)^\times$ has no p-torsion, which implies the claimed exactness. The assumptions (1) and (2) say that the right and left vertical map have finite cokernel, which implies the same for the middle vertical map. $\square$

The map (1) in Proposition 6.4.2 having finite cokernel is a necessary condition for Condition (Fin) to be satisfied:

Lemma 6.4.3. *Let X_S/k be as above and let w be a valuation on the function field K extending the p-adic valuation on k. If the map*

$$\mathcal{O}(X_S)^\times \to (K_w^{\mathrm{h}})^\times/(K_w^{\mathrm{h}})^{\times p}$$

has finite cokernel, then so does the map

$$\mathcal{O}(X_S)^\times \cap \mathcal{O}_w^\times \to \kappa(w)^\times/\kappa(w)^{\times p}.$$

Proof. Replacing the top row of diagram (6.4.1) above with its mod p version, we have a commutative diagram as follows:

$$\begin{tikzcd}
1 \arrow[r] & (\mathcal{O}(X_S)^\times \cap \mathcal{O}_w^\times)/p \arrow[r] \arrow[d] & \mathcal{O}(X_S)^\times/p \arrow[r, "w"] \arrow[d] & w(\mathcal{O}(X_S)^\times)/p \arrow[r] \arrow[d] & 0 \\
1 \arrow[r] & \mathcal{O}_w^{\mathrm{h}\,\times}/p \arrow[r] & K_w^{\mathrm{h}\,\times}/p \arrow[r, "w^{\mathrm{h}}"] & \Gamma_w/p \arrow[r] & 0.
\end{tikzcd}$$

The top row is still exact since $w(\mathcal{O}(X_S)^\times) \subseteq \Gamma_w$ is torsion-free. Since Γ_w has finite rational rank, so does its subgroup $w(\mathcal{O}(X_S)^\times)$. Thus, both terms on the right are finite by Lemma 6.4.1. By assumption, the middle vertical map has finite cokernel. The snake lemma implies that so does the left vertical map. It follows that the composition with the reduction map,

$$(\mathcal{O}(X_S)^\times \cap \mathcal{O}_w^\times)/p \to \mathcal{O}_w^{\mathrm{h}\,\times}/p \twoheadrightarrow \kappa(w)^\times/p$$

has finite cokernel as well. □

Remark 6.4.4. If $\mathcal{X}$ is a flat, proper model of X over the valuation ring $\mathcal{O}_k$ of k, then by the valuative criterion of properness, every valuation w on K with $\mathcal{O}_k \subseteq \mathcal{O}_w$ induces a unique morphism as follows:

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
\mathrm{Spec}(\mathcal{O}_w) \arrow[r] \arrow[ur, dashed, "\exists!"] & \mathrm{Spec}(\mathcal{O}_k).
\end{tikzcd}$$

The image of the closed point under the map $\mathrm{Spec}(\mathcal{O}_w) \to \mathcal{X}$ is called the **center** of w. The valuation w is uniquely determined by the family of its centers on all models. The valuations which extend the p-adic valuation on k are those whose center on every model lies on the special fibre. See [PS17, Appendix A] for a classification of the valuations on K with $\mathcal{O}_k \subseteq \mathcal{O}_w$.

Remark 6.4.5. For a rank 1 valuation w on K which extends the p-adic valuation on k, the residue field $\kappa(w)$ is an extension of the finite residue field $\mathbb{F}$ of k. If the extension $\kappa(w)/\mathbb{F}$ is algebraic, then the Frobenius $x \mapsto x^p$ is an automorphism of $\kappa(w)$ and we have $\kappa(w)^\times/\kappa(w)^{\times p} = 0$. In this case the surjectivity of the map

$$\mathcal{O}(X_S)^\times \cap \mathcal{O}_w^\times \to \kappa(w)^\times/\kappa(w)^{\times p}$$

is automatic. The necessary condition of Lemma 6.4.3 for a valuation w to satisfy (Fin) is thus only interesting in the case where $\kappa(w)$ is transcendental over $\mathbb{F}$. The Dimension Inequality 4.2.6 implies that $\mathrm{trdeg}(\kappa(w)/\mathbb{F}) = 1$ and $\mathrm{rr}(\Gamma_w) = 1$ in this case. Such a valuation belongs to an irreducible component of the special fibre of some model of X over $\mathcal{O}_k$.

7 Examples of good localisations

In this chapter we collect some examples of curves and sets of closed points which yield good localisations in the sense of Definition 5.1.1 and hence provide examples where the liftable section conjecture holds. Throughout this chapter, k denotes a finite extension of $\mathbb{Q}_p$ containing the p-th roots of unity.

7.1 The birational case

For any smooth, proper, geometrically connected curve X/k, we can choose the empty set of closed points $S = \emptyset$, in which case the localisation of X at S equals the generic point η_X, or in other words the spectrum $\operatorname{Spec}(K)$ of the function field of X. In this case we have $\mathcal{O}(X_S)^\times = K^\times$. The conditions for a good localisation are easy to verify: For Condition (Sep), let $x \neq y$ be two distinct k-rational points. Choose any element $a \in k^\times$ which is not a p-th power (for example, a uniformiser). By the Approximation Theorem 4.4.5, there exists a rational function f on X which satisfies $f(x) = a$ and $f(y) = 1$, so that $f(x)/f(y) = a$ is not a p-th power. Condition (Pic) follows from Hilbert's Theorem 90 since every connected finite étale cover of $\operatorname{Spec}(K)$ is the spectrum of a field. Condition (Rat) follows with Corollary 5.1.5 from the surjectivity of the map $\mathcal{O}_{X,x}^\times \twoheadrightarrow \kappa(x)^\times/\kappa(x)^{\times p}$ for every closed point x of X. For Condition (Fin) we can use the approximation criterion from Proposition 6.2.4 (d): For any rank 1 valuation w on K extending the p-adic valuation on k, the field K trivially satisfies approximation by $K^\times$ with respect to w.

In conclusion, the liftable section conjecture for good localisations X_S specialises to the birational liftable p-adic section conjecture from [Pop10, Theorem A] by choosing $S = \emptyset$. This is of course no coincidence as our proof was obtained by generalising the proof of loc. cit. from $\operatorname{Spec}(K)$ to X_S, while identifying a set of conditions which ensure that the arguments do in fact generalise.

7.2 Finite sets

Let X still be arbitrary and, generalising the case $S = \emptyset$ above, let $S \subseteq X_{\mathrm{cl}}$ be any finite set of closed points. Every finite étale cover of X_S has the form $Y_{f^{-1}(S)} \to X_S$ for some finite branched cover $f\colon Y \to X$ by Corollary 2.4.11. The set $f^{-1}(S)$ is again finite (of cardinality at most $\deg(f) \cdot |S|$), so it suffices to show Condition (Pic) for X_S rather than for every geometrically connected, finite $\mathbb{Z}/p\mathbb{Z}$-elementary abelian cover. For X_S, all four approximation conditions of Proposition 6.2.4 are satisfied: this follows from the Approximation Theorem 4.4.5, where we ensure that the sought-after function is invertible on points $s \in S$ by adding to the approximation problem finitely many conditions of the form $v_s(f-1) > 0$. For example, to show Condition (Rat), we have to check that the function field K satisfies approximation by $\mathcal{O}(X_{S\setminus\{x\}})^\times$ with respect to v_x for every non-rational closed point $x \in X_{\mathrm{cl}}$ with $p \nmid \deg(x)$. Let $f_x \in K$ and $\gamma \in \mathbb{Z}$ be given. Then, by the Approximation Theorem, there exists $f \in K$ satisfying the finite number of conditions $v_x(f - f_x) > \gamma$ and $v_s(f-1) > 0$ for all $s \in S \setminus \{x\}$, and the latter conditions ensure that f is an element of $\mathcal{O}(X_{S\setminus\{x\}})^\times$. The proof of the other three conditions goes along the same lines and is omitted.

7.3 Countable sets

Let X/k still be arbitrary and, further generalising the case of a finite set, suppose that $S \subseteq X_{\mathrm{cl}}$ is at most countable. We want to verify the approximation criteria of Proposition 6.2.4 as in the finite case, but the Approximation Theorem is unable to guarantee invertibility of functions at infinitely many points. We prove a new approximation theorem which accounts for this.

7.3.1 An approximation theorem with invertibility conditions

Theorem 7.3.1. *Let K be a field and $w_1, \ldots, w_n$ pairwise independent valuations of K. For $i = 1, \ldots, n$, let Γ_i be the value group of w_i and let $f_i \in K$ and $\gamma_i \in \Gamma_i$ be given. Let V be a set of valuations of K which are independent from the w_i and from each other. Assume that for any $f \in K^\times$, we have $v(f) = 0$ for all but finitely many $v \in V$. Assume moreover that there is a set $\Lambda \subseteq K$ of cardinality $|\Lambda| > |V|$ which is contained in $\mathcal{O}_{w_i}$ for all $i = 1, \ldots, n$ and in $\mathcal{O}_v^\times$ for all $v \in V$ and that the residue map $\Lambda \hookrightarrow \mathcal{O}_v^\times \to \kappa(v)^\times$ is injective for all $v \in V$. Then there exists $f \in K$ such that*

(1) $w_i(f - f_i) > \gamma_i$ for all $i = 1, \ldots, n$;

(2) $v(f) = 0$ for all $v \in V$.

The proof of Theorem 7.3.1 occupies the rest of this paragraph. Observe that if $V = \emptyset$, then condition (2) is vacuous and we can simply use the classical Approximation Theorem at the finitely many valuations $w_1, \ldots, w_n$. So assume $V \neq \emptyset$ in the following and fix some $v_0 \in V$. Note that none of the w_i is the trivial valuation because v_0 is assumed independent from the w_i.

Step 1: Avoiding zeros in V

We first construct $f \in K$ satisfying (1) and instead of (2) the weaker condition $v(f) \leq 0$ for $v \in V$. By the classical Approximation Theorem, there exists

$g \in K$ satisfying (1). We can ensure $g \neq 0$ by imposing the extra condition $v_0(g-1) > 0$, so that there are only finitely many $v \in V$ with $v(g) > 0$. The idea is now to modify g by adding a suitable function in such a way that the approximation conditions (1) are preserved while additionally avoiding any zeros in V. Using the classical Approximation Theorem again, we find $h \in K$ such that $w_i(h) > \gamma_i$ for all $i = 1, \ldots, n$ and $v(h-1) > 0$ (hence $v(h) = 0$) for all $v \in V$ with $v(g) > 0$. Consider the family of functions $g + \lambda h$ with $\lambda \in \Lambda$. They all satisfy the approximation conditions (1):

$$w_i(g + \lambda h - f_i) \geq \min(w_i(g - f_i), w_i(\lambda) + w_i(h)) > \gamma_i \quad \text{for } i = 1, \ldots, n.$$

We claim that for a suitable choice of λ we have $v(g + \lambda h) \leq 0$ for all $v \in V$. We use the following lemma:

Lemma 7.3.2. *Let K be a field and v a valuation on K. Let $g, h \in K$ be two elements such that $v(g), v(h)$ are not both positive. Let $\Lambda \subseteq K$ be a set of v-units such that the residue map $\Lambda \hookrightarrow \mathcal{O}_v^\times \twoheadrightarrow \kappa(v)^\times$ is injective. Then there is at most one $\lambda \in \Lambda$ for which $v(g + \lambda h) > 0$.*

Proof. If $v(g) \neq v(h)$, then we have $v(g + \lambda h) = \min(v(g), v(h)) \leq 0$ for all $\lambda \in \Lambda$. If $v(g) = v(h)$, then the common valuation is necessarily ≤ 0. We have $v(g/h + \lambda) \geq 0$, with strict inequality only if $\lambda(v) = -(g/h)(v)$ holds in the residue field $\kappa(v)$. By assumption, this happens for at most one $\lambda \in \Lambda$. For all others, we have

$$v(g + \lambda h) = v\big(\frac{g}{h} + \lambda\big) + v(h) = v(h) \leq 0. \qquad \square$$

Using the lemma and the assumption $|\Lambda| > |V|$, there exists some $\lambda \in \Lambda$ avoiding all bad choices, so that $f = g + \lambda h$ satisfies $v(f) \leq 0$ for all $v \in V$.

