NON-VARYING SUMS OF LYAPUNOV EXPONENTS OF ABELIAN DIFFERENTIALS IN LOW GENUS

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Abstract. We show that for many strata of Abelian differentials in low genus the sum of Lyapunov exponents for the Teichmüller geodesic flow is the same for all Teichmüller curves in that stratum, hence equal to the sum of Lyapunov exponents for the whole stratum. The reason for this behavior is the non-intersection property of Teichmüller curves with various geometrically defined divisors on moduli spaces of curves.

Contents

1. Introduction

Lyapunov exponents of dynamical systems are often hard to calculate explicitly. For the Teichmüller geodesic flow on the moduli space of Abelian differentials at least the *sum* of the positive Lyapunov exponents is accessible for two cases. The moduli space decomposes into various strata, each of which carries a finite invariant measure with full support. For these measures the sum of Lyapunov exponents can be calculated using [EMZ03] together with [EKZ]. On the other hand, the strata contain many Teichmüller curves, e.g. those generated by square-tiled surfaces. For Teichmüller curves an algorithm in [EKZ] calculates the sum of Lyapunov exponents, of course only one Teichmüller curve at a time.

On several occasions, one likes to have estimates, or even the precise values of Lyapunov exponents for all Teichmüller curves in the same stratum simultaneously. For example, it is shown in [DH] that Lyapunov exponents are responsible for the rate of diffusion in the wind-tree model, where the parameters of the obstacle correspond to picking a flat surface in a fixed stratum. One would like to know this escape rate not only for a specific choice of parameters nor for the generic value of parameters but for *all* parameters.

Zorich communicated to the authors, that, based on a limited number of computer experiments about a decade ago, Kontsevich and Zorich observed that the sum of Lyapunov exponents is non-varying among all the Teichmüller curves in a stratum roughly if the genus plus the number of zeros is less than seven, while the sum varies if this sum is greater than seven.

In this paper we show that a more precise version of this numerical observation indeed is true. More precisely, we treat the moduli space of genera less than or equal to five. For each of its strata – with three spin-related exceptions – we either exhibit an example showing that the sum is varying – the easy part – or prove that the sum is non-varying. The latter will be achieved by showing empty intersection of Teichmüller curves with various geometrically defined divisors on moduli spaces of curves. We remark that each stratum requires its own choice of divisor and its individual proof of disjointness, with varying complexity of the argument. In complement to our low genus results we mention a theorem of [EKZ] that shows that for all hyperelliptic loci the sum of Lyapunov exponents is non-varying.

We now give the precise statement of what emerged out of the observation by Kontsevich and Zorich. Let $(m_1, \ldots, m_k)$ be a partition of $2g-2$. Denote by $\Omega\mathcal{M}_g(m_1, \ldots, m_k)$ the stratum parameterizing genus g Riemann surfaces with Abelian differentials that have k distinct zeros of order $m_1, \ldots, m_k$. We say that the sum of Lyapunov exponents is *non-varying* in (a connected component of) a stratum $\Omega\mathcal{M}_g(m_1, \ldots, m_k)$, if for all Teichmüller curves generated by a flat surface in $\Omega\mathcal{M}_g(m_1, \ldots, m_k)$ its sum of Lyapunov exponents equals the sum for the finite invariant measure supported on (the area one hypersurface of) the whole stratum.

Theorem 1.1. *For all strata in genus $g = 3$ but the principal stratum the sum of Lyapunov exponents is non-varying.*

For the principal stratum, the sum of Lyapunov exponents is bounded above by 2. *This bound can be attained for Teichmüller curves in the hyperelliptic locus, e.g., for Teichmüller curves that are unramified double covers of genus two curves and also for Teichmüller curves that do not lie in the hyperelliptic locus.*

Theorem 1.2. *For the strata with signature* $(6)^{\mathrm{even}}$, $(6)^{\mathrm{odd}}$, $(5,1)$, $(3,3)$, $(3,2,1)$ *and* $(2,2,2)^{\mathrm{odd}}$ *as well as for the hyperelliptic strata in genus $g = 4$ the sum of Lyapunov exponents is non-varying.*

For all the remaining strata, except maybe $(4,2)^{\mathrm{odd}}$ *and* $(4,2)^{\mathrm{even}}$, *the sum of Lyapunov exponents is varying and bounded above by* $5/2$.

We give more precise upper bounds for the sum stratum by stratum in the text. We remark that e.g. for $\Omega\mathcal{M}_4(4,1,1)$ the sharp upper bound is $23/10$, which is attained for hyperelliptic curves, whereas for all non-hyperelliptic curves in this stratum the sum of Lyapunov exponents is bounded above by $21/10$. This special role of the hyperelliptic locus is visible throughout the paper.

For $g = 5$, since there are quite a lot of strata, we will not give a full discussion of upper bounds for varying sums, but restrict to the cases where the sum is non-varying.

Theorem 1.3. *For the strata with signature* $(8)^{\mathrm{even}}$, $(8)^{\mathrm{odd}}$ *and* $(5,3)$ *as well as for the hyperelliptic strata in genus $g = 5$ the sum of Lyapunov exponents is non-varying.*

For all the other strata, except maybe $(6,2)^{\mathrm{odd}}$, *the sum of Lyapunov exponents is varying.*

We also expect the three unconfirmed cases $(4,2)^{\mathrm{even}}$, $(4,2)^{\mathrm{odd}}$ and $(6,2)^{\mathrm{odd}}$ to be non-varying, but a proof most likely requires a good understanding of the moduli space of spin curves, on which much less is known than on the moduli space of curves.

The above theorems seem to be the end of this non-varying phenomenon. We cannot claim that there is not a single further stratum of genus greater than five and not hyperelliptic, where the sum is non-varying. But while the sum in strata with a single zero is always non-varying for $g \leq 5$, the sum does *vary* in both non-hyperelliptic components of the stratum $\Omega\mathcal{M}_6(10)$, as we show in Proposition 7.4.

As mentioned above, by [EKZ] for hyperelliptic strata in any genus the sum of Lyapunov exponents is non-varying. This has significance not only in dynamics, but also in the study of birational geometry of moduli spaces. In Theorem 8.1 we mention one application to the extremity of certain divisor classes on the moduli space of pointed curves, which answers a question posed by Harris and Morrison [HM98, Prob. (6.34)].

We now describe our strategy. One can associate three quantities 'slope', 'Siegel-Veech constant' and 'the sum of Lyapunov exponents' to a Teichmüller curve. Any one of the three determines the other two. Hence it suffices to verify the non-varying property for slopes. To do this, we exhibit a geometrically defined divisor on the moduli space of curves and show that Teichmüller curves in a stratum do not intersect this divisor. It implies that those Teichmüller curves have the same slope as that of the divisor.

The slope of the divisors, more generally their divisor classes in the Picard group of the moduli space, can be retrieved from the literature in most cases we need. In the remaining cases, we apply the standard procedure using test curves to calculate the divisor class.

Frequently, we also need to consider the moduli space of curves with marked points or spin structures, but the basic idea remains the same. For upper bounds of sums of Lyapunov exponents, they follow from the non-negative intersection property of Teichmüller curves with various divisors on moduli spaces.

Technically, some of the complications arise from the fact, that the empty intersection of a Teichmüller curve and a divisor is relatively easy to check in the interior of the moduli space, but requires extra care when dealing with stable nodal curves in the boundary.

The results stated above immediately trigger a number of questions. Just to mention the most obvious ones. What about quadratic differentials? What about measures supported on manifolds of intermediate dimension? What about the value distribution for the sums in a stratum where the sum is varying? We hope to treat these questions in a sequel to this paper.

This paper is organized as follows. In Sections 2 and 3 we give a background introduction to moduli spaces and their divisors, as well as to Lyapunov exponents and Teichmüller curves. In Section 4 we study the properties of Teichmüller curves that are needed in the proof. Our main results for $g = 3$, $g = 4$ and $g = 5$ are proved in Sections 5, 6 and 7, respectively. Finally in Section 8 we discuss an application of the Teichmüller curves in the hyperelliptic strata to the geometry of moduli spaces of pointed curves.

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2. Background on moduli spaces

2.1. **Strata of $\Omega\mathcal{M}_g$ and hyperelliptic loci.** Let $\Omega\mathcal{M}_g$ denote the vector bundle of holomorphic one-forms over the moduli space $\mathcal{M}_g$ of genus g curves minus the zero section and let $\mathbb{P}\Omega\mathcal{M}_g$ denote the associated projective bundle. The spaces $\Omega\mathcal{M}_g$ and $\mathbb{P}\Omega\mathcal{M}_g$ are stratified according to the zeros of one-forms. For $m_i \geq 1$

and $\sum_{i=1}^{k} m_i = 2g-2$, let $\Omega\mathcal{M}_g(m_1,\dots,m_k)$ denote the stratum parameterizing one-forms that have k distinct zeros of order $m_1,\dots,m_k$.

Denote by $\overline{\mathcal{M}}_g$ the Deligne-Mumford compactification of $\mathcal{M}_g$. The bundle of holomorphic one-forms extends over $\overline{\mathcal{M}}_g$, parameterizing *stable one-forms* or equivalently sections of the dualizing sheaf. We denote the total space of this extension by $\Omega\overline{\mathcal{M}}_g$.

Points in $\Omega\mathcal{M}_g$, called *flat surfaces*, are usually written as (X,ω) for a one-form ω on X. For a stable curve X, denote the dualizing sheaf by ω_X. We will stick to the notation that points in $\Omega\overline{\mathcal{M}}_g$ are given by a pair (X,ω) with $\omega \in H^0(X,\omega_X)$.

For $d_i \geq -1$ and $\sum_{i=1}^{s} d_i = 4g-4$, let $\mathcal{Q}(d_1,\dots,d_s)$ denote the *moduli space of quadratic differentials* that have s distinct zeros or poles of order $d_1,\dots,d_s$. The condition $d_i \geq -1$ ensures that the quadratic differentials in $\mathcal{Q}(d_1,\dots,d_s)$ have at most simple poles. Namely, $\mathcal{Q}(d_1,\dots,d_s)$ parameterizes pairs (X,q) of a Riemann surface X and a meromorphic section q of $\omega_X^{\otimes 2}$ with the prescribed type of zeros and poles.

If the quadratic differential is not a global square of a one-form, there is a natural double covering $\pi : Y \to X$ such that $\pi^* q = \omega^2$. This covering is ramified precisely at the zeros of odd order of q and at its poles. It gives a map

$$\phi : \mathcal{Q}(d_1,\dots,d_s) \to \Omega\mathcal{M}_g(m_1,\dots,m_k),$$

where the signature $(m_1,\dots,m_k)$ is determined by the ramification type (see [KZ03] for more details).

If the domain and the range of the map ϕ have the same dimension for some signature, we call the image a *component of hyperelliptic flat surfaces* of the corresponding stratum. This can only happen, if the domain of ϕ parameterizes genus zero curves. More generally, if the domain of ϕ parameterizes genus zero curves, we call the image a *locus of hyperelliptic flat surfaces* in the corresponding stratum. These loci are often called hyperelliptic loci, e.g. in [KZ03] and [EKZ]. We prefer to reserve *hyperelliptic locus* for the subset of $\mathcal{M}_g$ (or its closure in $\overline{\mathcal{M}}_g$) parameterizing hyperelliptic curves and thus specify with 'flat surfaces' if we speak of subsets of $\Omega\mathcal{M}_g$.

2.2. **Spin structures and connected components of strata.** A *spin structure* (or *theta characteristic*) on a smooth curve X is a line bundle $\mathcal{L}$ whose square is the canonical bundle, i.e. $\mathcal{L}^{\otimes 2} \sim K_X$. The *parity of a spin structure* is given by $h^0(\mathcal{L}) \mod 2$. This parity is well-known to be deformation invariant. There is a notion of spin structure on a stable curve, extending the smooth case (see [Cor89], also recalled in [Far09]). We only need the following consequence. The *moduli space of spin curves* $\overline{\mathcal{S}}_g$ parameterizes pairs (X,η), where η is a theta characteristic of X. It has two components $\overline{\mathcal{S}}_g^-$ and $\overline{\mathcal{S}}_g^+$ distinguished by the parity of the spin structure. The spin structures on stable curves are defined such that the morphisms $\pi : \overline{\mathcal{S}}_g^- \to \overline{\mathcal{M}}_g$ and $\pi : \overline{\mathcal{S}}_g^+ \to \overline{\mathcal{M}}_g$ are finite of degree $2^{g-1}(2^g-1)$ and $2^{g-1}(2^g+1)$, respectively, cf. loc. cit.

Recall the classification of connected components of strata in $\Omega\mathcal{M}_g$.

Theorem 2.1 ([KZ03])**.** *The strata of $\Omega\mathcal{M}_g$ have up to three connected components, distinguished by the parity of the spin structure and by being hyperelliptic or not. For $g \geq 4$, the strata $\Omega\mathcal{M}_g(2g-2)$ and $\Omega\mathcal{M}_g(2k,2k)$ with an integer $k = (g-1)/2$ have three components, the component of hyperelliptic flat surfaces* *and two components with odd or even parity of the spin structure but not consisting exclusively of hyperelliptic curves.*

The stratum $\Omega\mathcal{M}_3(4)$ has two components, $\Omega\mathcal{M}_3(4)^{\mathrm{hyp}}$ and $\Omega\mathcal{M}_3(4)^{\mathrm{odd}}$. The stratum $\Omega\mathcal{M}_3(2,2)$ also has two components, $\Omega\mathcal{M}_3(2,2)^{\mathrm{hyp}}$ and $\Omega\mathcal{M}_3(2,2)^{\mathrm{odd}}$.

Each stratum $\Omega\mathcal{M}_g(2k_1,\dots,2k_r)$ for $r\geq 3$ or $r=2$ and $k_1\neq (g-1)/2$ has two components determined by even and odd spin structures.

Each stratum $\Omega\mathcal{M}_g(2k-1,2k-1)$ for $k\geq 2$ has two components, the component of hyperelliptic flat surfaces $\Omega\mathcal{M}_g(2k-1,2k-1)^{\mathrm{hyp}}$ and the other component $\Omega\mathcal{M}_g(2k-1,2k-1)^{\mathrm{non-hyp}}$.

In all the other cases, the stratum is connected.

Consider the partition $(2,\dots,2)$. For $(X,\omega)\in\Omega\mathcal{M}_g(2,\dots,2)^{\mathrm{odd}}$ with $\mathrm{div}(\omega)=2\sum_{i=1}^{g-1}p_i$, the line bundle $\eta=\mathcal{O}_X(\sum_{i=1}^{g-1}p_i)$ is an odd theta characteristic. Therefore, we have a natural morphism

$$f:\Omega\mathcal{M}_g(2,\dots,2)^{\mathrm{odd}}\to\overline{\mathcal{S}}_g^-.$$

Note that f contracts the locus where $h^0(\eta)>1$. Similarly one can define such a morphism for even spin structures.

2.3. **Picard groups of moduli spaces.** Let $\mathcal{M}_{g,n}$ be the moduli space of genus g curves with n ordered marked points and let $\mathcal{M}_{g,[n]}$ be the moduli space of genus g curves with n unordered marked points. We write $\mathrm{Pic}(\cdot)$ for the rational Picard group $\mathrm{Pic}_{\mathrm{fun}}(\cdot)_{\mathbb{Q}}$ of a moduli stack (see [HM98] for more details). Since the quantities we are interested in, the sum of Lyapunov exponents and slopes, are invariant under coverings unramified in the interior of $\overline{\mathcal{M}}_g$, this is the group we want to calculate intersections with, not the Picard group of the coarse moduli space.

We fix some standard notation for elements in the Picard group. Let λ denote the first Chern class of the Hodge bundle. Let δ_i, $i=1,\dots,\lfloor g/2\rfloor$ be the boundary divisor of $\overline{\mathcal{M}}_g$ whose generic element is a smooth curve of genus i joined at a node to a smooth curve of genus $g-i$. The generic element of the boundary divisor δ_0 is an irreducible nodal curve of geometric genus $g-1$. In the literature sometimes δ_0 is denoted by δ_{irr}. We write δ for the total boundary class.

For moduli spaces with marked points we denote by ω_{rel} the relative dualizing sheaf of $\overline{\mathcal{M}}_{g,1}\to\overline{\mathcal{M}}_g$ and $\omega_{i,\mathrm{rel}}$ its pullback to $\overline{\mathcal{M}}_{g,n}$ via the map forgetting all but the i-th marked point. For a set $S\subset\{1,\dots,n\}$ we let $\delta_{i;S}$ denote the boundary divisor whose generic element is a smooth curve of genus i joined at a node to a smooth curve of genus $g-i$ and the sections in S lying on the first component.

Theorem 2.2 ([AC87]). *The rational Picard group of $\overline{\mathcal{M}}_g$ is generated by λ and the boundary classes δ_i, $i=0,\dots,\lfloor g/2\rfloor$.*

More generally, the rational Picard group of $\overline{\mathcal{M}}_{g,n}$ is generated by λ, $\omega_{i,\mathrm{rel}}$, $i=1,\dots,n$, by δ_0 and by $\delta_{i;S}$, $i=0,\dots,\lfloor g/2\rfloor$, where $|S|>1$ if $i=0$ and $1\in S$ if $i=g/2$.

Alternatively, we define $\psi_i\in\mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$ to be the class with value $-\pi_*(\sigma_i^2)$ on the universal family $\pi:\mathcal{X}\to C$ with section σ_i corresponding to the i-th marked point. We have the relation

$$\omega_{i,\mathrm{rel}}=\psi_i-\sum_{i\in S}\delta_{0;S}.$$

Consequently, a generating set of $\mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$ can also be formed by the ψ_i, λ and boundary classes.