Step 2: Avoiding poles in V

We now construct $f \in K^\times$ satisfying (1) and $v(f) \geq 0$ for all $v \in V$. We reduce this problem to Step 1 by setting $f = 1/f'$ for some suitable $f' \in K^\times$. Then the poles of f are precisely the zeros of f':

$$v\big(1/f'\big) < 0 \;\Leftrightarrow\; v(f') > 0 \quad \text{for } v \in V.$$

We can assume that the f_i are all different from zero by adding some $\delta_i \in K^\times$ with $w_i(\delta_i) > \gamma_i$ if necessary (such δ_i exists since w_i is not the trivial valuation). For each $i = 1, \ldots, n$, the inversion map on $K^\times$ is continuous with respect to the topology induced by w_i by Proposition 4.4.2, so there exist $\gamma_i' \in \Gamma_i$ for which we have the implication:

$$w_i\big(f' - \frac{1}{f_i}\big) > \gamma_i' \;\Rightarrow\; w_i\big(\frac{1}{f'} - f_i\big) > \gamma_i.$$

By Step 1, there exists $f' \in K$ satisfying $w_i\big(f' - \frac{1}{f_i}\big) > \gamma_i'$ for $i = 1, \ldots, n$ and without zeros in V. We have $f' \neq 0$ since $V \neq \emptyset$, so that we can take $f = 1/f'$ and have the claimed properties.

Step 3: Avoiding zeros and poles in V

Finally, we want to construct $f \in K$ satisfying conditions (1) and (2). Using Step 2, choose $g \in K^\times$ satisfying (1) and having no poles in V. Using Step 2 again, choose $h \in K^\times$ such that $w_i(h) > \gamma_i$ for all $i = 1, \dots, n$, such that $v(h-1) > 0$ (hence $v(h) = 0$) for all $v \in V$ with $v(g) > 0$, and without poles in $V \setminus \{v \in V : v(g) > 0\}$. Then h and hence all functions $g + \lambda h$ with $\lambda \in \Lambda$ have no poles in V. As in Step 1, they all satisfy (1). Using Lemma 7.3.2 and $|\Lambda| > |V|$, there exists some $\lambda \in \Lambda$ such that $f \coloneqq g + \lambda h$ has no zeros in V either, and hence satisfies (2). This finishes the proof of Theorem 7.3.1.

7.3.2 Proof of main theorems for countable sets

We use the approximation theorem of the previous paragraph to verify the conditions of a good localisation in the case of a countable set of points:

Theorem 7.3.3 ($\Rightarrow$ Theorem B (a))**.** *Let k be a finite extension of $\mathbb{Q}_p$ with $\mu_p \subseteq k$, let X/k be a smooth, proper, geometrically connected curve and let $S \subseteq X_{\mathrm{cl}}$ be a set of closed points which is at most countable. Then X_S is a good localisation.*

Proof. We verify the approximation criteria of Proposition 6.2.4. Note that for Condition (Pic) it suffices to consider only X_S itself rather than all geometrically connected, finite $\mathbb{Z}/p\mathbb{Z}$-elementary abelian covers since those are again localisations at sets of at most countably many closed points. All four approximation criteria demand the existence of functions in the function field K satisfying one or two approximation conditions and being invertible at all points of a subset $S' \subseteq S$. We therefore choose V in Theorem 7.3.1 to be the set of valuations v_s with $s \in S'$. For Λ we choose the set $\mathcal{O}_k \setminus \{0\}$ of nonzero scalars in the valuation ring of k. This is contained in $\mathcal{O}_w$ for every valuation w of K whose restriction to k is trivial or equals the p-adic valuation, and it is contained in $\mathcal{O}_{X,s}^\times$ for all $s \in S'$, with the reduction map $\Lambda \hookrightarrow \mathcal{O}_{X,s}^\times \to \kappa(s)^\times$ being injective since the residue field $\kappa(s)$ is an extension of k. The valuation ring $\mathcal{O}_k$ is uncountable whereas S is assumed at most countable, hence we have $|\Lambda| > |S|$. So Theorem 7.3.1 applies and shows that the approximation criteria of Proposition 6.2.4 are all satisfied for X_S. Thus, X_S is a good localisation as claimed. $\square$

For base fields not necessarily containing the p-th roots of unity we have the following version of the liftable section conjecture:

Theorem 7.3.4 ($=$ Theorem D (a))**.** *Let k be a finite extension of $\mathbb{Q}_p$ and ℓ/k a finite Galois extension with $\mu_p \subseteq \ell$. Let X/k be a smooth, proper, geometrically connected curve and let $S \subseteq X_{\mathrm{cl}}$ be a set of closed points which is at most countable. Then for every liftable section $s' \colon \mathrm{Gal}(\ell'/k) \to \mathrm{Gal}((X_S \otimes_k \ell)'/X_S)$ there exists a unique k-rational point x of X such that $\mathrm{res}_{\ell/k}(s')$ lies over x. If moreover one of the following holds:*

- *p does not divide $[\ell : k]$; or*
- *s' is twice-liftable;*

then s' itself lies over x.

Proof. Denote by $S \otimes \ell$ the preimage of S under the projection $X \otimes \ell \to X$. We have $X_S \otimes \ell = (X \otimes \ell)_{S \otimes \ell}$ since localisation commutes with base change along closed morphisms (Lemma 2.2.5). The set $S \otimes \ell$ is still at most countable. By Theorem 7.3.3, $X_S \otimes \ell$ is a good localisation and thus satisfies the liftable section conjecture. Theorem 1.4.4 shows that for every liftable section s' as above there exists a unique k-rational point x of X such that $\mathrm{res}_{\ell/k}(s')$ lies over x. It also shows that s' itself lies over x if p does not divide $[\ell : k]$. To apply Theorem 1.4.4 (b), consider a geometrically connected, finite étale subcover $(X_S \otimes \ell)' \to W \to X_S$. By Corollary 2.4.11, $W \to X_S$ is of the form $Y_{f^{-1}(S)} \to X_S$ for some finite morphism $f\colon Y \to X$. In particular, W is itself a localisation at a set of at most countably many closed points, and the same holds for the base change $W \otimes \ell$. By Theorem 7.3.3, every such $W \otimes \ell$ is a good localisation and hence satisfies the liftable section conjecture. Theorem 1.4.4 (b) now implies that if s' is twice-liftable, then s' itself lies over x. □

Theorem 7.3.5 (= Theorem E (a))**.** *Let k be a finite extension of $\mathbb{Q}_p$, let X/k be a smooth, proper, geometrically connected curve and let $S \subseteq X_{\mathrm{cl}}$ be a set of closed points which is at most countable. Then X_S satisfies the section conjecture.*

Proof. Set $\ell := k(\mu_p)$. Every geometrically connected, finite étale cover of X_S is of the form $Y_{f^{-1}(S)} \to X_S$ for some finite morphism $f\colon Y \to X$ (Corollary 2.4.11), hence is again a localisation at a set of at most countably many closed points. The same holds for the base change $Y_{f^{-1}(S)} \otimes \ell$. Hence, by Theorem 7.3.3, $W \otimes \ell$ is a good localisation and thus satisfies the liftable section conjecture, for every geometrically connected, finite étale cover $W \to X_S$. This implies by Theorem 1.4.4 that X_S satisfies the section conjecture. □

7.4 Transcendental points

Let k be a finite extension of $\mathbb{Q}_p$ with $\mu_p \subseteq k$, let $k_0 \subseteq k$ be an arbitrary subfield, let X_0/k_0 be a smooth, proper, geometrically connected curve and set $X = X_0 \otimes_{k_0} k$. Denote by K_0 and K the function fields of X_0 and X, respectively. We have an inclusion $K_0 \subseteq K$, reflecting the fact that rational functions on X_0 pull back to rational functions on X.

Definition 7.4.1. A closed point $x \in X_{\mathrm{cl}}$ is called **algebraic over** k_0 (relative to X_0) if its image under the projection $X \to X_0$ is a closed point. The non-algebraic closed points are called **transcendental over** k_0.

Example 7.4.2. Let $X_0 = \mathbb{P}^1_{k_0}$, so that $X = \mathbb{P}^1_k$. The point ∞ is mapped to the closed point ∞ under the projection $\mathbb{P}^1_k \to \mathbb{P}^1_{k_0}$, hence this point is algebraic over k_0. A k-rational point a of the affine part $\mathbb{A}^1_k = \mathrm{Spec}(k[t])$ is algebraic over k_0 if and only if $(t-a) \cap k_0[t] \neq (0)$, that is if there exists a nonzero polynomial f with coefficients in k_0 such that $f(a) = 0$. Hence, the algebraic points over k_0 in $\mathbb{A}^1(k)$ are those elements of k which are algebraic over the subfield k_0 in the sense of commutative algebra.

Remark 7.4.3. Algebraicity of a closed point of X/k over a subfield $k_0 \subseteq k$ is not an intrinsic property but depends on the choice of model X_0/k_0. For example, in the case $X = \mathbb{P}^1_k$ there are k-automorphisms (such as translations by transcendental numbers) which do not preserve algebraicity.

The first aim of this section 7.4 is to prove the following:

Theorem 7.4.4 ($\Rightarrow$ Theorem B (b))**.** *With the notation as above, let $S \subseteq X_{\mathrm{cl}}$ be a set of closed points containing only finitely many points which are algebraic over k_0. Then X_S is a good localisation.*

After proving Theorem 7.4.4, we deduce the consequences for the liftable section conjecture without p-th roots of unity and for the full section conjecture.

7.4.1 Verification of the conditions

For the purpose of proving Theorem 7.4.4, we can assume that $k_0 \subseteq k$ is dense with respect to the p-adic topology. To see this, we use the following consequence of Krasner's Lemma:

Lemma 7.4.5. *Let k be a finite extension of $\mathbb{Q}_p$. Then k contains a number field which is p-adically dense in k.*

Proof. Let $\alpha \in k$ be a generator of the extension $k/\mathbb{Q}_p$ and let $f(t) \in \mathbb{Q}_p[t]$ be its minimal polynomial. For a polynomial $g(t) \in \mathbb{Q}[t]$ of the same degree with coefficients p-adically close to those of f, there exists a root β of $g(t)$ in an algebraic closure $\overline{k}$ which is p-adically close to α. By Krasner's Lemma [NSW08, Lemma (8.1.6)], k is contained in $\mathbb{Q}_p(\beta)$, and hence equal to $\mathbb{Q}_p(\beta)$ since g has the same degree as f. The number field $\mathbb{Q}(\beta)$ is then dense in k. $\square$

The set of points of X which are algebraic over k_0 does not change under extending scalars from k_0 to a finite field extension inside k. Thus, using Lemma 7.4.5, we can assume in the following that the subfield k_0 is p-adically dense in k.

The three conditions (Sep), (Rat), (Fin) for a good localisation have in common that they all stipulate the existence of a sufficient supply of rational functions on X which are invertible on X_S. We have a way of producing rational functions which are guaranteed to be invertible at all transcendental points:

Lemma 7.4.6. *Nonzero rational functions defined on X_0 are invertible at all transcendental points over k_0 of X.*

Proof. If $x \in X_{\mathrm{cl}}$ is transcendental over k_0, then its image in X_0 equals the generic point $\mathrm{Spec}(K_0)$, which implies that $K_0^\times$ is contained in $\mathcal{O}_{X,x}^\times$. □

Condition (Sep)

Proposition 7.4.7. *If $S \subseteq X_{\mathrm{cl}}$ contains only finitely many algebraic points over k_0, then X_S satisfies Condition (Sep).*

Proof. Let $x \neq y$ be distinct points in $X(k)$. We have to construct a function f' in $\mathcal{O}(X_{S\cup\{x,y\}})^\times$ for which $f'(x)/f'(y)$ is not a p-th power. Let $U_0 \subseteq X_0$ an affine open subscheme which contains the image of $S \cup \{x, y\}$ under the projection $X \to X_0$. This is possible since this image contains only finitely closed points of X_0, and any open subset of X_0 which is not the whole curve is affine. Let $t_1, \ldots, t_n$ be a system of affine coordinates on U_0, i.e. generators of the coordinate ring $\mathcal{O}(U_0)$ as a k_0-algebra. Set $U \coloneqq U_0 \otimes_{k_0} k$. Then U is an open affine subscheme of X containing $S \cup \{x, y\}$, and the t_i are a system of affine coordinates on $\mathcal{O}(U)$ over k. Since $x \neq y$ by assumption, $t_i(x) \neq t_i(y)$ for at least one i. Set $t \coloneqq t_i$. Then, for any $c_1, c_2 \in k$, the system of linear equations

$$\begin{aligned} at(x) + b &= c_1, \\ at(y) + b &= c_2, \end{aligned}$$

is uniquely solvable for $a, b \in k$. By taking c_1 to be a uniformiser of k and $c_2 = 1$, for example, we find a and b in k such that, upon setting $f \coloneqq at + b$, the value $f(x)/f(y) = c_1/c_2$ is not a p-th power in k. Now, using that k_0 is dense in k, choose elements a_0 and b_0 in k_0 which are p-adically close to a and b, and set $f_0 \coloneqq a_0 t + b_0$. The group $k^{\times p}$ of p-th powers is open in $k^\times$ of finite index, hence its complement is also open and for a_0 and b_0 sufficiently close to a and b, the value $f_0(x)/f_0(y)$ is still not a p-th power. By replacing b_0 with $b_0 + \varepsilon_0$ for some p-adically small number $\varepsilon_0 \in k_0$ (for example $\varepsilon_0 = p^N$ with $N \gg 0$), we can moreover avoid that f_0 vanishes at any of the finitely many points in S which are algebraic over k_0, so that f_0 is invertible at those points. By construction, f_0 is an element of $\mathcal{O}(U_0)$, i.e. defined already on X_0, and therefore automatically invertible at all transcendental points over k_0 by Lemma 7.4.6. Thus, f_0 is in $\mathcal{O}(X_S)^\times$ and satisfies the requirements. □

Condition (Rat)

Proposition 7.4.8. *If $S \subseteq X_{\mathrm{cl}}$ contains only finitely many algebraic points over k_0, then X_S satisfies Condition (Rat).*

Proof. Let $x \in X_{\mathrm{cl}}$ be any closed point. We show that the map

$$\mathcal{O}(X_{S\cup\{x\}})^\times \to \kappa(x)^\times/\kappa(x)^{\times p}$$

given by $f \mapsto f(x)$, is surjective, so that X_S satisfies Condition (Rat) by Corollary 5.1.5. Let $U_0 \subseteq X_0$ be an open affine subscheme which contains the image of $S \cup \{x\}$, let $t_1, \ldots, t_n$ be a system of affine coordinates on U_0 and set $U \coloneqq U_0 \otimes_{k_0} k$. The residue field $\kappa(x)$ is generated by $t_1(x), \ldots, t_n(x)$ as a k-algebra. Let $a \in \kappa(x)^\times$ be arbitrary and write $a = P(t_1(x), \ldots, t_n(x))$ for some polynomial $P \in k[X_1, \ldots, X_n]$. Using that k_0 is dense in k, choose $P_0 \in k_0[X_1, \ldots, X_n]$ with coefficients approximating those of P. Since $\kappa(x)^{\times p}$ is open in $\kappa(x)$, we have

$$P_0(t_1(x), \ldots, t_n(x)) \in a \cdot \kappa(x)^{\times p}$$

for P_0 sufficiently close to P. By replacing P_0 with $P_0 + \varepsilon_0$ for some p-adically small number $\varepsilon_0 \in k_0$ we can avoid that $P_0(t_1, \ldots, t_n)$ vanishes at any of the finitely many algebraic points over k_0 in S. Then the rational function $f_0 \coloneqq P_0(t_1, \ldots, t_n)$ satisfies $f_0(x) \in a \cdot \kappa(x)^{\times p}$ by construction, is invertible on the algebraic points in S over k_0 and, since it is already defined on X_0, is invertible at all points which are transcendental over k_0. □

Condition (Fin)

Proposition 7.4.9. *If $S \subseteq X_{\mathrm{cl}}$ contains only finitely many algebraic points over k_0, then X_S satisfies Condition (Fin).*

Proof. We show that $\mathcal{O}(X_S)^\times$ is dense in K with respect to every rank one valuation w which extends the p-adic valuation v_k on k, so that X_S satisfies Condition (Fin) by Proposition 6.2.4 (d). Let w be one such valuation, let $\gamma \in \Gamma_w$ be an element of its value group, and let $f \in K$. We have to construct a function $f_0 \in \mathcal{O}(X_S)^\times$ satisfying $w(f_0 - f) > \gamma$. Let $U_0 \subseteq X_0$ be an affine dense open subscheme containing the image of S under the projection $X \to X_0$, let $t = (t_1, \ldots, t_n)$ be affine coordinates on U_0 and set $U \coloneqq U_0 \otimes_{k_0} k$. Write $f = g(t)/h(t)$ as a rational function with polynomials $g, h \in k[X_1, \ldots, X_n]$. Using that k_0 is dense in k, choose $g_0, h_0 \in k_0[X_1, \ldots, X_n]$ with coefficients approximating those of g and h. Explicitly, write $g = \sum_i a_i X^i$ in multi-index notation with $a_i \in k$, choose $b_i \in k_0$ with $v_k(a_i - b_i) \gg 0$, and set $g_0 \coloneqq \sum_i b_i X^i$ (and similarly for h_0). For each multi-index $i \in \mathbb{N}_0^n$, the value

$$w(a_i t^i - b_i t^i) = v_k(a_i - b_i) + w(t^i)$$

becomes arbitrarily large as b_i approaches a_i since the rank one value group Γ_w embeds into $\mathbb{R}$ as an ordered group. Thus, $g_0(t)$ and $h_0(t)$ can be chosen arbitrarily close to $g(t)$ and $h(t)$ with respect to the valuation w. By replacing g_0 with $g_0 + \delta_0$ and h_0 with $h + \varepsilon_0$ for some p-adically small numbers $\delta_0, \varepsilon_0 \in k_0$

we can achieve that $g_0(t)$ and $h_0(t)$ are not constant zero on X and do not vanish at any of the finitely many algebraic points in S over k_0. Then, setting $f_0 \coloneqq g_0(t)/h_0(t)$, the function f_0 is invertible on the algebraic points in S over k_0; it is also invertible at all transcendental points over k_0 since it it defined already on X_0, and it satisfies $w(f_0 - f) > \gamma$ for g_0 and h_0 sufficiently close to g and h by the continuity of the field operations with respect to the w-adic topology. □

Condition (Pic)

The remaining Condition (Pic) involves the Picard group of X_S (and of finite covers of X_S). Functions which are invertible on X_S have limited use in proving statements about $\mathrm{Pic}(X_S)$. Therefore, instead of approximating functions on X by functions defined on X_0, we will use approximation of divisors on X by divisors supported on algebraic points over k_0.

Proposition 7.4.10. *If $S \subseteq X_{\mathrm{cl}}$ contains only finitely many algebraic points over k_0, then X_S satisfies Condition (Pic).*

Proof. Denote by T the complement of S. By assumption, T contains all but finitely many of the algebraic points over k_0. We show that T is uniformly dense in X (Definition 6.3.8), which implies that X_S satisfies Condition (Pic) by Proposition 6.3.11. Let ℓ/k be a finite extension. We have to show that $T(\ell)$ is dense in $X(\ell)$. The situation is stable under extending scalars, so it suffices to treat the case $\ell = k$. Let $x \in X(k)$ and let $U \subseteq X(k)$ be a p-adically open neighbourhood of x. We have to show that $U \cap T(k) \neq \emptyset$. Since X_0 is smooth, there exists an affine open subset $V_0 \subseteq X_0$ containing the image of x such that there exists an étale morphism $\pi_0 \colon X_0 \to \mathbb{A}^1_{k_0}$ [Stacks, Lemma 054L]. By [Poo17, Prop. 3.5.73], the image $\pi_0(U \cap V_0(k)) \subseteq k$ is open. Since k_0 is dense in k, there exists a point $y_0 \in \pi_0(U \cap V_0(k)) \cap k_0$. By replacing y_0 with $y_0 + \varepsilon_0$ for some p-adically small number $\varepsilon_0 \in k_0$, we can ensure that y_0 is not the image of one of the finitely many algebraic points over k_0 outside T under π_0. Let $x_0 \in U \cap V_0(k)$ with $\pi_0(x_0) = y_0$. Then, since π_0 is étale, the residue field $\kappa(x_0)$ is a finite extension of $\kappa(y_0) = k_0$, which implies that x_0 is algebraic over k_0. By construction, x_0 is not equal to one of the finitely many algebraic points over k_0 outside T, so we have $x_0 \in T(k)$ and hence $U \cap T(k) \neq \emptyset$. □

This finishes the proof of Theorem 7.4.4.

7.4.2 Base fields without p-th roots of unity and full section conjecture

Having proved the liftable section conjecture in the case where S contains only finitely many algebraic points over k_0, we deduce the consequences for the case where the base field does not necessarily contain the p-th roots of unity and for the section conjecture with the full fundamental group. So let k be an arbitrary finite extension of $\mathbb{Q}_p$ and, as above, let $k_0 \subseteq k$ be a subfield, let X_0/k_0 be a smooth, proper, geometrically connected curve and set $X = X_0 \otimes_{k_0} k$. Denote by $\pi\colon X \to X_0$ the canonical projection. The generic points of X and X_0 are denoted by η_X and η_{X_0}.

The part of the descent statement in Theorem 1.4.4 concerning twice-liftable sections as well as Theorem 1.4.6 for the full section conjecture require that certain covers of $X_S \otimes \ell$ satisfy the liftable section conjecture. In order to have a situation which is stable under finite étale covers, S needs to contains all transcendental points over k_0. In order to prove this, we start with a few lemmas.

In the following statement it is more natural to work with localisations at subsets containing the generic point rather than at sets of closed points:

Lemma 7.4.11. *Let $S_0 \subseteq (X_0)_{\mathrm{cl}}$ be a set of closed points of X_0, and let $\widetilde{S}_0 := S_0 \cup \{\eta_{X_0}\} \subseteq |X_0|$. Then the canonical map is an isomorphism*

$$X_{\pi^{-1}(\widetilde{S}_0)} \cong (X_0)_{\widetilde{S}_0} \otimes_{k_0} k.$$

Proof. The canonical map is given by

$$X_{\pi^{-1}(\widetilde{S}_0)} = \varprojlim_{U \supseteq \pi^{-1}(\widetilde{S}_0)} U \longrightarrow \varprojlim_{U_0 \supseteq \widetilde{S}_0} \pi^{-1}(U_0) = (X_0)_{\widetilde{S}_0} \otimes_{k_0} k,$$

where we use that fibre products commute with limits. We are claiming that the open subsets of the form $\pi^{-1}(U_0)$ with $U_0 \supseteq \widetilde{S}_0$ are cofinal among all open subsets $U \supseteq \pi^{-1}(\widetilde{S}_0)$ of X. Let U be an open subset of X containing $\pi^{-1}(\widetilde{S}_0)$. Every point in the complement $X \setminus U$ is mapped to a point of X_0 outside $\widetilde{S}_0$. Since the generic point η_{X_0} is contained in $\widetilde{S}_0$, we have that $\pi(X \setminus U)$ is a finite set of closed points. Hence, the set $U_0 := X_0 \setminus \pi(X \setminus U)$ is open. It satisfies $U_0 \supseteq \widetilde{S}_0$ and $\pi^{-1}(U_0) \subseteq U$. This shows the claimed cofinality. $\square$

Lemma 7.4.12. *Suppose $S_0 \subseteq (X_0)_{\mathrm{cl}}$ is a set of closed points of X_0 and set $S := \pi^{-1}(S_0 \cup \{\eta_{X_0}\}) \setminus \{\eta_X\}$, so that we have $X_S = (X_0)_{S_0} \otimes_{k_0} k$ by Lemma 7.4.11. Then every finite étale cover of X_S arises by base change from a finite étale cover of $(X_0)_{S_0}$.*

Proof. We are claiming the essential surjectivity of the functor

$$\mathrm{Cov}((X_0)_{S_0}) \to \mathrm{Cov}(X_S) \tag{$*$}$$

sending a finite étale cover $W_0 \to (X_0)_{S_0}$ to the base change $W_0 \otimes_{k_0} k \to X_S$. Choose compatible fibre functors F and F_0 on the two Galois categories and write

$$\pi_1(X_S) := \pi_1(X_S, F) \quad \text{and} \quad \pi_1((X_0)_{S_0}) := \pi_1((X_0)_{S_0}, F_0)$$

for the respective fundamental groups. Via the anti-equivalence between profinite groups and Galois categories with fibre functor (Theorem 2.3.1), the essential surjectivity of $(*)$ translates into the injectivity of the homomorphism $\pi_1(X_S) \to \pi_1((X_0)_{S_0})$. Choose algebraic closures $\overline{k}/k$ and $\overline{k_0}/k_0$ with $\overline{k_0} \subseteq \overline{k}$, defining absolute Galois group $G_k \coloneqq \mathrm{Gal}(\overline{k}/k)$ and $G_{k_0} \coloneqq \mathrm{Gal}(\overline{k_0}/k_0)$. The two fundamental exact sequences for X_S and $(X_0)_{S_0}$ fit into a commutative diagram as follows:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(X_S \otimes_k \overline{k}) \arrow[r] \arrow[d] & \pi_1(X_S) \arrow[r] \arrow[d] & G_k \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \pi_1((X_0)_{S_0} \otimes_{k_0} \overline{k_0}) \arrow[r] & \pi_1((X_0)_{S_0}) \arrow[r] & G_{k_0} \arrow[r] & 1
\end{tikzcd}$$

The Künneth formula for fundamental groups [SGA 1, Exp. XIII, Prop. 4.6 (b)] applied to the fibre product

$$X_S \otimes_k \overline{k} = ((X_0)_{S_0} \otimes_{k_0} \overline{k_0}) \times_{\overline{k_0}} \mathrm{Spec}(\overline{k})$$

implies that the left vertical map is an isomorphism. The hypotheses of the Künneth formula are satisfied since the fields have characteristic zero, so that resolution of singularities is known. We claim that also the right vertical map is injective. Any automorphism $\sigma \in \mathrm{Gal}(\overline{k}/k)$ is continuous with respect to the unique extension of the p-adic absolute value on k to $\overline{k}$. As a consequence of Krasner's Lemma, the subfield $\overline{\mathbb{Q}} \subseteq \overline{k}$ and hence also $\overline{k_0}$ is dense in $\overline{k}$. Thus, if σ restricts to the identity on $\overline{k_0}$, then necessarily $\sigma = \mathrm{id}$.