For a divisor class $D = a\lambda - \sum_{i=0}^{\lfloor g/2 \rfloor} b_i\delta_i$ in $\mathrm{Pic}(\overline{\mathcal{M}}_g)$, define its *slope* to be

$$s(D) = \frac{a}{b_0}.$$

For our purpose the higher boundary divisors need not to be considered, as Teichmüller curves do not intersect δ_i for $i > 0$ (Corollary 4.2).

2.4. **Linear series on curves.** Many divisors on moduli spaces are related to the geometry of *linear series*. Here we review some basic properties of linear series (see [ACGH85] for a comprehensive introduction).

Let X be a genus g curve and $\mathcal{L}$ a line bundle of degree d on X. Denote by $|\mathcal{L}|$ the linear system parameterizing sections of $\mathcal{L}$ mod scalar, i.e.

$$|\mathcal{L}| = \{\mathrm{div}(s) \mid s \in H^0(\mathcal{L})\}.$$

If $H^0(\mathcal{L}) = n$, then $|\mathcal{L}| \cong \mathbb{P}^{n-1}$. For a (projective) r-dimensional linear subspace V of $|\mathcal{L}|$, call $(\mathcal{L}, V)$ a linear series g^r_d. If $\mathcal{L} \sim \mathcal{O}_X(D)$ for a divisor D on X, we also denote by $|\mathcal{O}_X(D)|$ or simply by $|D|$ the linear system.

If all divisors parameterized in a linear series contain a common point p, then p is called a *base point*. Otherwise, this linear series is called *base-point-free*. A base-point-free g^r_d induces a morphism $X \to \mathbb{P}^r$. The divisors in this g^r_d correspond to (the pullback of) hyperplane sections of the image curve. For instance, a hyperelliptic curve admits a g^1_2, i.e. a double cover of $\mathbb{P}^1$. The following fact will be used frequently when we prove the disjointness of Teichmüller curves with a geometrically defined divisor.

Proposition 2.3. *A point p is not a base point of the linear system $|\mathcal{L}|$ if and only if $h^0(\mathcal{L}) - 1 = h^0(\mathcal{L}(-p))$, where $\mathcal{L}(-p) = \mathcal{L} \otimes \mathcal{O}_X(-p)$.*

The canonical linear system is a g^{g-1}_{2g-2}, which induces an embedding to $\mathbb{P}^{g-1}$ for a non-hyperelliptic curve. The image of this embedding is called a *canonical curve*. The following geometric version of the Riemann-Roch theorem is useful for the study of canonical curves. Let D be an effective divisor of degree d on X. Denote by $\overline{\phi_K(D)}$ the linear subspace in $\mathbb{P}^{g-1}$ spanned by the images of points in D under the canonical map ϕ_K.

Theorem 2.4 (Geometric Riemann-Roch)**.** *In the above setting, we have*

$$\dim |D| = d - 1 - \dim \overline{\phi_K(D)}.$$

We will focus on the geometry of low genus canonical curves. Curves of genus 2 are always hyperelliptic. For non-hyperelliptic curves of genus 3, their canonical images correspond to plane quartics.

For $g = 4$, a non-hyperelliptic canonical curve X in $\mathbb{P}^3$ is a complete intersection cut out by a quadric and a cubic. Any divisor $D = p + q + r$ in a g^1_3 of X spans a line in $\mathbb{P}^3$, by Geometric Riemann-Roch. This line intersects X at p, q, r, hence it is contained in the quadric by Bézout. If the quadric is smooth, it is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. It has two families of lines, called two *rulings*. Any line in a ruling intersects X at three points (with multiplicity), hence X has two different linear systems g^1_3 corresponding to the two rulings. If the quadric is singular, then it is a quadric cone with a unique ruling, hence X has a unique g^1_3.

For $g = 5$, a general canonical curve is cut out by three quadric hypersurfaces in $\mathbb{P}^4$ and it does not have any g^1_3. On the other hand, a genus 5 curve with a g^1_3, i.e. a *trigonal curve*, has canonical image contained in a cubic scroll surface. By Geometric Riemann-Roch, divisors in the g^1_3 span lines that sweep out the surface.

Recall that on a nodal curve X, Serre duality and Riemann-Roch hold with the dualizing sheaf ω_X in place of the canonical bundle (see e.g. [HM98] for more details). We also need the following generalized Clifford's theorem for Deligne-Mumford stable curves.

Theorem 2.5 ([Cap])**.** *Let X be a stable curve and D an effective divisor on X with* $\deg(D) \leq 2g-1$*. Then we have*

$$h^0(\mathcal{O}_X(D)) - 1 \leq \deg(D)/2$$

if one of the following conditions holds: (i) X is smooth; (ii) more generally, X has at most two components; (iii) X does not have separating nodes and $\deg(D) \leq 4$*.*

2.5. **Special divisors on moduli spaces.** In the application for Teichmüller curves we do not care about the coefficients of δ_i for $i \geq 1$ in the divisor classes in $\mathrm{Pic}(\overline{\mathcal{M}}_g)$, since Teichmüller curves do not intersect these components (see Corollary 4.2). As shorthand, we use δ_{other} to denote some linear combination of δ_i for $i \geq 1$. Similarly, in $\overline{\mathcal{M}}_{g,n}$ we use δ_{other} to denote some linear combination of all boundary divisors but δ_0. By the same reason we do not distinguish between $\omega_{i,\mathrm{rel}}$ and ψ_i for a divisor class, since they only differ by boundary classes in δ_{other}.

The hyperelliptic locus in $\overline{\mathcal{M}}_3$. Denote by $H \subset \overline{\mathcal{M}}_g$ the closure of locus of genus g hyperelliptic curves. We call H the hyperelliptic locus in $\overline{\mathcal{M}}_g$. Note that H is a divisor if and only if $g = 3$. A stable curve X lies in the boundary of H if there is an *admissible cover* of degree two $\tilde{X} \to \mathbb{P}^1$, for some nodal curve $\tilde{X}$ whose stabilization is X. We refer to [HM98] for the definition of admissible covers.

The class of the hyperelliptic locus $H \subset \overline{\mathcal{M}}_3$ calculated e.g. in [HM98, (3.165)] is given as follows:

$$H = 9\lambda - \delta_0 - 3\delta_1, \tag{1}$$

hence it has slope $s(H) = 9$.

Divisors of Weierstrass points. Let $W \subset \overline{\mathcal{M}}_{g,1}$ be the divisor parameterizing a curve with a Weierstrass point. In [Cuk89], the class of W was calculated for all g, which specializes as follows:

$$W = 6\omega_{\mathrm{rel}} - \lambda - \delta_{\mathrm{other}} \quad \text{for} \quad g = 3. \tag{2}$$

The theta-null divisor. Consider the divisor $\Theta \subset \overline{\mathcal{M}}_{g,1}$ parameterizing (X, p) such that X admits an odd theta characteristic whose support contains p. The class of Θ was calculated in [Far10], which specializes as follows:

$$\Theta = 30\lambda + 60\omega_{\mathrm{rel}} - 4\delta_0 - \delta_{\mathrm{other}} \quad \text{for} \quad g = 4. \tag{3}$$

The Brill-Noether divisors. The Brill-Noether locus BN^r_d in $\overline{\mathcal{M}}_g$ parameterizes curves X that possesses a g^r_d. If the Brill-Noether number

$$\rho(g, r, d) = g - (r+1)(g - d + r) = -1,$$

then BN^r_d is indeed a divisor.

There are pointed versions of this divisor. Let $\underline{w} = (w_1, \ldots, w_n)$ be a tuple of integers. Let $BN^r_{d,\underline{w}}$ be the locus in $\overline{\mathcal{M}}_{g,n}$ of pointed curves $(X, p_1, \ldots, p_n)$ with a line bundle $\mathcal{L}$ of degree d such that $\mathcal{L}$ admits a g^r_d and $h^0(\mathcal{L}(-\sum w_i p_i)) \geq r$. This Brill-Noether locus is a divisor, if the generalized Brill-Noether number

$$\rho(g, r, d, \underline{w}) = g - (r+1)(g - d + r) - r(|\underline{w}| - 1) = -1.$$

The hyperelliptic divisor and the Weierstrass divisor could also be interpreted as Brill-Noether divisors, but we stick to the traditional notation for them.

The class of these pointed divisors has been calculated in many special cases, in particular in [Log03] and later in [Far09]. We collect the results that are needed here and perform the computations in some further cases.