As the left and right vertical maps are both injective, it follows that the middle vertical map is injective as well. □

Lemma 7.4.13. *Let $S \subseteq X_{\mathrm{cl}}$ be a set of closed points. Assume that k_0 is relatively algebraically closed in k. Then the following are equivalent:*

(i) S contains all transcendental points over k_0;

(ii) there exists a set of closed points $S_0 \subseteq (X_0)_{\mathrm{cl}}$ with $\pi(S) \subseteq S_0 \cup \{\eta_{X_0}\}$ such that the canonical map is an isomorphism $X_S \cong (X_0)_{S_0} \otimes_{k_0} k$.

Proof. For the implication (ii)⇒(i) assume that $X_S \cong (X_0)_{S_0} \otimes_{k_0} k$ for some set of closed points $S_0 \subseteq (X_0)_{\mathrm{cl}}$ with $\pi(S) \subseteq S_0 \cup \{\eta_{X_0}\}$. Then by Lemma 7.4.11 we have $X_S = X_{\pi^{-1}(\widetilde{S}_0)}$ with $\widetilde{S}_0 \coloneqq S_0 \cup \{\eta_{X_0}\}$. Taking the underlying spaces, we find

$$S \cup \{\eta_X\} = \pi^{-1}(\widetilde{S}_0)$$

as subspaces of $|X|$. The transcendental points of X are the closed points which are mapped to the generic point in X_0 under π, so they are all contained in $\pi^{-1}(\widetilde{S}_0)$. Hence, S contains all transcendental points over k_0.

Conversely, assume that S contains all transcendental points over k_0. Let $\widetilde{S} \coloneqq S \cup \{\eta_X\}$. We claim that we have $\widetilde{S} = \pi^{-1}(\pi(\widetilde{S}))$. This is equivalent to

$\widetilde{S}$ being a union of fibres of π. The fibre over the generic point η_{X_0} consists of the generic point of X and all transcendental closed points over k_0, hence this fibre is completely contained in $\widetilde{S}$ by assumption. The fact that k_0 is relatively algebraically closed in k implies that the fibre over each closed point of X_0 consists only of a single point. This shows the equality $\widetilde{S} = \pi^{-1}(\pi(\widetilde{S}))$. Let $S_0 := \pi(\widetilde{S}) \setminus \{\eta_{X_0}\}$. Then we have $X_S \cong (X_0)_{S_0} \otimes_{k_0} k$ by Lemma 7.4.11. □

Remark 7.4.14. In Lemma 7.4.13 (ii), the set of closed points $S_0 \subseteq (X_0)_{\mathrm{cl}}$ with $X_S \cong (X_0)_{S_0} \otimes_{k_0} k$ is unique: it satisfies $\pi^{-1}(S_0 \cup \{\eta_{X_0}\}) = S \cup \{\eta_X\}$ by Lemma 7.4.11, and by taking images under the surjective map π we find

$$S_0 = \pi(S) \setminus \{\eta_{X_0}\}.$$

Proposition 7.4.15. *Assume $\mu_p \subseteq k$ and let $S \subseteq X_{\mathrm{cl}}$ be a set of closed points containing all transcendental points and only finitely many algebraic points over k_0. Then every geometrically connected, finite étale cover of X_S is a good localisation.*

Proof. We may replace k_0 with any algebraic extension in k without changing the property of closed points of X to be algebraic over k_0. We can therefore assume that k_0 is relatively algebraically closed in k. Then, as S contains all transcendental points over k_0 by assumption, we have $X_S = (X_0)_{S_0} \otimes_{k_0} k$ by Lemma 7.4.13, with $S_0 \subseteq (X_0)_{\mathrm{cl}}$ consisting of the closed points in $\pi(S)$. By Lemma 7.4.12, every geometrically connected, finite étale cover of X_S arises by base change from a finite étale cover of $(X_0)_{S_0}$. Such a cover has the form $(Y_0)_{T_0} \to (X_0)_{S_0}$ with $f_0\colon Y_0 \to X_0$ unramified over S_0 and $T_0 = f_0^{-1}(S_0)$. Let $Y := Y_0 \otimes_{k_0} k$ and let $T \subseteq Y_{\mathrm{cl}}$ be the set of closed points mapping to T_0 or the generic point in Y_0, so that $(Y_0)_{T_0} \otimes_{k_0} \otimes k = Y_T$ by Lemma 7.4.11. The fact that Y_T is geometrically connected over k implies that $(Y_0)_{T_0}$ is geometrically connected over k_0. We have to show that Y_T is a good localisation. We have a cartesian diagram as follows:

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ Y_0 & \xrightarrow{f_0} & X_0. \end{array}$$

The horizontal morphisms f and f_0 are finite and hence send closed points to closed points. As a consequence, a closed point $y \in Y_{\mathrm{cl}}$ is transcendental over k_0 (relative to Y_0) if and only if $f(y)$ is transcendental over k_0 (relative to X_0). From $T_0 = f_0^{-1}(S_0)$ we have $T = f^{-1}(S)$. By assumption, S contains only finitely many points which are algebraic over k_0. As f has finite fibres, also T contains only finitely many points which are algebraic over k_0. We are thus in the situation of Theorem 7.4.4 and conclude that Y_T is a good localisation. □

We can now deduce the version of the liftable section conjecture over base fields without p-th roots of unity and the section conjecture for the full fundamental groups:

Theorem 7.4.16 (= Theorem D (b))**.** *Let $X = X_0 \otimes_{k_0} k$ as above, let ℓ/k be a finite Galois extension with $\mu_p \subseteq \ell$ and let $S \subseteq X_{\mathrm{cl}}$ be a set of closed points containing only finitely many points which are algebraic over k_0. Then for every liftable section $s'\colon \mathrm{Gal}(\ell'/k) \to \mathrm{Gal}((X_S \otimes_k \ell)'/X_S)$, there exists a unique k-rational point x of X such that the restriction $\mathrm{res}_{\ell/k}(s')$ lies over $x \otimes_k \ell$. If moreover one of the following holds:*

- *p does not divide $[\ell : k]$; or*
- *every transcendental point of X over k_0 is contained in S and s' is twice-liftable;*

then s' itself lies over x.

Proof. Denote by $S \otimes \ell \subseteq (X \otimes \ell)_{\mathrm{cl}}$ the preimage of S under the projection $X \otimes \ell \to X$. We have $X_S \otimes \ell = (X \otimes \ell)_{S \otimes \ell}$ since localisation commutes with base change along closed morphisms (Lemma 2.2.5). A closed point in $X \otimes \ell$ is algebraic over k_0 if and only if its image in X is so. Since S contains only finitely many points which are algebraic over k_0, the same holds for $S \otimes \ell$. By Theorem 7.4.4, $X_S \otimes \ell$ is a good localisation, so it satisfies the liftable section conjecture. With Theorem 1.4.4 we conclude that for every liftable section $s'\colon \mathrm{Gal}(\ell'/k) \to \mathrm{Gal}((X_S \otimes_k \ell)'/X_S)$ there exists a unique rational point x of X such that $\mathrm{res}_{\ell/k}(s')$ lies over $x \otimes \ell$. Theorem 1.4.4 (a) also yields that s' itself lies over x if p does not divide $[\ell : k]$. Assume that every transcendental point of X over k_0 is contained in S and that s' is twice-liftable. Proposition 7.4.15 shows that $W \otimes \ell$ is a good localisation for every geometrically connected, finite étale subcover $(X_S \otimes \ell)' \to W \to X_S$. Hence, Theorem 1.4.4 (b) applies and s' itself lies over x. □

Theorem 7.4.17 (= Theorem E (b))**.** *Let $X = X_0 \otimes_{k_0} k$ as above and let $S \subseteq X_{\mathrm{cl}}$ be a set of closed points which contains all transcendental points and only finitely many algebraic points over k_0. Then X_S satisfies the section conjecture.*

Proof. Let $\ell = k(\mu_p)$. Proposition 7.4.15 implies that $W \otimes \ell$ is a good localisation for every geometrically connected, finite étale cover $W \to X_S$. So Theorem 1.4.6 applies and shows that X_S satisfies the section conjecture. □

Outlook

We see several potential directions to continue the line of research of this dissertation:

- *More examples of good localisations.* We have considered here only localisations at countable sets and at sets of transcendental points over a subfield. It would be interesting to find further examples. Complements of p-adic discs would be possible candidates to study. Another idea is to generalise our approximation theorem with invertibility conditions. In the present form it relies on a comparison of cardinalities; one could try and modify it to use a comparison of p-adic volumes instead. The case of countable sets could then be generalised to sets of small p-adic volume. The larger the set of points, the closer we get to the section conjecture for the complete curve, which is as yet unproved. On the other hand, also negative examples showing the limitations of the method would be interesting.

- *Higher dimensions.* The starting point of our investigation was Pop's proof of the birational liftable p-adic section conjecture for curves. He has shown in [Pop17] that the method of proof generalises to higher-dimensional varieties. We expect that our generalisation for localisations of curves should similarly extend to localisations of higher-dimensional varieties. We have introduced localisations of schemes and their fundamental groups in great generality partly with this generalisation in mind. The task will be to find the correct conditions on the localisation of a variety under which the proof of the liftable section conjecture can be adapted to work.

- *Sections on larger quotients.* The liftable section conjecture is remarkable in that the existence of a rational point is deduced from the existence of a section for a very small quotient of the fundamental group. However, one may ask whether the conditions of a good localisation can be weakened at the cost of working with sections on larger quotients such as the pro-p quotient. Having weaker conditions could lead to more examples of localisations satisfying the p-adic section conjecture.

Bibliography

[AB59] Maurice Auslander and D. A. Buchsbaum. "Unique Factorization in Regular Local Rings". In: *Proceedings of the National Academy of Sciences* 45.5 (1959), pp. 733–734. ISSN: 0027-8424. DOI: `10.1073/pnas.45.5.733`. eprint: `https://www.pnas.org/content/45/5/733.full.pdf`. URL: `https://www.pnas.org/content/45/5/733`.

[Bou87] N. Bourbaki. *Topological vector spaces. Chapters 1–5*. Elements of Mathematics (Berlin). Translated from the French by H. G. Eggleston and S. Madan. Springer-Verlag, Berlin, 1987, pp. viii+364. ISBN: 3-540-13627-4. DOI: `10.1007/978-3-642-61715-7`. URL: `https://doi.org/10.1007/978-3-642-61715-7`.

[Del89] P. Deligne. "Le groupe fondamental de la droite projective moins trois points". In: *Galois groups over* $\mathbf{Q}$ *(Berkeley, CA, 1987)*. Vol. 16. Math. Sci. Res. Inst. Publ. Springer, New York, 1989, pp. 79–297. DOI: `10.1007/978-1-4613-9649-9_3`.

[EGA II] Alexander Grothendieck. "Éléments de géométrie algébrique : II. Étude globale élémentaire de quelques classes de morphismes". fr. In: *Publications Mathématiques de l'IHÉS* 8 (1961), pp. 5–222. URL: `http://www.numdam.org/item/PMIHES_1961__8__5_0`.

[EGA IV_2] A. Grothendieck. "Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II". In: *Inst. Hautes Études Sci. Publ. Math.* 24 (1965), p. 231. ISSN: 0073-8301. URL: `http://www.numdam.org/item?id=PMIHES_1965__24__231_0`.

[EGA IV_3] Alexander Grothendieck. "Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III". In: *Inst. Hautes Études Sci. Publ. Math.* 28 (1966), p. 255. ISSN: 0073-8301.

[EGA IV_4] A. Grothendieck. "Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV". In: *Inst. Hautes Études Sci. Publ. Math.* 32 (1967), p. 361. ISSN: 0073-8301. URL: `http://www.numdam.org/item?id=PMIHES_1967__32__361_0`.

[End72] Otto Endler. *Valuation theory.* To the memory of Wolfgang Krull (26 August 1899–12 April 1971), Universitext. Springer-Verlag, New York-Heidelberg, 1972, pp. xii+243.

[EP05] Antonio J. Engler and Alexander Prestel. *Valued fields.* Berlin: Springer, 2005, pp. x + 205. ISBN: 3-540-24221-X/hbk. DOI: 10.1007/3-540-30035-X.

[Gro66] Alexander Grothendieck. "Le groupe de Brauer : II. Théories cohomologiques". fr. In: *Séminaire Bourbaki : années 1964/65 1965/66, exposés 277-312.* Séminaire Bourbaki 9. talk:297. Société mathématique de France, 1966, pp. 287–307.

[Gro97] Alexander Grothendieck. "Brief an G. Faltings". In: *Geometric Galois actions, 1.* Vol. 242. London Math. Soc. Lecture Note Series. With an English translation on pp. 285–293. Cambridge: Cambridge University Press, 1997, pp. 49–58.

[GS17] Philippe Gille and Tamás Szamuely. *Central simple algebras and Galois cohomology.* Vol. 165. Cambridge Studies in Advanced Mathematics. Second edition. Cambridge University Press, Cambridge, 2017, pp. xi+417. ISBN: 978-1-316-60988-0; 978-1-107-15637-1.

[Kle05] Steven L. Kleiman. "The Picard scheme". In: *Fundamental algebraic geometry.* Vol. 123. Math. Surveys Monogr. Amer. Math. Soc., Providence, RI, 2005, pp. 235–321.

[Koe05] Jochen Koenigsmann. "On the 'Section Conjecture' in anabelian geometry". In: *Journal für die reine und angewandte Mathematik* 2005.588 (2005), pp. 221–235. URL: https://www.degruyter.com/view/journals/crll/2005/588/article-p221.xml.

[Lic69] Stephen Lichtenbaum. "Duality theorems for curves over p-adic fields". In: *Inventiones mathematicae* 7.2 (1969), pp. 120–136.

[Liu02] Qing Liu. *Algebraic geometry and arithmetic curves.* Vol. 6. Oxford Graduate Texts in Mathematics. Oxford: Oxford University Press, 2002, pp. xvi+576. ISBN: 0-19-850284-2.

[Mat55] Arthur Mattuck. "Abelian varieties over p-adic ground fields". In: *Ann. of Math. (2)* 62 (1955), pp. 92–119. ISSN: 0003-486X. DOI: 10.2307/2007101.

[Mat89] Hideyuki Matsumura. *Commutative ring theory.* Second. Vol. 8. Cambridge Studies in Advanced Mathematics. Translated from the Japanese by M. Reid. Cambridge University Press, Cambridge, 1989, pp. xiv+320. ISBN: 0-521-36764-6.

[MS82] A. S. Merkurjev and A. A. Suslin. "K-cohomology of Severi-Brauer varieties and the norm residue homomorphism". In: *Izv. Akad. Nauk SSSR Ser. Mat.* 46.5 (1982), pp. 1011–1046, 1135–1136. ISSN: 0373-2436.

[NSW08] J. Neukirch, A. Schmidt, and K. Wingberg. *Cohomology of Number Fields*. Grundlehren der mathematischen Wissenschaften. Springer, 2008. ISBN: 9783540378884.

[Poo17] B. Poonen. *Rational Points on Varieties:* Graduate Studies in Mathematics. American Mathematical Society, 2017. ISBN: 9781470437732.

[Pop10] Florian Pop. "On the birational p-adic section conjecture". In: *Compos. Math.* 146.3 (2010), pp. 621–637. ISSN: 0010-437X. DOI: 10.1112/S0010437X09004436. URL: https://doi-org.proxy.ub.uni-frankfurt.de/10.1112/S0010437X09004436.

[Pop17] Florian Pop. "The $\mathbb{Z}/p$ metabelian birational p-adic section conjecture for varieties". In: *Compos. Math.* 153.7 (2017), pp. 1433–1445. ISSN: 0010-437X. DOI: 10.1112/S0010437X1700714X. URL: https://doi-org.proxy.ub.uni-frankfurt.de/10.1112/S0010437X1700714X.

[Pop88] Florian Pop. "Galoissche Kennzeichnung p-adisch abgeschlossener Körper". In: *J. Reine Angew. Math.* 392 (1988), pp. 145–175. ISSN: 0075-4102. DOI: 10.1515/crll.1988.392.145. URL: https://doi-org.proxy.ub.uni-frankfurt.de/10.1515/crll.1988.392.145.

[PS17] Florian Pop and Jakob Stix. "Arithmetic in the fundamental group of a p-adic curve. On the p-adic section conjecture for curves". In: *J. Reine Angew. Math.* 725 (2017), pp. 1–40. ISSN: 0075-4102. DOI: 10.1515/crelle-2014-0077. URL: https://doi-org.proxy.ub.uni-frankfurt.de/10.1515/crelle-2014-0077.

[Roq66] Peter Roquette. "Splitting of algebras by function fields of one variable". In: *Nagoya Math. J.* 27.2 (1966), pp. 625–642.

[Ser02] Jean-Pierre Serre. *Galois cohomology.* English. Springer Monographs in Mathematics. Translated from the French by Patrick Ion and revised by the author. Springer-Verlag, Berlin, 2002, pp. x+210. ISBN: 3-540-42192-0.

[Ser79] Jean-Pierre Serre. *Local fields*. Vol. 67. Graduate Texts in Mathematics. Translated from the French by Marvin Jay Greenberg. Springer-Verlag, New York-Berlin, 1979, pp. viii+241. ISBN: 0-387-90424-7.

[SGA 1] Alexander Grothendieck. *Revêtements étales et groupe fondamental (SGA 1)*. Vol. 224. Lecture notes in mathematics. Springer-Verlag, 1971.

[SGA 4-1] *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos*. Lecture Notes in Mathematics, Vol. 269. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat. Springer-Verlag, Berlin-New York, 1972, pp. xix+525.

[Stacks] The Stacks Project Authors. *Stacks Project*. `http://stacks.math.columbia.edu`.

[Sti10] Jakob Stix. "On the period-index problem in light of the section conjecture". In: *Amer. J. Math.* 132.1 (2010), pp. 157–180. ISSN: 0002-9327. DOI: `10.1353/ajm.0.0092`. URL: `https://doi.org/10.1353/ajm.0.0092`.

[Sti12] Jakob Stix. "On cuspidal sections of algebraic fundamental groups". In: *Galois-Teichmüller theory and arithmetic geometry*. Vol. 63. Adv. Stud. Pure Math. Math. Soc. Japan, Tokyo, 2012, pp. 519–563. DOI: `10.2969/aspm/06310519`. URL: `https://doi-org.proxy.ub.uni-frankfurt.de/10.2969/aspm/06310519`.

[Sti13] Jakob Stix. *Rational points and arithmetic of fundamental groups*. Vol. 2054. Lecture Notes in Mathematics. Evidence for the section conjecture. Springer, Heidelberg, 2013, pp. xx+249. ISBN: 978-3-642-30673-0; 978-3-642-30674-7. DOI: `10.1007/978-3-642-30674-7`.

[Sza09] Tamás Szamuely. *Galois groups and fundamental groups*. Vol. 117. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2009, pp. x+270. ISBN: 978-0-521-88850-9. DOI: `10.1017/CBO9780511627064`.

Zusammenfassung

In dieser Arbeit geht es um die Schnittvermutung für sogenannte Lokalisierungen von Kurven über p-adischen Körpern, was eine Interpolation zwischen der birationalen und der vollen Schnittvermutung darstellt. Wir erklären im Folgenden einige Varianten der Schnittvermutung. In jedem Fall geht es darum, die rationalen Punkte einer Kurve rein gruppentheoretisch mittels ihrer Fundamentalgruppe zu beschreiben. Die Vermutung fügt sich ein in einen weiteren Ideenkreis, die sogenannte *anabelsche Geometrie*, wo allgemeiner untersucht wird, inwieweit arithmetisch-geometrische Information aus zugeordneten Fundamentalgruppen rekonstruiert werden kann. Dieses Gebiet geht auf einen Brief von Grothendieck an Faltings [Gro97] zurück, der weitgehende Vermutungen dieser Art enthält, die zu einem großen Teil bis heute unbewiesen sind.

Die Schnittvermutung

Wir wollen zunächst die Grundideen kurz erläutern.

Rationale Punkte als Schnitte.

Sei k ein Körper. Angenommen, wir suchen Elemente $a_1, \dots, a_n \in k$, die einer Liste von polynomiellen Gleichungen mit Koeffizienten in k genügen sollen:

$$F_j(a_1, \dots, a_n) = 0 \quad \text{für } j = 1, \dots, m \tag{$*$}$$

mit Polynomen $F_1, \dots, F_m \in k[T_1, \dots, T_n]$. Ein Tupel $(a_1, \dots, a_n) \in k^n$ ist mittels der Vorschrift $T_i \mapsto a_i$ äquivalent zu einem Ringhomomorphismus $k[T_1, \dots, T_n] \to k$, der zur Inklusion linksinvers ist. Das Tupel erfüllt die Gleichungen $(*)$ genau dann, wenn der entsprechende Homomorphismus über den Quotienten

$$A \coloneqq k[T_1, \dots, T_n]/(F_1, \dots, F_m)$$

faktorisiert. Mit anderen Worten: eine k-rationale Lösung der Gleichungen $(*)$ ist äquivalent zu einer Retraktion:

$$\begin{tikzcd} A \arrow[d, dashed, bend right] \\ k \arrow[u] \end{tikzcd}$$

Das arithmetische Problem lässt sich in der Sprache der Schemata geometrisch formulieren. Elemente des Körpers k werden als Funktionen auf einem Raum $\operatorname{Spec}(k)$ aufgefasst. Gleichermaßen werden Elemente von A als Funktionen auf $X \coloneqq \operatorname{Spec}(A)$ aufgefasst. Die Abbildung $k \to A$ definiert einen Morphismus $X \to \operatorname{Spec}(k)$ in umgekehrter Richtung. Eine Lösung $(a_1, \ldots, a_n) \in k^n$ der Gleichungen $(*)$ übersetzt sich zunächst wie oben in eine Retraktion und damit unter dem Funktor Spec in einen Schnitt:

$$
\begin{tikzcd}
X \arrow[d] \\
\operatorname{Spec}(k) \arrow[u, dashed, bend right]
\end{tikzcd}
$$

Schnitte auf Fundamentalgruppen.