The class of the classical Brill-Noether divisor was calculated in [HM82], in particular

$$BN^1_3 = 8\lambda - \delta_0 - \delta_{\text{other}} \quad \text{for} \quad g = 5. \tag{4}$$

If $|\underline{w}| = g = d$ and $r = 1$ the class of the Brill-Noether divisor was calculated in [Log03, Thm. 5.4]. It has class

$$BN^1_{g,\underline{w}} = -\lambda + \sum_{i=1}^{k} \frac{w_i(w_i+1)}{2}\omega_{i,\text{rel}} - \delta_{\text{other}}.$$

In particular for $\underline{w} = (1, 2)$, it specializes as follows:

$$BN^1_{3,(1,2)} = -\lambda + \omega_{1,\text{rel}} + 3\omega_{2,\text{rel}} - \delta_{\text{other}} \quad \text{for} \quad g = 3, \tag{5}$$

For $\underline{w} = (1, 1, 2)$, it specializes as follows:

$$BN^1_{4,(1,1,2)} = -\lambda + \omega_{1,\text{rel}} + \omega_{2,\text{rel}} + 3\omega_{3,\text{rel}} - \delta_{\text{other}} \quad \text{for} \quad g = 4. \tag{6}$$

If $r = 1$ and $\underline{w} = (2)$, the class of the divisor was also calculated in [Log03]. It specializes to

$$BN^1_{3,(2)} = 4\omega_{\text{rel}} + 8\lambda - \delta_0 - \delta_{\text{other}} \quad \text{for} \quad g = 4. \tag{7}$$

If all $w_i = 1$ and $n = r + 1$ the Brill-Noether divisor specializes to the divisor Lin calculated in [Far09, Sec. 4.2]. In particular [Far09, Thm. 4.6] gives

$$\text{Lin}^1_3 = BN^1_{3,(1,1)} = -\omega_{1,\text{rel}} - \omega_{2,\text{rel}} + 8\lambda - \delta_0 - \delta_{\text{other}} \quad \text{for} \quad g = 4. \tag{8}$$

Generalizing the calculation of Logan for $r = 1$ and $n = 1$ to arbitrary weight w_1, one obtains the divisor called $\text{Nfold}^1_d(1)$ in the proof of [Far09, Thm. 4.9]. From the proof one deduces

$$\text{Nfold}^1_4(1) = BN^1_{4,(3)} = 7\lambda + 15\omega_{\text{rel}} - \delta_0 - \delta_{\text{other}} \quad \text{for} \quad g = 5. \tag{9}$$

Nfold(1) is a degeneration of the divisor Nfold in [Far09]. A partial degeneration is $\text{Nfold}(2) = BN^1_{4,(1,2)}$ in $\overline{\mathcal{M}}_{5,2}$. We will compute its class below to show

$$\text{Nfold}^1_4(2) = BN^1_{4,(1,2)} = 7\lambda + 7\omega_{1,\text{rel}} + 2\omega_{2,\text{rel}} - \delta_0 - \delta_{\text{other}} \quad \text{for} \quad g = 5. \tag{10}$$

We will illustrate a useful method of *test curves* for calculating the class of a divisor (see e.g. [HM98] for more examples). For instance, using certain test curves including a Teichmüller curve we can determine (partially, but sufficiently for our purposes) the class of Lin^1_3. Given that this class was already determined in (8), the following proposition is not strictly necessary but serves as an illustration of the method. The reader who is familiar with this may skip over it and proceed to the next section.

Proposition 2.6. *The class of* Lin_3^1 *equals*

$$\mathrm{Lin}_3^1 = k(-\omega_{1,\mathrm{rel}} - \omega_{2,\mathrm{rel}} + 8\lambda - \delta_0 - \delta_{\mathrm{other}})$$

for some constant k.

Proof. Suppose that

$$\mathrm{Lin}_3^1 = a_1\omega_{1,\mathrm{rel}} + a_2\omega_{2,\mathrm{rel}} + b\lambda - c\delta_0 - \delta_{\mathrm{other}}$$

for some unknown coefficients a_1, a_2, b, c. By symmetry we have $a_1 = a_2$.

As the first test curve, let B be a general pencil of curves of class $(3,3)$ on a smooth quadric Q in $\mathbb{P}^3$. There are 18 base points in the pencil. Choose two of them as the marked points p, q. Since B is general, p, q are not contained in any ruling of Q, namely there is no section of a linear series g_3^1 that contains both p and q. It implies that B and Lin_3^1 are disjoint in $\overline{\mathcal{M}}_{4,2}$, i.e.

$$B \cdot \mathrm{Lin}_3^1 = 0.$$

Blowing up the 18 base points, we obtain a surface $S \subset \mathbb{P}^1 \times Q$, which is a one-parameter family of genus 4 curves over $B \cong \mathbb{P}^1$. Let $\chi_{\mathrm{top}}(\cdot)$ denote the topological Euler characteristic. We have

$$\chi_{\mathrm{top}}(S) = \chi_{\mathrm{top}}(\mathbb{P}^1) \cdot \chi_{\mathrm{top}}(X) + \text{ the number of nodal fibers,}$$

where X has genus equal to 4. We know that

$$\chi_{\mathrm{top}}(\mathbb{P}^1) = -2, \quad \chi_{\mathrm{top}}(X) = -6,$$

$$\chi_{\mathrm{top}}(S) = \chi_{\mathrm{top}}(Q) + 18 = \chi_{\mathrm{top}}(\mathbb{P}^1) \times \chi_{\mathrm{top}}(\mathbb{P}^1) + 18 = 22.$$

All together it implies there are 34 irreducible nodal curves in the family B, namely,

$$B \cdot \delta_0 = 34.$$

Let $\omega_{S/B}$ denote the relative dualizing sheaf of S over B. Since S has class $(1;3,3)$ in $\mathbb{P}^1 \times Q$, one checks that

$$\begin{aligned}\omega_{S/B} &= \omega_S - \pi^*\omega_B \\ &= (\omega_{\mathbb{P}^1\times Q} + S)|_S - \pi^*\omega_B \\ &= (-2;-2,-2) + (1;3,3) - (-2;0,0) \\ &= (1;1,1)\end{aligned}$$

on S, where $\pi : S \to B$ is the projection. Then $\pi_*(c_1^2(\omega_{S/B}))$ on B is equal to the top intersection

$$(1;1,1) \cdot (1;1,1) \cdot (1;3,3) = 14$$

on $\mathbb{P}^1 \times Q$. Using the Noether formula $12\lambda = \pi_*(c_1^2(\omega_{S/B})) + \delta$, we get

$$B \cdot \lambda = \frac{1}{12}(34 + 14) = 4.$$

Moreover, let Γ_p and Γ_q be the exceptional curves corresponding to the blow-up of the two marked points. For $i = 1, 2$ we have $B \cdot \omega_{i,\mathrm{rel}} = -1$, since this is the self-intersection of Γ_p (resp. Γ_q). Note that B does not intersect any boundary divisors except δ_0. Putting the above intersection numbers together, we obtain a relation

$$-a_1 + 2b + 17c = 0.$$

As the second test curve, we take a Teichmüller curve C generated by a flat surface in the stratum $\Omega\mathcal{M}_4(3,3)^{\mathrm{non-hyp}}$, e.g. the square-tiled surface given by the permutations ($\pi_r = (123456789\,10), \pi_u = (19568)$). Using the algorithm in [EKZ] along with (12), (13) in Section 3.4, we find for this particular curve that the sum of Lyapunov exponents equals $L(C) = 2$, i.e. the slope $s(C) = 33/4$. Using Proposition 4.5 with $\kappa = 5/8$, we obtain another relation

$$-a_1 + 4b + 33c = 0.$$

The two relations imply that

$$b = -8c,\ a_1 = c$$

and this concludes the proof. □

In the following proof we use properties of sums of Lyapunov exponents and Teichmüller curves that are explained later. The reader is encouraged to verify that we do not use a circular reasoning.

Proof of Equation (10). Using the same logic in the proof of [Far09, Thm. 4.6, 4.9], $\lambda, \delta_0, \psi_1$ have non-varying coefficients in Nfold_4^1, which is $BN^1_{4,(1,1,1)}$ in our notation, and in $\mathrm{Nfold}_4^1(2)$. Hence we have

$$\mathrm{Nfold}_4^1(2) = 7\lambda - \delta_0 + 2\psi_1 + c\psi_2 - \delta_{\mathrm{other}}.$$

Now we need to determine c. Take a Teichmüller curve generated by a square-tiled surface in the stratum $\Omega\mathcal{M}_5(5,3)$, e.g. the one generated by

$$(\pi_r = (123456789\,10\,11), \pi_u = (1359687)).$$

Using Theorem 3.2 one calculates that for this Teichmüller curve C we have $L(C) = 9/4$. Together with Proposition 4.5 we can solve for c and obtain that $c = 7$. □

Remark 2.7. As cross-check and to give a proof of Equation (10) without using an explicit Teichmüller curve we perform a classical test curve calculation. Assume that

$$\mathrm{Nfold}_4^1(2) = 7\lambda - \delta_0 + 2\psi_1 + c\psi_2 - e\delta_{0;\{1,2\}} - \delta_{\mathrm{other}}.$$

We have to take $\delta_{0;\{1,2\}}$ into account, for the test curves used below intersect $\delta_{0;\{1,2\}}$. Let X be a general genus 5 curve. Take a fixed general point x_2 on X and move another point x_1 along C. Call this family B_1. We have

$$B_1 \cdot \lambda = 0,\ B_1 \cdot \delta_0 = 0,\ B_1 \cdot \psi_1 = 9,\ B_1 \cdot \psi_2 = 1,\ B_1 \cdot \delta_{0;\{1,2\}} = 1.$$

The intersection number $B_1 \cdot \mathrm{Nfold}_4^1(2)$ was settled in [Log03, Prop. 3.4] by setting $a_1 = 2, a_2 = 1, g = 5, h = 1$, hence it equals 10. Note that Logan counts the number of pairs (p_2, q_1), which equals 5, but for our purpose x_1 can be either p_2 or q_1, so we double the counting. We obtain a relation

$$c - e + 8 = 0.$$

Now fix a general point x_1 and move another point x_2 along X. Call this family B_2. We have

$$B_2 \cdot \lambda = 0,\ B_2 \cdot \psi_1 = 1,\ B_2 \cdot \psi_2 = 9,\ B_2 \cdot \delta_{0;\{1,2\}} = 1.$$

The intersection number $B_2 \cdot \mathrm{Nfold}(2)$ was calculated in [Log03, Prop. 3.4] by setting $a_1 = 1, a_2 = 2, g = 5, h = 1$, hence it equals 50. This equals Logan's counting, since in the pair (p_2, q_1) now p_2 has weight 2, which distinguishes it from q_1. We obtain another relation

$$9c - e - 48 = 0.$$

Combining the two relations we obtain that $c = 7, e = 15$, which completes the cross-check.

Gieseker-Petri divisors. Consider a linear series $(\mathcal{L}, V) \in G^r_d(X)$ for a linear subspace $V \subset H^0(\mathcal{L})$ of dimension $r+1$, $\deg(\mathcal{L}) = d$ and the multiplication map

$$\mu : V \otimes H^0(\omega_X \otimes \mathcal{L}^{-1}) \to H^0(\omega_X).$$

Define the Gieseker-Petri locus

$$GP^r_{g,d} = \{[X] \in \overline{\mathcal{M}}_g, \exists \,\text{base-point-free}\, (\mathcal{L}, V) \in G^r_d(X) \,\text{such that}\, \mu \,\text{is not injective}\}.$$

The divisor class of the Gieseker-Petri locus was calculated in [EH87]. It specializes to

$$GP = 17\lambda - 2\delta_0 + \delta_{\text{other}} \quad \text{for} \quad g = 4. \tag{11}$$

Alternatively, one can describe $GP^1_{4,3}$ in $\mathcal{M}_4$ as follows. The canonical image of a genus 4 non-hyperelliptic curve is contained in a quadric surface in $\mathbb{P}^3$. Then $GP^1_{4,3}$ is the closure of the locus where this quadric is singular.

3. Background on Lyapunov exponents and Teichmüller curves

3.1. **Lyapunov exponents.** Fix an $\mathrm{SL}_2(\mathbb{R})$-invariant, ergodic measure μ on $\Omega\mathcal{M}_g$. The Lyapunov exponents for the Teichmüller geodesic flow on $\Omega\mathcal{M}_g$ measure the logarithm of the growth rate of the Hodge norm of cohomology classes during parallel transport along the geodesic flow. More precisely, let V be the restriction of the real Hodge bundle (i.e. the bundle with fibers $H^1(X, \mathbb{R})$) to the support M of μ. Let S_t be the lift of the geodesic flow to V via the Gauss-Manin connection. Then Oseledec's theorem shows the existence of a filtration

$$V = V_{\lambda_1} \supset \cdots \supset V_{\lambda_k} \supset 0$$

by measurable vector subbundles with the property that, for almost all $m \in M$ and all $v \in V_m \setminus \{0\}$, one has

$$||S_t(v)|| = \exp(\lambda_i t + o(t)),$$

where i is the maximum value such that v is in the fiber of V_i over m, i.e. $v \in (V_i)_m$. The numbers λ_i for $i = 1, \ldots, k \leq \mathrm{rank}(V)$ are called the *Lyapunov exponents of* S_t. Note that these exponents are unchanged if we replace the support of μ by a finite unramified covering with a lift of the flow and the pullback of V. We adopt the convention to repeat the exponents according to the rank of V_i/V_{i+1} such that we will always have $2g$ of them, possibly some of them equal. Since V is symplectic, the spectrum is symmetric, i.e. $\lambda_{g+k} = -\lambda_{g-k+1}$. The reader may consult [For06] or [Zor06] for a more detailed introduction to this subject.

Most of our results will be about the *sum* of Lyapunov exponents defined as

$$L = \sum_{i=1}^{g} \lambda_i.$$

3.2. **Teichmüller curves as fibered surfaces.** A *Teichmüller curve* $C \to \mathcal{M}_g$ is an algebraic curve in the moduli space of curves that is totally geodesic with respect to the Teichmüller metric. There exists a finite unramified cover $B \to C$ such that the monodromies around the punctures of B are unipotent and such that the universal family over some level covering of $\mathcal{M}_g$ pulls back to a family of curves $f : \mathcal{X} \to B$. We denote by $f : \overline{\mathcal{X}} \to \bar{B}$ a relatively minimal semistable model of a fibered surface of fiber genus g with smooth total space. Let $\Delta \subset \bar{B}$ be the set of points with singular fibers, hence $B = \bar{B} \setminus \Delta$. See e.g. [Möl06] for more on this setup. By a further finite unramified covering (outside Δ) we may suppose that the zeros of ω on X extend to sections σ_i of f. We denote by $S_i \subset \overline{\mathcal{X}}$ the images of these sections.

Teichmüller curves are generated by flat surfaces or half-translation surfaces. Aiming to understand $\Omega\mathcal{M}_g$ we focus here exclusively on the first kind. We denote by (X, ω) a *generating flat surface*. Teichmüller curves come with a uniformization $C = \mathbb{H}/\mathrm{SL}(X, \omega)$, where $\mathrm{SL}(X, \omega)$ is the *affine group* (or *Veech group*) of the flat surface (X, ω). Let $K = \mathbb{Q}(\mathrm{tr}(\gamma), \gamma \in \mathrm{SL}(X, \omega))$ denote the trace field of the affine group and let $L/\mathbb{Q}$ denote the Galois closure of $K/\mathbb{Q}$.

The variation of Hodge structure (VHS) over a Teichmüller curve decomposes into sub-VHS

$$R^1 f_* \mathbb{C} = (\oplus_{\sigma \in \mathrm{Gal}(L/\mathbb{Q})/\mathrm{Gal}(K/\mathbb{Q})} \mathbb{L}^\sigma) \oplus \mathbb{M},$$

where $\mathbb{L}$ is the VHS with the standard 'affine group' representation, $\mathbb{L}^\sigma$ are the Galois conjugates and $\mathbb{M}$ is just some representation ([Möl06, Prop. 2.4]). One of the purposes of our work is to shed some light on what possibilities for the numerical data of $\mathbb{M}$ can occur.

The bridge between the 'dynamical' definition of Lyapunov exponents and the 'algebraic' method applied in the sequel is given by the following result. Note that if the VHS splits into direct summands one can apply Oseledec's theorem to the summands individually. The full set of Lyapunov exponents is the union (with multiplicity) of the Lyapunov exponents of the summands.

Theorem 3.1 ([Kon97], [KZ], [BM10])**.** *If the VHS over the Teichmüller curve contains a sub-VHS $\mathbb{W}$ of rank $2k$, then the sum of the k corresponding non-negative Lyapunov exponents equals*

$$\sum_{i=1}^{k} \lambda_i^{\mathbb{W}} = \frac{2 \deg \mathbb{W}^{(1,0)}}{2g(\bar{B}) - 2 + |\Delta|},$$

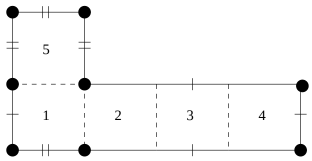
where $\mathbb{W}^{(1,0)}$ is the $(1,0)$-part of the Hodge-filtration of the vector bundle associated with $\mathbb{W}$. In particular, we have

$$\sum_{i=1}^{g} \lambda_i = \frac{2 \deg f_* \omega_{\overline{\mathcal{X}}/\bar{B}}}{2g(\bar{B}) - 2 + |\Delta|}.$$

3.3. **Square-tiled surfaces.** A *square-tiled surface* is a flat surface (X, ω), where X is obtained as a covering of a torus ramified over one point only and ω is the pullback of the holomorphic one-form on the torus. It is well-known that in this case $\mathrm{SL}(X, \omega)$ is commensurable to $\mathrm{SL}_2(\mathbb{Z})$, hence $\mathbb{L}$ has no Galois conjugates or equivalently, the rank of $\mathbb{M}$ is $2g - 2$.

In order to specify a square-tiled surface, with say d squares, it suffices to specify the monodromy of the covering. This monodromy is given by two permutations (π_u, π_r) on d letters, corresponding to going up and going to the right, respectively.

The surface below corresponds to a degree 5, genus 2, branched cover of the standard torus with a unique ramification point marked by $\bullet$:



It is easy to see that the monodromy permutations for this square-tiled surface are given by $(\pi_u = (15)(2)(3)(4), \pi_r = (1234)(5))$.