Gegeben sei eine stetige Abbildung von topologischen Räumen $f\colon X \to B$. Wir fassen dies als eine Familie von Räumen $X_b \coloneqq f^{-1}(b)$ auf, die über den Basisraum B parametrisiert ist. Angenommen, wir suchen für jedes $b \in B$ einen Punkt in der Faser $x_b \in X_b$, der stetig mit b variiert, in dem Sinne, dass die resultierende Abbildung $B \to X$, gegeben durch $b \mapsto x_b$, stetig ist. Mit anderen Worten, wir suchen stetige Schnitte:

$$
\begin{tikzcd}
X \arrow[d] \\
B. \arrow[u, dashed, bend right]
\end{tikzcd}
$$

Eine wichtige Invariante eines topologischen Raums ist seine Fundamentalgruppe. Sie kann benutzt werden, um die vorliegende Situation zu untersuchen. Dafür muss man mit f kompatible Basispunkte $b_0 \in B$ und $x_0 \in f^{-1}(b_0)$ wählen und erhält einen von f induzierten Gruppenhomomorphismus

$$f_*\colon \pi_1(B, b_0) \to \pi_1(X, x_0)$$

Sei nun ein Schnitt $x\colon B \to X$ von f gegeben. Zu jeder Schleife im Basisraum B mit Anfangs- und Endpunkt b_0 haben wir dann mittels x eine Schleife in X mit Anfangs- und Endpunkt $x(b_0)$. Dies definiert den Homomophismus $x_*\colon \pi_1(B, b_0) \to \pi_1(X, x(b_0))$. Wegen der Verschiedenheit der Basispunkte ist x_* nicht direkt ein Schnitt von f_*. Wenn wir aber annehmen, dass die Faser $f^{-1}(b_0)$ wegzusammenhängend ist, dann können wir einen Pfad $\gamma\colon x(b_0) \rightsquigarrow x_0$ in $f^{-1}(b_0)$ wählen und erhalten dadurch einen Homomorphismus

$$s_x\colon \pi_1(B, b_0) \xrightarrow{x_*} \pi_1(X, x(b_0)) \xrightarrow{\gamma(-)\gamma^{-1}} \pi_1(X, x_0).$$

Dieser ist nun in der Tat ein Schnitt von f_*, da der Pfad γ unter f auf den konstanten Pfad in b_0 abgebildet wird. Wählt man anstelle von γ einen anderen Pfad $\gamma'\colon x(b_0) \rightsquigarrow x_0$ in $f^{-1}(b_0)$, dann unterscheiden sich die beiden um eine Schleife $\delta \coloneqq \gamma' \circ \gamma^{-1}$ in $f^{-1}(b_0)$ mit Anfangs- und Endpunkt x_0. In der Definition von s_x wirkt sich dies als eine Konjugation mit $i_*([\delta])$ aus, wobei $i_*\colon \pi_1(f^{-1}(b_0), x_0) \to \pi_1(X, x_0)$ von der Inklusion der Faser induziert ist. Der Schnitt $x\colon B \to X$ liefert somit auf Fundamentalgruppen eine wohldefinierte $\pi_1(f^{-1}(b_0), x_0)$-Konjugationsklasse von Schnitten $[s_x]$. Die Zuordnung $x \mapsto [s_x]$ ist also eine Abbildung

$$\left(\,\text{Schnitte } x\colon B \to X \text{ von } f\,\right) \longrightarrow \begin{pmatrix}\text{Konjugationsklassen von}\\ \text{Schnitten von } f_*\end{pmatrix}.$$

Insbesondere kann man die Nicht-Existenz von Schnitten $x\colon B \to X$ von f dadurch zeigen, dass man die Nicht-Existenz von Schnitten $\pi_1(B, b_0) \to \pi_1(X, x_0)$ von f_* nachweist. Diese letzte Beobachtung gilt auch dann, wenn der Raum X (anstelle der Faser $f^{-1}(b_0)$) wegzusammenhängend ist und $\pi_1(B, b_0)$ abelsch ist.

Um das an einem Beispiel zu illustrieren, betrachten wir etwa die Quadratabbildung $\mathbb{C}^\times \to \mathbb{C}^\times$, $z \mapsto z^2$. Auf Fundamentalgruppen induziert dies die Multiplikation mit 2 auf $\mathbb{Z}$. Dies lässt keinen Schnitt zu, somit kann man schließen, dass es unmöglich ist, jeder von Null verschiedenen komplexen Zahl $z \in \mathbb{C}^\times$ auf stetige Weise eine Quadratwurzel zuzuordnen.

Die étale Fundamentalgruppe.

Wir haben oben erläutert, dass man Lösungen von Polynomgleichungen über einem Körper k geometrisch als Schnitte einer Abbildung $X \to \operatorname{Spec}(k)$ von Schemata auffassen kann. Man ist also in einer ähnlichen Situation wie im vorherigen Abschnitt: X kann als parametrisierte Familie von Räumen über dem Basisraum $\operatorname{Spec}(k)$ angesehen werden. Auf den ersten Blick erscheint diese Sichtweise unergiebig, da der unterliegende topologische Raum von $\operatorname{Spec}(k)$ aus nur einem einzigen Punkt besteht, entsprechend dem Nullideal in k. Aber generell ist der unterliegende Zariski-Raum eines Schemas gänzlich ungeeignet, um die Fundamentalgruppe mittels Homotopieklassen von Schleifen zu definieren. Man verfolgt daher gemäß [SGA 1, Exp. V] einen anderen Ansatz und definiert die étale Fundamentalgruppe eines Schemas stattdessen, indem man die Rolle, die die topologische Fundamentalgruppe in der Überlagerungstheorie spielt, in die Welt der Schemata überträgt.

Um dies zu erklären, betrachten wir ein zusammenhängendes Schema X. Als Basispunkt wählen wir einen geometrischen Punkt x_0 von X, d.h. einen Morphismus $\operatorname{Spec}(\Omega) \to X$ mit einem separabel abgeschlossenen Körper Ω. Die Rolle, die (endliche) Überlagerungen in der Topologie spielen, wird von den endlich étalen Morphismen $f\colon Y \to X$ übernommen. Die Faser $f^{-1}(x_0) \to x_0$ ist eine endliche disjunkte Vereinigung von Kopien von $\operatorname{Spec}(\Omega)$, kann somit einfach als endliche Menge betrachtet werden. In der Topologie würde man (unter schwachen Voraussetzungen an den topologischen Raum X) eine Gruppenwirkung von $\pi_1(X, x_0)$ auf der Faser $f^{-1}(x_0)$ durch das Hochheben von Pfaden definieren: gegeben eine Schleife γ in X mit Anfangs- und Endpunkt x_0, und gegeben einen Punkt $y \in f^{-1}(x_0)$, lässt sich γ auf eindeutige Weise zu einem Pfad $\tilde{\gamma}$ in Y mit Anfangspunkt y hochheben, und man definiert $\gamma.y \in f^{-1}(x_0)$ als den Endpunkt von $\tilde{\gamma}$. In der algebraischen Welt wird ein Element von $\pi_1(X, x_0)$ einfach *definiert* als ein System von Permutationen von $f^{-1}(x_0)$ für jede endlich étale Überlagerung $f\colon Y \to X$. Das System muss einzig eine Natürlichkeitsbedingung erfüllen, d.h. eine Verträglichkeit mit Morphismen zwischen endlich étalen Überlagerungen: Für zwei endlich étale Überlagerungen $f_i\colon Y_i \to X$ und einen Morphismus $g\colon Y_1 \to Y_2$ über X soll für $\gamma \in \pi_1(X, x_0)$ das folgende Quadrat kommutieren:

$$\begin{array}{ccc} f_1^{-1}(x_0) & \xrightarrow{g} & f_2^{-1}(x_0) \\ \big\downarrow{\scriptstyle\gamma} & & \big\downarrow{\scriptstyle\gamma} \\ f_1^{-1}(x_0) & \xrightarrow{g} & f_2^{-1}(x_0). \end{array}$$

Kompakter ausgedrückt: Wir haben eine Kategorie $\mathrm{Cov}(X)$ der endlich étalen Überlagerungen von X; diese ist ausgestattet mit einem Faserfunktor $\mathrm{Fib}_{x_0}\colon \mathrm{Cov}(X) \to \mathrm{FinSet}$ in die Kategorie der endlichen Mengen, gegeben durch $(f\colon Y \to X) \mapsto f^{-1}(x_0)$; und die étale Fundamentalgruppe $\pi_1(X, x_0)$ ist definiert als die Automorphismengruppe dieses Faserfunktors:

$$\pi_1(X, x_0) := \mathrm{Aut}(\mathrm{Fib}_{x_0}).$$

Man versieht $\pi_1(X, x_0)$ mit der gröbsten Topologie, so dass die Wirkungen auf allen Fasern $f^{-1}(x_0)$ stetig sind. Damit wird $\pi_1(X, x_0)$ zu einer proendlichen Gruppe.

Betrachten wir den Spezialfall, dass es sich bei X um das Spektrum eines Körpers handelt, $X = \mathrm{Spec}(k)$. Die Wahl eines separablen Abschlusses $\overline{k}/k$ definiert einen geometrischen Basispunkt $x_0\colon \mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(k)$. Die zusammenhängenden endlich étalen Überlagerungen von $\mathrm{Spec}(k)$ sind von der Form $\mathrm{Spec}(\ell) \to \mathrm{Spec}(k)$ mit einer endlich separablen Körpererweiterung ℓ/k. Die Faser über x_0 lässt sich mit der Menge der k-Einbettungen $\mathrm{Hom}_k(\ell, \overline{k})$ gleichsetzen. Die absolute Galoisgruppe $\mathrm{Gal}(\overline{k}/k)$ wirkt auf natürliche Weise auf $\mathrm{Hom}_k(\ell, \overline{k})$, und man kann sich überlegen, dass jedes kompatible System von Permutationen der Mengen $\mathrm{Hom}_k(\ell, \overline{k})$ durch ein Element von $\mathrm{Gal}(\overline{k}/k)$ definiert ist. Daraus folgt, dass die étale Fundamentalgruppe von $\mathrm{Spec}(k)$ gerade die absolute Galoisgruppe ist:

$$\pi_1(\mathrm{Spec}(k), \mathrm{Spec}(\overline{k})) = \mathrm{Gal}(\overline{k}/k).$$

Insbesondere enthält die étale Fundamentalgruppe von $\mathrm{Spec}(k)$ viel mehr Information als der unterliegende Zariski-Raum vermuten lässt. Dies ist ein Ausdruck der Tatsache, dass die étale Topologie eines Schemas (die keine Topologie im klassischen Sinne ist, sondern eine Grothendieck-Topologie) viel feiner ist als die Zariski-Topologie.

Die Schnittvermutung für eigentliche Kurven.

Sei nun k ein Körper der Charakteristik 0 und sei X/k eine glatte, eigentliche, geometrisch zusammenhängende Kurve, gegeben etwa als Nullstellenmenge homogener Polynome in einem projektiven Raum. Sei $\overline{k}/k$ ein algebraischer Abschluss und sei $\overline{x}_0$ ein geometrischer Punkt in $X_{\overline{k}} := X \otimes_k \overline{k}$. Wir haben dann eine *fundamentale exakte Sequenz* wie folgt:

$$1 \longrightarrow \pi_1(X_{\overline{k}}, \overline{x}_0) \longrightarrow \pi_1(X, \overline{x}_0) \longrightarrow \mathrm{Gal}(\overline{k}/k) \longrightarrow 1. \tag{8.1}$$

Ein k-rationaler Punkt $x \in X(k)$ ist per Definition ein Schnitt der Strukturabbildung $X \to \mathrm{Spec}(k)$ und induziert mittels Funktorialität einen Schnitt auf Fundamentalgruppen

$$s_x \colon \mathrm{Gal}(\overline{k}/k) \to \pi_1(X, \overline{x}_0).$$

Genau wie im topologischen Fall muss man dazu einen (étalen) Pfad $\overline{x} \rightsquigarrow \overline{x}_0$ (mit dem durch x induzierten geometrischen Punkt $\overline{x} \colon \mathrm{Spec}(\overline{k}) \to X_{\overline{k}}$) auf dem zusammenhängenden Schema $X_{\overline{k}}$ wählen, um die Verschiedenheit der Basispunkte zu berücksichtigen. Je zwei solcher Pfade unterscheiden sich um eine Schleife, also ein Element von $\pi_1(X_{\overline{k}}, \overline{x}_0)$. In der Folge ist der Schnitt s_x wohldefiniert bis auf $\pi_1(X_{\overline{k}}, \overline{x}_0)$-Konjugation. In dem Fall, dass der Grundkörper k endlich erzeugt über seinem Primkörper $\mathbb{Q}$ ist, besagt die Grothendiecksche *Schnittvermutung* [Gro97], dass die so erhaltene Abbildung $x \mapsto [s_x]$ eine Bijektion

$$X(k) \longrightarrow \left(\begin{array}{c} \text{Konjugationsklassen von} \\ \text{Schnitten von (8.1)} \end{array} \right) \tag{8.2}$$

ist, vorausgesetzt das Geschlecht von X ist mindestens zwei (d.h. X ist *hyperbolisch*). Sowohl die Schnittvermutung als auch ihr Äquivalent über p-adischen Grundkörpern (die *p-adische Schnittvermutung*) sind noch offen (siehe [Sti13] für Teilresultate und Indizien).

Die Schnittvermutung für offene Kurven

Grothendieck hat auch eine Variante der Schnittvermutung für nicht notwendigerweise eigentliche Kurven aufgestellt. Dem liegt die Beobachtung zugrunde, dass auch k-rationale Punkte "im Unendlichen" (genannt *Spitzen*), d.h. die Punkte in $\overline{X} \setminus X$, wo $\overline{X}$ die glatte Kompaktifizierung von X bezeichnet, zu Schnitten der Fundamentalgruppensequenz (8.1) führen (*Spitzenschnitte*). Die Schnittvermutung für offene Kurven besagt, dass jeder Schnitt $\mathrm{Gal}(\overline{k}/k) \to \pi_1(X, \overline{x}_0)$ von einem k-rationalen Punkt von $\overline{X}$ herkommt, also entweder von einer Spitze oder einem Punkt in X. Für nicht notwendigerweise eigentliche Kurven muss die Hyperbolizitätsbedingung als $\chi(X) < 0$ formuliert werden, d.h. die Eulercharakteristik von X soll negativ sein. Bezeichnet g das Geschlecht von $\overline{X}$ und $r = \#(\overline{X} \setminus X)(\overline{k})$ die geometrische Anzahl der Spitzen, dann ist die Eulercharakteristik durch

$$\chi(X) = 2 - 2g - r. \tag{8.3}$$

gegeben. Die Kurve X ist also genau dann hyperbolisch, wenn es sich bei $X_{\overline{k}}$ entweder um den $\mathbb{P}^1$ ohne mindestens drei Punkte, um eine Kurve vom Geschlecht eins ohne mindestens einen Punkt oder um eine Kurve höheren Geschlechts mit einer beliebigen Zahl herausgenommener Punkte handelt.

Die birationale Schnittvermutung

Die Formel (8.3) für die Eulercharakteristik legt nahe, dass X umso "hyperbolischer" wird, je mehr abgeschlossene Punkte man entfernt. Dies motiviert eine birationale Variante der Schnittvermutung, wo die Kurve vollständig auf ihren generischen Punkt reduziert ist, d.h. auf das Spektrum ihres Funktionenkörpers K. Die fundamentale exakte Sequenz (8.1) wird dann zu der folgenden kurzen exakten Sequenz von absoluten Galoisgruppen:

$$1 \longrightarrow G_{K\overline{k}} \longrightarrow G_K \longrightarrow G_k \longrightarrow 1. \tag{8.4}$$

Jeder rationale Punkt $x \in X(k)$ führt auf folgende Weise zu birationalen Schnitten: Sei $\widetilde{X}$ die Normalisierung von X im algebraischen Abschluss $\overline{K}/K$ und sei $\tilde{x}$ ein gewählter Punkt über x in $\widetilde{X}$. Die G_K-Wirkung auf $\overline{K}$ induziert eine Wirkung auf $\widetilde{X}$. Der Stabilisator $\mathrm{D}_{\tilde{x}|x} \subseteq G_K$ des Punktes $\tilde{x}$ wird die *Zerlegungsgruppe* von $\tilde{x}|x$ genannt. Die Gruppe $\mathrm{D}_{\tilde{x}|x}$ operiert auf dem Restklassenkörper $\kappa(\tilde{x})$, der kanonisch zu $\overline{k}$ isomorph ist. Der resultierende Homomorphismus $\mathrm{D}_{\tilde{x}|x} \to G_k$ ist surjektiv und sein Kern $\mathrm{I}_{\tilde{x}|x}$ ist die *Trägheitsgruppe* von $\tilde{x}|x$. Ein Schnitt $s\colon G_k \to G_K$ wird *Schnitt über* x genannt, falls sein Bild in einer Zerlegungsgruppe $\mathrm{D}_{\tilde{x}|x}$ für ein $\tilde{x}$ über x enthalten ist.

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{I}_{\tilde{x}|x} \arrow[r] \arrow[d, hook] & \mathrm{D}_{\tilde{x}|x} \arrow[r] \arrow[d, hook] & G_k \arrow[r] \arrow[d, equal] \arrow[l, bend right, dashed, "s"'] & 1 \\
1 \arrow[r] & G_{K\overline{k}} \arrow[r] & G_K \arrow[r] & G_k \arrow[r] & 1
\end{tikzcd} \tag{8.5}$$

Es existieren Schnitte über jedem k-rationalen Punkt von x: Sei nämlich $\mathrm{T}^{\circ}_{X,x}$ der Tangentialraum ohne Ursprung von X an x, aufgefasst als k-Schema. Somit ist $\mathrm{T}^{\circ}_{X,x}$ nichtkanonisch isomorph zum Schema $\mathbb{G}_m$ über k. Deligne zeigt in seiner Theorie der *tangentialen Basispunkte* [Del89, §15], dass die obere Zeile des Diagramms (8.5) isomorph zur fundamentalen exakten Sequenz von $\mathrm{T}^{\circ}_{X,x}/k$ ist. Insbesondere induziert jeder von Null verschiedene Tangentialvektor an x

eine $\mathrm{I}_{\tilde{x}|x} \cong \hat{\mathbb{Z}}(1)$-Konjugationsklasse von Schnitten über x, deren Bild in der Zerlegungsgruppe $\mathrm{D}_{\tilde{x}|x}$ enthalten ist.

Die *birationale Schnittvermutung* ist für die eigentliche Kurve X/k erfüllt, wenn jeder Schnitt von (8.4) über genau einem k-rationalen Punkt von X liegt. Über Zahlkörpern ist die Vermutung offen, aber das p-adische Äquivalent über endlichen Erweiterungen von $\mathbb{Q}_p$ wurde von Königsmann mittels Modelltheorie p-adisch abgeschlossener Körper bewiesen [Koe05]. Pop hat hiervon eine "minimalistische" Variante gezeigt, wo G_K durch einen sehr kleinen Quotienten ersetzt wird [Pop10]. Im Fall, dass k die p-ten Einheitswurzeln enthält, genügt es etwa, mit dem $\mathbb{Z}/p\mathbb{Z}$-metabelschen Quotienten von G_K zu arbeiten. Diese letztere Variante wird in dieser Arbeit verallgemeinert.

Die Schnittvermutung für Lokalisierungen von Kurven

Die vorliegende Arbeit beschäftigt sich mit der Schnittvermutung für Lokalisierungen von Kurven. Es geht dabei um Zwischenversionen zwischen der birationalen Schnittvermutung und der Schnittvermutung für die volle Kurve. Sei X/k weiterhin eine glatte, eigentliche, geometrisch zusammenhängende Kurve über einem Körper k der Charakteristik 0.

Definition. Für eine beliebige Menge abgeschlossener Punkte $S \subseteq X_{\mathrm{cl}}$, definieren wir die **Lokalisierung von X bei S** als das pro-(offene Unterschema) von X

$$X_S := \bigcap \big\{ U \subseteq X \text{ dicht offen mit } S \subseteq U \big\}. \tag{8.6}$$

Der Durchschnitt soll im schematheoretischen Sinn in X gebildet werden, d.h. als Faserprodukt von Schemata über X.

Wir zeigen, dass der Limes (8.6) stets existiert. Anschaulich wird X_S aus X gewonnen, indem man alle abgeschlossenen Punkte außerhalb von S entfernt. Der unterliegende topologische Raum $|X_S|$ von X_S ist der Unterraum von $|X|$, der aus dem generischen Punkt η_X und den Punkten in S besteht. Zum Beispiel: für $S = X_{\mathrm{cl}}$ ist $X_S = X$ die volle Kurve; im Fall $S = \emptyset$ ist $X_S = \eta_X$ der generische Punkt. Im Allgemeinen liegt X_S zwischen η_X and X.

Sei $\overline{k}/k$ wieder ein algebraischer Abschluss und $\overline{x}_0$ ein geometrischer Punkt von $X_S \otimes_k \overline{k}$. Sei $X_S^{\mathrm{univ}} \to X_S$ die zugehörige universelle proendlich étale Überlagerung und sei $\widetilde{X} \to X$ die Normalisierung von X im Funktionenkörper von X_S^{univ}. Gegeben einen k-rationalen Punkt $x \in X(k)$ und einen Punkt $\tilde{x}$ in $\widetilde{X}$ über x, haben wir eine Zerlegungsgruppe $\mathrm{D}_{\tilde{x}|x}$ (den Stabilisator von $\tilde{x}$ unter der $\pi_1(X_S, \overline{x}_0)$-Wirkung auf $\widetilde{X}$) und eine Trägheitsgruppe $\mathrm{I}_{\tilde{x}|x}$ wie im birationalen Fall, und wir haben das folgende Diagramm:

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{I}_{\tilde{x}|x} \arrow[r] \arrow[d, hook] & \mathrm{D}_{\tilde{x}|x} \arrow[r] \arrow[d, hook] & \mathrm{Gal}(\overline{k}/k) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \pi_1(X_S \otimes_k \overline{k}, \overline{x}_0) \arrow[r] & \pi_1(X_S, \overline{x}_0) \arrow[r] & \mathrm{Gal}(\overline{k}/k) \arrow[r] & 1.
\end{tikzcd} \tag{8.7}$$

Wieder sagen wir, dass ein Schnitt der Abbildung $\pi_1(X_S, \overline{x}_0) \to \mathrm{Gal}(\overline{k}/k)$ ein *Schnitt über x* sei, falls sein Bild in einer Zerlegungsgruppe $\mathrm{D}_{\tilde{x}|x}$ für ein $\tilde{x}|x$ in $\widetilde{X}$ enthalten ist. Wir zeigen, dass über jedem k-rationalen Punkt von X Schnitte existieren.

Definition. Wir sagen, die Lokalisierung X_S/k **erfüllt die Schnittvermutung**, falls jeder Schnitt $s\colon \mathrm{Gal}(\overline{k}/k) \to \pi_1(X_S, \overline{x}_0)$ über genau einem k-rationalen Punkt von X liegt.

Eines unser Hauptergebnisse besteht darin, dass wir im Fall eines p-adischen Grundkörpers hinreichende Bedingungen an X_S identifizieren, welche die Gültigkeit der Schnittvermutung für X_S implizieren. Wir zeigen auf diese Weise, dass die Schnittvermutung beispielsweise erfüllt ist, wenn S höchstens abzählbar ist (siehe den Abschnitt unten über unsere Hauptergebnisse).

Die hebbare Schnittvermutung

Diese Arbeit beschäftigt sich hauptsächlich mit einer Variante der Schnittvermutung für Lokalisierungen von Kurven, die mit sehr kleinen Quotienten der Fundamentalgruppe arbeitet. Um die Aussage zu formulieren, fixieren wir eine Primzahl p und führen die folgende Notation ein:

Notation. Sei Π eine proendliche Gruppe. Sei $\Pi = \Pi^{(0)} \trianglerighteq \Pi^{(1)} \trianglerighteq \dots$ die **$\mathbb{Z}/p\mathbb{Z}$-abgeleitete Reihe**:

$$\Pi^{(0)} := \Pi, \quad \Pi^{(i+1)} = [\Pi^{(i)}, \Pi^{(i)}]\big(\Pi^{(i)}\big)^p.$$

Wir bezeichnen mit

$$\begin{aligned} \Pi' &:= \Pi/\Pi^{(1)} = \Pi^{\mathrm{ab}} \otimes \mathbb{Z}/p\mathbb{Z}, \\ \Pi'' &:= \Pi/\Pi^{(2)} \end{aligned}$$

den maximalen $\mathbb{Z}/p\mathbb{Z}$-abelschen bzw. $\mathbb{Z}/p\mathbb{Z}$-metabelschen Quotienten von Π.

Man beobachte, dass die Zuordnungen $\Pi \mapsto \Pi'$ und $\Pi \mapsto \Pi''$ funktoriell sind und dass jeder surjektive Homomorphismus proendlicher Gruppen unter $(-)'$ und $(-)''$ surjektiv bleibt.

Definition. Sei $\Pi \twoheadrightarrow G$ ein surjektiver Homomorphismus proendlicher Gruppen. Ein Schnitt $s'\colon G' \to \Pi'$ werde **hebbar** genannt, wenn ein Schnitt $s''\colon G'' \to \Pi''$ existiert, so dass das folgende Diagramm kommutiert:

$$
\begin{tikzcd}
\Pi'' \arrow[r, two heads] \arrow[d, two heads] & G'' \arrow[l, dashed, bend right, "s''"'] \arrow[d, two heads] \\
\Pi' \arrow[r, two heads] & G'. \arrow[l, bend right, "s'"']
\end{tikzcd}
$$

Sei X_S/k die Lokalisierung einer Kurve an einer Menge abgeschlossener Punkte wie im vorherigen Abschnitt. Sei $G_k := \mathrm{Gal}(\overline{k}/k)$ die absolute Galoisgruppe von k und sei $\pi_1(X_S)$ die Fundamentalgruppe von X_S bezüglich eines geometrischen Basispunkts auf $X_S \otimes_k \overline{k}$, so dass wir einen surjektiven Homomorphismus $\pi_1(X_S) \to G_k$ haben. Wir können wieder definieren, wann ein hebbarer Schnitt $s' \colon G_k' \to \pi_1(X_S)'$ über einem k-rationalen Punkt von X liegt, und es existieren hebbare Schnitte über jedem solchen.