3.4. **Siegel-Veech constants, slopes and the sum of Lyapunov exponents.** Write $\mu = (m_1, \ldots, m_k)$ for a partition of $2g-2$. Let c_μ denote the *Siegel-Veech (area) constant* of (the connected component of) the stratum $\Omega\mathcal{M}_g(\mu)$. Roughly speaking, c_μ measures the average number of weighted horizontal cylinders on flat surfaces (X, ω) in $\Omega\mathcal{M}_g(\mu)$. The weight for each horizontal cylinder is given by its height/length. Similarly, one can define the *Siegel-Veech constant* $c(C)$ for a Teichmüller curve C, or more generally for any $\mathrm{SL}_2(\mathbb{R})$-invariant suborbifold in $\Omega\mathcal{M}_g(\mu)$. See [EKZ] and [EMZ03] for a comprehensive introduction to Siegel-Veech constants.

Let κ_μ be a constant

$$\kappa_\mu = \frac{1}{12}\sum_{i=1}^{k}\frac{m_i(m_i+2)}{m_i+1}$$

determined by the signature of the stratum. The Siegel-Veech constant and the sum of Lyapunov exponents are related as follows.

Theorem 3.2 ([EKZ]). *For any $\mathrm{SL}_2(\mathbb{R})$-invariant finite measure m on the stratum $\Omega\mathcal{M}_g(\mu)$ we have*

$$L(m) = \kappa_\mu + c(m). \tag{12}$$

In particular this holds for the measure with support equal to a connected component of $\Omega\mathcal{M}_g(\mu)$ and for the measure supported on a Teichmüller curve.

For any given Teichmüller curve at a time this theorem allows to calculate the sum of Lyapunov exponents. It suffices to calculate the cusps (in practice, e.g. for a square-tiled surface, this amounts to calculating the Veech group) and to evaluate the Siegel-Veech (area) contribution of the cusp.

Let $s(C)$ be the *slope* of a Teichmüller curve C defined by

$$s(C) = \frac{C\cdot\delta}{C\cdot\lambda}.$$

Recall the slope of a divisor defined in Section 2.3. These two slope definitions are dual to each other via the natural intersection pairing between a Teichmüller curve and a divisor on $\overline{\mathcal{M}}_g$.

Given a Teichmüller curve C, one has to understand *only one* of the quantities $L(C)$, $c(C)$ and $s(C)$, because of the relation in Theorem 3.2 and since they are related by

$$s(C) = \frac{12c(C)}{L(C)} = 12 - \frac{12\kappa_\mu}{L(C)}. \tag{13}$$

This is a consequence of the Noether formula

$$12\lambda = \delta + f_*(c_1^2(\omega_{\mathcal{X}/C})),$$

as shown in [Che10b, Thm. 1.8].

Let us briefly explain how it works. By Theorem 3.1 we know $\chi \cdot L(C) = 2\deg\lambda$, where $\chi = 2g(C) - 2 + |\Delta|$. Using the Noether formula, the class of $\omega_{\mathcal{X}/C}$ in the proof of Proposition 4.5 and Theorem 3.2, we can derive that $6\chi \cdot c(C) = \deg\delta$. Hence the relation (13) follows immediately.

Suppose that *all* Teichmüller curves C in a stratum $\Omega\mathcal{M}_g(\mu)$ are disjoint from a divisor D in $\overline{\mathcal{M}}_g$. Then $C \cdot D = 0$ for all these Teichmüller curves, hence $s(C) = s(D)$. Since the slopes are non-varying, so are the sums of Lyapunov exponents and the Siegel-Veech constants for these Teichmüller curves. In general, we also need to consider the moduli spaces of curves with marked points or spin structures. But the idea still relies on the non-intersection property of the Teichmüller curves with certain divisors on those moduli spaces.

If L is non-varying for all Teichmüller curves (or just those generated by square-tiled surfaces) in a stratum $\Omega\mathcal{M}_g(\mu)$, it implies that the sum of Lyapunov exponents for the whole stratum is equal to L.

Proposition 3.3. *As the area (i.e. the degree of the torus coverings) approaches infinity, the limit of sums of Lyapunov exponents for Teichmüller curves generated by square-tiled surfaces in a stratum is equal to the sum of Lyapunov exponents for that stratum.*

This is due to the fact that square-tiled surfaces in a stratum are parameterized by 'lattice points' under the period map coordinates. Hence, their asymptotic behavior reveals information for the whole stratum. See e.g. [Che10b, App. A] for a proof.

3.5. **Lyapunov exponents for loci of hyperelliptic flat surfaces.** We recall a result of [EKZ] that deals with the sum of Lyapunov exponents for Teichmüller curves generated by hyperelliptic curves, more generally for any invariant measure on loci of hyperelliptic flat surfaces. It implies immediately that hyperelliptic strata are non-varying.

Theorem 3.4 ([EKZ])**.** *Suppose that M is a regular $\mathrm{SL}_2(\mathbb{R})$-invariant suborbifold in a locus of hyperelliptic flat surfaces of some stratum $\Omega\mathcal{M}_g(m_1, \ldots, m_k)$. Denote by $(d_1, \ldots, d_s)$ the orders of singularities of the underlying quadratic differentials on the quotient projective line.*

Then the sum of Lyapunov exponents for M is

$$L(M) \;=\; \frac{1}{4} \cdot \sum_{\substack{j \text{ such that} \\ d_j \text{ is odd}}} \frac{1}{d_j + 2}.$$

where, as usual, we associate the order $d_i = -1$ to simple poles.

Corollary 3.5. *Hyperelliptic strata are non-varying. For a Teichmüller curve C generated by (X,ω) we have*

$$
\begin{aligned}
L(C) &= \frac{g^2}{2g-1} \quad \textit{and} \quad s(C) = 8 + \frac{4}{g} \quad \textit{if} \quad (X,\omega) \in \Omega\mathcal{M}_g^{\mathrm{hyp}}(2g-2), \\
L(C) &= \frac{g+1}{2} \quad \textit{and} \quad s(C) = 8 + \frac{4}{g} \quad \textit{if} \quad (X,\omega) \in \Omega\mathcal{M}_g^{\mathrm{hyp}}(g-1,g-1).
\end{aligned}
\tag{14}
$$

4. Properties of Teichmüller curves

Here we collect the properties of Teichmüller curves that are needed in the proofs in the subsequent sections.

Proposition 4.1. *Suppose that C is a Teichmüller curve generated by an Abelian differential in $\Omega\mathcal{M}_g(\mu)$ and let μ' be a degeneration of the signature μ. Then the canonical lift of C to $\mathbb{P}\Omega\overline{\mathcal{M}}_g(\mu)$ is disjoint from $\mathbb{P}\Omega\overline{\mathcal{M}}_g(\mu')$.*

Proof. The claim is obvious over the interior of the moduli space. We only need to check the disjointness over the boundary. The cusps of Teichmüller curves are obtained by applying the Teichmüller geodesic flow $\mathrm{diag}(e^t, e^{-t})$ to the direction of the flat surface (X,ω) in which (X,ω) decomposes completely into cylinders. The stable surface at the cusp is obtained by 'squeezing' the core curves of these cylinders. This follows from the explicit description in [Mas75]. Since the zeros of ω are located away from the core curves of the cylinders, the claim follows. □

Corollary 4.2. *The section ω of the canonical bundle of each smooth fiber over a Teichmüller curve extends to a section ω_∞ for each singular fiber X_∞ over the closure of a Teichmüller curve. The signature of zeros of ω_∞ is the same as that of ω. Moreover, X_∞ does not have separating nodes.*

Proof. The first statement follows from the description in the preceding proof. The second statement is a consequence of the topological fact that a core curve of a cylinder can never disconnect a flat surface. In particular it implies that Teichmüller curves do not intersect the boundary divisors δ_i for $i>0$ on $\overline{\mathcal{M}}_g$, because a curve parameterized in δ_i for $i>0$ possesses at least one separating node. □

An immediate case-by-case study implies the following.

Corollary 4.3. *For Teichmüller curves generated by a flat surface in $\Omega\mathcal{M}_g(2g-2)$ the degenerate fibers are irreducible.*

For Teichmüller curves generated by a flat surface in $\Omega\mathcal{M}_g(k_1,k_2)$, with $k_1 \geq k_2$ both odd, the degenerate fibers are irreducible or consist of two components joined at n nodes for an odd number n and $1 < n \leq 2k_2 - 1$.

Proposition 4.4. *Let C be a Teichmüller curve generated by an Abelian differential (X,ω) in $\Omega\mathcal{M}_g(\mu)$. Suppose that an irreducible degenerate fiber X_∞ over a cusp of C is hyperelliptic. Then X is hyperelliptic, hence the whole Teichmüller curve lies in the locus of hyperelliptic flat surfaces.*

Moreover, if $\mu \in \{(4),(3,1),(6),(5,1),(3,3),(3,2,1),(8),(5,3)\}$ and (X,ω) is not hyperelliptic, then no degenerate fiber of the Teichmüller curve is hyperelliptic.