Definition. Wir sagen, X_S/k erfüllt die **hebbare Schnittvermutung**, falls jeder hebbare Schnitt $s' \colon G_k' \to \pi_1(X_S)'$ über einem eindeutigen k-rationalen Punkt von X liegt.

Hauptergebnisse

Die hebbare Schnittvermutung für gute Lokalisierungen

Sei k eine endliche Erweiterung von $\mathbb{Q}_p$, welche die p-ten Einheitswurzeln enthält. Unser Ausgangspunkt ist Pops Beweis der birationalen hebbaren Schnittvermutung über k [Pop10, Theorem A]. Das Ziel unserer Arbeit ist, dieses Ergebnis und dessen Beweis auf Lokalisierungen von Kurven zu verallgemeinern und somit einen Schritt vom birationalen Fall in Richtung offener oder eigentlicher Kurven zu gehen, wo die Vermutung noch offen ist. Unser Hauptergebnis ist das Ausmachen von Bedingungen an die Lokalisierung einer Kurve, unter denen sich Pops Beweis verallgemeinern lässt und die hebbare Schnittvermutung demzufolge erfüllt ist. Wir führen zu diesem Zweck den Begriff der **guten Lokalisierung** ein. Gute Lokalisierungen sind durch vier Bedingungen definiert, die grob gesprochen besagen, dass es hinreichend viele invertierbare Funktionen auf X_S gibt. Unser Hauptsatz lautet damit wie folgt:

Satz A. *Sei k eine endliche Erweiterung von $\mathbb{Q}_p$ mit $\mu_p \subseteq k$. Sei X/k eine glatte, eigentliche, geometrisch zusammenhängende Kurve und sei $S \subseteq X_{\mathrm{cl}}$ eine Menge abgeschlossener Punkte. Wenn X_S eine gute Lokalisierung ist, dann erfüllt X_S/k die hebbare Schnittvermutung.*

Um die Nützlichkeit des Satzes zu demonstrieren, weisen wir die Bedingungen für eine gute Lokalisierung in einigen Fällen nach und erhalten so konkrete Beispiele von Lokalisierungen von Kurven, für welche die hebbare Schnittvermutung gilt:

Satz B. *Sei k eine endliche Erweiterung von $\mathbb{Q}_p$ mit $\mu_p \subseteq k$. Sei X/k eine glatte, eigentliche, geometrisch zusammenhängende Kurve und sei $S \subseteq X_{\mathrm{cl}}$ eine Menge abgeschlossener Punkte. Angenommen, es gilt eines der folgenden:*

(a) $S \subseteq X_{\mathrm{cl}}$ ist höchstens abzählbar; oder

(b) X ist über einem Unterkörper $k_0 \subseteq k$ definiert und $S \subseteq X_{\mathrm{cl}}$ enthält nur endlich viele über k_0 algebraische Punkte

Dann erfüllt X_S/k die hebbare Schnittvermutung.

Um den Nachweis für höchstens abzählbare Punktmengen S zu führen, beweisen wir einen neuen Approximationssatz mit Invertierbarkeitsbedingungen für allgemeine Bewertungen.

Die hebbare Schnittvermutung ohne p-te Einheitswurzeln

Falls k eine endliche Erweiterung von $\mathbb{Q}_p$ ist, welche nicht die p-ten Einheitswurzeln enthält, dann ist die hebbare Schnittvermutung über k im Allgemeinen falsch. Es kann jedoch sehr allgemein gezeigt werden, dass die Gültigkeit der hebbaren Schnittvermutung über einer Körpererweiterung ℓ/k eine Variante der hebbaren Schnittvermutung über k impliziert.

Um das genaue Ergebnis zu formulieren, sei k erst einmal ein beliebiger Körper der Charakteristik 0 und sei X_S eine Lokalisierung einer glatten, eigentlichen, geometrisch zusammenhängenden Kurve über k. Sei ℓ/k eine endliche Galoiserweiterung. Bezeichne

$$(X_S \otimes_k \ell)'' \to (X_S \otimes_k \ell)' \to X_S \otimes_k \ell$$

den Anfang des Turms von Überlagerungen, welcher durch die $\mathbb{Z}/p\mathbb{Z}$-abgeleitete Reihe von $\pi_1(X_S \otimes \ell)$ definiert ist. Es handelt sich also um die maximale $\mathbb{Z}/p\mathbb{Z}$-elementar abelsche Überlagerung $(X_S \otimes_k \ell)' \to X_S \otimes_k \ell$ und die maximale $\mathbb{Z}/p\mathbb{Z}$-metabelsche Überlagerung $(X_S \otimes_k \ell)'' \to X_S \otimes_k \ell$. Die entsprechenden Grundkörpererweiterungen seien mit $\ell''/\ell'/\ell$ bezeichnet. Man bemerke, dass die beiden Überlagerungen auch über X_S galoissch sind, da sie charakteristische Überlagerungen der Galoisüberlagerung $X_S \otimes_k \ell \to X_S$ sind. Die beiden Körpererweiterungen ℓ' und ℓ'' sind gleichermaßen galoissch über k.

Definition. Ein Schnitt $s'\colon \mathrm{Gal}(\ell'/k) \to \mathrm{Gal}((X_S \otimes_k \ell)'/X_S)$ heiße **hebbar**, wenn es einen Schnitt s'' gibt, so dass das folgende Diagramm kommutiert:

$$\begin{tikzcd}
\mathrm{Gal}((X_S \otimes_k \ell)''/X_S) \arrow[r, two heads] \arrow[d, two heads] & \mathrm{Gal}(\ell''/k) \arrow[d, two heads] \arrow[l, bend right, dashed, "s''"'] \\
\mathrm{Gal}((X_S \otimes_k \ell)'/X_S) \arrow[r, two heads] & \mathrm{Gal}(\ell'/k) \arrow[l, bend right, "s'"']
\end{tikzcd}$$

Im Fall $\ell = k$ haben wir $\pi_1(X_S)' = \mathrm{Gal}(X'_S/X_S)$ und $G'_k = \mathrm{Gal}(k'/k)$, und diese Definition von hebbaren Schnitten spezialisiert sich zur zuvor formulierten Definition. Wir zeigen, dass für alle ℓ/k auch in diesem allgemeineren Sinne hebbare Schnitte über jedem k-rationalen Punkt von X existieren.

Satz C. *Sei k ein Körper der Charakteristik 0, sei X/k eine glatte, eigentliche, geometrisch zusammenhängende Kurve und sei $S \subseteq X_{\mathrm{cl}}$ eine Menge abgeschlossener Punkte. Sei ℓ/k eine endliche Galoiserweiterung, so dass $X_S \otimes_k \ell$ die hebbare Schnittvermutung erfüllt. Dann existiert für jeden hebbaren Schnitt*

$$s' \colon \mathrm{Gal}(\ell'/k) \to \mathrm{Gal}((X_S \otimes_k \ell)'/X_S)$$

genau ein k-rationaler Punkt x von X, so dass der eingeschränkte Schnitt

$$s'|_{\mathrm{Gal}(\ell'/\ell)} \colon \mathrm{Gal}(\ell'/\ell) \to \mathrm{Gal}((X_S \otimes_k \ell)'/(X_S \otimes_k \ell))$$

über $x \otimes_k \ell$ liegt.

Im Zusammenhang mit diesem Satz untersuchen wir außerdem die Frage, wann schon der uneingeschränkte Schnitt s' über x liegt. Wir geben zwei verschiedene Bedingungen an, unter denen dies der Fall ist. Zum Beispiel zeigen wir mit Methoden der nicht-abelschen Galoiskohomologie, dass schon s' über x liegt, wenn der Grad $[\ell : k]$ nicht durch die Primzahl p teilbar ist.

Die volle Schnittvermutung für Lokalisierungen von Kurven

Die hebbare Schnittvermutung ist insofern interessant, als man Informationen über rationale Punkte schon aus sehr kleinen Quotienten der Fundamentalgruppen gewinnt. Man kann aber sogar auf die Schnittvermutung für die vollen Fundamentalgruppen schließen, wenn die hebbare Variante für alle Überlagerungen erfüllt ist. Im folgenden Satz wird die Gültigkeit der hebbaren Schnittvermutung über gewissen zusammenhängenden endlich étalen Überlagerungen von X_S angenommen. Wie wir zeigen, sind all solche selbst wieder Lokalisierungen von Kurven.

Satz D. *Sei k ein Körper der Charakteristik 0, sei X/k eine glatte, eigentliche, geometrisch zusammenhängende Kurve und sei $S \subseteq X_{\mathrm{cl}}$ eine Menge abgeschlossener Punkte. Angenommen, es existiert eine endliche Galoiserweiterung ℓ/k, so dass für jede geometrisch zusammenhängende endlich étale Überlagerung $W \to X_S$ die hebbare Schnittvermutung für $W \otimes_k \ell$ erfüllt ist. Dann erfüllt X_S die Schnittvermutung, d.h. jeder Schnitt $s \colon G_k \to \pi_1(X_S)$ liegt über einem eindeutigen k-rationalen Punkt von X.*

Durch Anwendung dieses Satzes erhalten wir neue Beispiele von Lokalisierungen von Kurven, welche die Schnittvermutung erfüllen:

Satz E. *Sei k eine endliche Erweiterung von $\mathbb{Q}_p$, sei X/k eine glatte, eigentliche, geometrisch zusammenhängende Kurve und sei $S \subseteq X_{\mathrm{cl}}$ eine Menge abgeschlossener Punkte. Angenommen, es gilt eines der folgenden:*

(a) S ist höchstens abzählbar; oder

(b) X ist über einem Unterkörper $k_0 \subseteq k$ definiert und S enthält alle über k_0 transzendenten und nur endlich viele über k_0 algebraische Punkte.

Dann gilt für X_S die Schnittvermutung.

Lebenslauf

Martin Wolfgang Lüdtke

Kontakt	Institut für Mathematik Johann-Wolfgang-Goethe-Universität Robert-Mayer-Str. 6–8 60325 Frankfurt am Main `luedtke@math.uni-frankfurt.de`
Geboren	28. September 1990 in Bielefeld
Staatsangehörigkeit	Deutsch
Ausbildung	**seit 2015: Goethe-Universität Frankfurt** Promotionsstudium Betreuer: Jakob Stix **2013–2015: Ruprecht-Karls-Universität Heidelberg** Masterstudium, Abschlussnote: *Sehr gut* (1,0) Masterarbeit: Birational Anabelian Geometry of Curves over Algebraically Closed Fields Betreuer: Alexander Schmidt **2012–2013: University of Cambridge** Part III of the Mathematical Tripos (*Honours*) Part III Essay: The Grunwald–Wang Theorem Betreuer: Tom Fisher **2009–2012: Ruprecht-Karls-Universität Heidelberg** Bachelorstudium, Abschlussnote: *Sehr gut* (1,0) Bachelorarbeit: p-adische L-Funktionen und die Leopoldt-Vermutung Betreuer: Kay Wingberg
Veröffentlichungen	• A Birational Anabelian Reconstruction Theorem for Curves over Algebraically Closed Fields in Arbitrary Characteristic *Israel Journal of Mathematics* (2018) DOI: `10.1007/s11856-018-1757-2`

Lehrerfahrung	**Lehrassistenzen an der Goethe-Universität Frankfurt**	
	WS 2019/20	Lineare Algebra (Lehramt)
	SS 2019	Grundlagen der Algebra
	WS 2018/19	Lineare Algebra
	SS 2018	Elementarmathematik 2
	SS 2017	Grundlagen der Algebra
	SS 2017	Seminar: Buch der Beweise
	2015–2017	Lernzentrum
	Tutorien an der Ruprecht-Karls-Universität Heidelberg	
	SS 2015	Funktionalanalysis
	WS 2014/15	Algebraische Geometrie
	SS 2014	Algebra 2
	WS 2013/14	Algebra 1
	SS 2012	Theoretische Informatik
	WS 2010/11	Einführung in die Praktische Informatik
Sprachkenntnisse	Deutsch, Englisch, Spanisch, Französisch	
Stipendien	Studienstiftung des Deutschen Volkes (2010–2015)	