The last conclusion does not hold for all strata. For instance, Teichmüller curves generated by a non-hyperelliptic flat surface in the stratum $\Omega\mathcal{M}_3(2,1,1)$ always intersect the hyperelliptic locus at the boundary, as we will see later in the discussion for that stratum.

As motivation for the proof, recall why a Teichmüller curve generated by (X, ω) with X hyperelliptic stays within the corresponding locus of hyperelliptic flat surfaces. The hyperelliptic involution acts as (-1) on all one-forms, hence on ω. In the flat coordinates of X given by $\mathrm{Re}(\omega)$ and $\mathrm{Im}(\omega)$, the hyperelliptic involution acts by the matrix $-\mathrm{Id}$. The Teichmüller curve is the $\mathrm{SL}(2,\mathbb{R})$-orbit of (X, ω) and $-\mathrm{Id}$ is in the center of $\mathrm{SL}(2,\mathbb{R})$. So if (X, ω) admits a hyperelliptic involution, so does $A \cdot (X, \omega)$ for any $A \in \mathrm{SL}(2,\mathbb{R})$.

Proof. Suppose that the stable model X_∞ of the degenerate fiber is irreducible of geometric genus h with $(g-h)$ pairs of points (p_i, q_i) identified. A semi-stable model of X_∞ admits a degree two admissible cover of the projective line if and only if the normalization X_n of X_∞ is branched at $2h+2$ branch points over a main component with covering group generated by an involution ϕ and, moreover, for each of the $2(g-h)$ nodes there is a projective line intersecting X_n in p_i and $q_i = \phi(p_i)$ with two branch points.

In the flat coordinates of X_n given by ω, the surface consists of a compact surface X_0 with boundary of genus h and $2(g-h)$ half-infinite cylinders (corresponding to the nodes) attached to the boundary of X_0. We may define X_0 canonically, by sweeping out the half-infinite cylinder at p_i (or q_i) with lines of slope equal to the residue (considered as element in $\mathbb{R}^2$) of ω at p_i until such a line hits a zero of ω, i.e. a singularity of the flat structure.

With this normalization, the above discussion shows that for irreducible stable curves the hyperelliptic involution exchanges the half-infinite cylinders corresponding to p_i and q_i and it defines an involution ϕ of X_0. As in the smooth case, ϕ acts as $-\mathrm{Id}$ on X_0.

To obtain smooth fibers over the Teichmüller curve (in a neighborhood of X_∞) one has to glue cylinders of finite (large) height in place of the half-infinite cylinders of appropriate ratios of moduli. The hypothesis on ϕ acting on X_0 and on the half-infinite cylinders implies that ϕ is a well-defined involution on the smooth curves. Moreover, ϕ has two fixed points in each of the finite cylinders and $2h+2$ fixed points on X_0, making $2g+2$ fixed points in total. This shows that the smooth fibers of the Teichmüller curve are hyperelliptic.

To complete the proof we have to consider the two-component degenerations for $\mu \in \{(3,1),(5,1),(5,3)\}$. In both cases, the hyperelliptic involutions can neither exchange the components (since the zeros are of different order) nor fix the components (since the zeros are of odd order).

For $\mu = (3,3)$ a hyperelliptic involution ϕ cannot fix the component, since 3 is odd. It cannot exchange the two components and exchange a pair of half-infinite cylinders that belong to different nodes, since ϕ could then be used to define a non-trivial involution for each component. This involution fixes the zeros and this contradicts that 3 is odd. If ϕ exchanges all pairs of half-infinite cylinders that belong to the same node, ϕ has two fixed points in each cylinder on the smooth 'opened up' surface. Now we can apply the same argument as in the irreducible case to conclude that the 'opened up' flat surfaces are hyperelliptic as well.

For $\mu = (3,2,1)$ a hyperelliptic involution can neither fix the component with the (unique) zero of order three, since 3 is odd, nor map it elsewhere, since the zeros are of different order. □

Let C be a Teichmüller curve generated by $(X,\omega) \in \Omega\mathcal{M}_g(m_1,\ldots,m_k)$. Let $B \to C$ be a finite unramified cover such that the m_i-fold zero defines a section σ_i (not only a multi-section) with image S_i of the pullback family $f : \mathcal{X} \to \bar{B}$.

Proposition 4.5. *If $f : \bar{B} \to \overline{\mathcal{M}}_{g,1}$ is the lift of a Teichmüller curve using σ_i, then*

$$\sigma_i^2 = \frac{-\chi}{2(m_i+1)},$$

where $\chi = 2g(\bar{B}) - 2 + |\Delta|$ and Δ is the set of cusps in $\bar{B}$. In particular the intersection number with $\omega_{i,\mathrm{rel}}$, which is by definition equal to $-S_i^2$, is given by

$$\bar{B} \cdot \omega_{i,\mathrm{rel}} = \frac{\bar{B}\cdot\lambda - (\bar{B}\cdot\delta)/12}{(m_i+1)\kappa_\mu},$$

where $\kappa_\mu = \frac{1}{12}\sum_{j=1}^k \frac{m_j(m_j+2)}{m_j+1}$.

Proof. Let $\mathcal{L} \subset f_*\omega_{\mathcal{X}/\bar{B}}$ be the ('maximal Higgs', see [Möl06]) line bundle whose fiber over the point corresponding to $[X]$ is $\mathbb{C}\cdot\omega$, the generating differential of the Teichmüller curve. The property 'maximal Higgs' says by definition that

$$\deg(\mathcal{L}) = \chi/2. \tag{15}$$

Let S be the union of the sections $S_1,\ldots,S_k$. Pulling back the above inclusion to $\mathcal{X}$ gives an exact sequence

$$0 \to f^*\mathcal{L} \to \omega_{\mathcal{X}/\bar{B}} \to \mathcal{O}_S\left(\sum_{j=1}^k m_j S_j\right) \to 0,$$

since the multiplicities of the vanishing locus of the generating differential of the Teichmüller curve are constant along the whole compactified Teichmüller curve. This implies that $\omega_{\mathcal{X}/\bar{B}}$ is numerically equal to

$$f^*\mathcal{L} + \sum_{j=1}^k m_j S_j.$$

By the adjunction formula we get

$$S_i^2 = -\omega_{\mathcal{X}/\bar{B}} \cdot S_i = -m_i S_i^2 - \deg(\mathcal{L})$$

since the intersection product of two fibers of f is zero. Together with (15) we thus obtain the desired self-intersection formula.

By Theorem 3.1 and the relation (13), we have

$$\bar{B}\cdot\lambda = \frac{\chi}{2}\cdot L,$$

$$\bar{B}\cdot\delta = \frac{\chi}{2}\cdot(12L - 12\kappa_\mu).$$

Hence the second claimed formula follows, for $-S_i^2 = \bar{B}\cdot\omega_{i,\mathrm{rel}}$ by definition. □

Remark 4.6. For square-tiled surfaces the self-intersection number of a section on the elliptic surface is not hard to calculate ([Kod63], recalled in [Möl], and also in [Che10a, Thm. 1.15]). Pullback introduces the coefficient $m_i + 1$ in the denominator. This shows the formula in the square-tiled case. The general case of the formula can also be shown by adapting the argument given in [Bai07, Theorem 12.2], since there are $m_i + 1$ ways to split a singularity of order m_i.

Degrees of zeros $(d_1, \ldots, d_n)$	Hyperelliptic or spin structure	Lyapunov exponents				
		Component		Teichmüller curves		
		$\approx$	$\sum_{j=1}^{g} \lambda_j$	$\approx$	$\sum_{j=1}^{g} \lambda_j$	Reference
(4)	*hyperelliptic*	1.80000	$\frac{9}{5}$	Non-varying		Thm. 3.4
(4)	*odd*	1.60000	$\frac{8}{5}$	Non-varying		Sec. 5.1
$(3,1)$	–	1.75000	$\frac{7}{4}$	Non-varying		Sec. 5.2
$(2,2)$	*hyperelliptic*	1.50000	$\frac{3}{2}$	Non-varying		Thm. 3.4
$(2,2)$	*odd*	1.66666	$\frac{5}{3}$	Non-varying		Sec. 5.3
$(2,1,1)$	–	1.83333	$\frac{11}{6}$	Non-varying		Sec. 5.4
$(1,1,1,1)$	–	1.89285	$\frac{53}{28}$	2	2	$\mathcal{Q}(2,2,-1^8)$

FIGURE 1. Varying and non-varying sums in genus three

5. Genus three

In genus 3 all the strata have non-varying sums of Lyapunov exponents except the principal stratum. We summarize the behavior in Table 1. We also give a sharp upper bound of the sum for the principal stratum.

5.1. The stratum $\Omega\mathcal{M}_3(4)^{\mathrm{odd}}$. In the case $\Omega\mathcal{M}_3(4)^{\mathrm{odd}}$ the algorithm of [EMZ03] to calculate Siegel-Veech constants for components of strata gives

$$L_{(4)^{\mathrm{odd}}} = 8/5, \quad s_{(4)^{\mathrm{odd}}} = 9, \quad c_{(4)^{\mathrm{odd}}} = 6/5.$$

Proof of Theorem 1.1, Case $\Omega\mathcal{M}_3(4)^{\mathrm{odd}}$. The connected components $\Omega\mathcal{M}_3(4)^{\mathrm{odd}}$ and $\Omega\mathcal{M}_3(4)^{\mathrm{hyp}}$ are not only disjoint in $\Omega\mathcal{M}_3$, by Proposition 4.4 they are also disjoint in $\Omega\overline{\mathcal{M}}_3$. Hence the Teichmüller curves in this stratum do not intersect the hyperelliptic locus H in $\overline{\mathcal{M}}_3$. Recall the divisor class of H in (1). By $s(H) = 9$ and $C \cdot H = 0$, we obtain that $s(C) = 9$, hence $c(C) = 6/5$ and $L(C) = 8/5$ for all Teichmüller curves in this stratum using (12) and (13). □

5.2. The stratum $\Omega\mathcal{M}_3(3,1)$. In the case $\Omega\mathcal{M}_3(3,1)$ we have

$$L_{(3,1)} = 7/4, \quad s_{(3,1)} = 9, \quad c_{(3,1)} = 21/16.$$

Proof of Theorem 1.1, Case $\Omega\mathcal{M}_3(3,1)$. As in the case of the stratum $\Omega\mathcal{M}_3(4)^{\mathrm{odd}}$, a Teichmüller curve C in the stratum $(3,1)$ does not intersect the hyperelliptic locus, not even at the boundary by Proposition 4.4. Consequently we can apply the same disjointness argument as in the preceding case. Since $s(H) = 9$, we obtain that $s(C) = 9$, hence $c(C) = 21/16$ and $L(C) = 7/4$ using (12) and (13).

□

5.3. **The stratum $\Omega\mathcal{M}_3(2,2)^{\rm odd}$.** In the case $\Omega\mathcal{M}_3(2,2)^{\rm odd}$, we have

$$L_{(2,2)^{\rm odd}} = 5/3, \quad s_{(2,2)^{\rm odd}} = 44/5, \quad c_{(2,2)^{\rm odd}} = 11/9.$$

We first consider the general strata $\Omega\mathcal{M}_g(2,\ldots,2)^{\rm odd}$. Let $\overline{Z}_g$ be the divisor on $\overline{\mathcal{S}}_g^-$ parameterizing (X,η) such that the odd theta characteristic η satisfies

$$\eta \sim \mathcal{O}_X(2p_1 + p_2 + \cdots + p_{g-2}).$$

The class of $\overline{Z}_g$ was calculated in [FV]:

$$\overline{Z}_g = (g+8)\lambda - \frac{g+2}{4}\alpha_0 - 2\beta_0 - \sum_{i=1}^{[g/2]} 2(g-i)\alpha_i - \sum_{i=1}^{[g/2]} 2i\beta_i,$$

where λ is the pullback of the λ-class on $\overline{\mathcal{M}}_g$ and α_i, β_i are the boundary classes of $\overline{\mathcal{S}}_g^-$ over δ_i.

Let C be a Teichmüller curve in $\Omega\overline{\mathcal{M}}_g(2,\ldots,2)^{\rm odd}$ generated by a flat surface (X,ω) such that $\operatorname{div}(\omega) = 2\sum_{i=1}^{g-1} p_i$ for distinct points p_i. Using $\eta = \sum_{i=1}^{g-1} p_i$ as an odd theta-characteristic, we can map C to $\overline{\mathcal{S}}_g^-$. We are interested in the case when an odd theta-characteristic η always satisfies $h^0(\eta) = 1$. In this situation, $\overline{Z}_g$ can be identified as the divisorial stratum $\Omega\overline{\mathcal{M}}_g(4,2,\ldots,2)^{\rm odd}$. Consequently by Proposition 4.1, the image of C in $\overline{\mathcal{S}}_g^-$ does not intersect $\overline{Z}_g$. Then we have

$$0 = C \cdot \overline{Z}_g = C \cdot \Big((g+8)\lambda - \frac{g+2}{4}\alpha_0\Big),$$

since C does not intersect any boundary components except α_0. Note that $\pi^*\lambda = \lambda$ and $\pi^*\delta_0 = \alpha_0 + 2\beta_0$, where $\pi : \overline{\mathcal{S}}_g^- \to \overline{\mathcal{M}}_g$ is the natural projection morphism. By the projection formula we have

$$0 = (\pi_* C) \cdot \Big((g+8)\lambda - \frac{g+2}{4}\delta_0\Big),$$

which implies that

$$s(C) = \frac{4(g+8)}{g+2} = 4 + \frac{24}{g+2}.$$

We remark that the assumption $h^0(\eta) = 1$ for all η only holds in low genus, as a consequence of Clifford's theorem.

Proof of Theorem 1.1, Case $\Omega\mathcal{M}_3(2,2)^{\rm odd}$. For $g = 3$ an odd theta characteristic cannot have three or more sections by Clifford's theorem (Theorem 2.5). Hence the preceding considerations apply and we obtain that $s(C) = 44/5$. □

Note that this argument does not distinguish between hyperelliptic and non-hyperelliptic curves and for double covers of $\mathcal{Q}(1,1,-1^6)$ the result is in accordance with Theorem 3.4.

5.4. **The stratum $\Omega\mathcal{M}_3(2,1,1)$.** In the case $\Omega\mathcal{M}_3(2,1,1)$ we have

$$L_{(2,1,1)} = 11/6, \quad s_{(2,1,1)} = 98/11, \quad c_{(2,1,1)} = 49/36.$$

Note that the effective divisor of lowest slope in $\overline{\mathcal{M}}_3$ is the hyperelliptic locus with slope equal to 9 [HM90]. Since $s_{(2,1,1)} = 98/11 < 9$, we will not be able to argue by disjointness in $\overline{\mathcal{M}}_3$ as in the preceding cases. Nevertheless, this issue can be settled by passing to the moduli space of pointed curves $\overline{\mathcal{M}}_{3,2}$.

First we can lift a Teichmüller curve generated by a flat surface in this stratum to $\overline{\mathcal{M}}_{3,2}$ by marking the double zero p as the first point and one of the simple zeros q as the second point. In fact, we can do so after passing to a double covering of C where the simple zeros q and r can be distinguished. This double covering is unramified, since by definition of a Teichmüller curve the zeros never collide. Moreover, the slope and hence the sum of Lyapunov exponents are unchanged by passing to an unramified double covering. For simplicity we will continue to call C the Teichmüller curve we work with.

Proposition 5.1. *In the above setup, the Teichmüller curve C does not intersect the pointed Brill-Noether divisor $BN^1_{3,(1,2)}$ on $\overline{\mathcal{M}}_{3,2}$.*

Proof. Recall that $BN^1_{3,(1,2)}$ parameterizes pointed curves (X,p,q) that possess a g^1_3 containing $p+2q$ as a section. suppose that (X,ω) is in the intersection of C and $BN^1_{3,(1,2)}$. Since $h^0(\mathcal{O}_X(p+2q)) = 2$, we obtain that $h^0(\mathcal{O}_X(p+r-q)) = 1$ by Riemann-Roch and $h^0(\mathcal{O}_X(p+r)) = 2$ for $q \neq p, r$. If X is smooth, then X is hyperelliptic and p,r are conjugate. But $\omega_X \sim \mathcal{O}_X(2p+q+r)$, so p,q are also conjugate, contradiction. For singular X we deduce from $h^0(\mathcal{O}_X(p+r)) = 2$ that p and r are in the same component X_0 of X. This component admits an involution ϕ that acts on the set of zeros of $\omega_X|_{X_0}$. But p and r have different orders, so they cannot be conjugate under ϕ, contradiction. □

Proof of Theorem 1.1, Case $\Omega\mathcal{M}_3(2,1,1)$. By the proposition and the divisor class of $BN^1_{3,(1,2)}$ in (5), we obtain that

$$C \cdot (-\lambda + \omega_{1,\mathrm{rel}} + 3\omega_{2,\mathrm{rel}}) = 0.$$

Since $\kappa_{(2,1,1)} = 17/36$, using Proposition 4.5, we have

$$C \cdot \omega_{1,\mathrm{rel}} = \frac{C\cdot\lambda - (C\cdot\delta)/12}{17/12},$$
$$C \cdot \omega_{2,\mathrm{rel}} = \frac{C\cdot\lambda - (C\cdot\delta)/12}{17/18}.$$

Plugging in the above, we obtain that $s(C) = 98/11$ and the values of $L(C), c(C)$ follow from (12) and (13). □

Remark 5.2. Alternatively, the theorem can be deduced by showing that for a Teichmüller curve C its intersection loci with the hyperelliptic locus H in $\overline{\mathcal{M}}_3$ and with the Weierstrass divisor W in $\overline{\mathcal{M}}_{3,1}$ are the same. We show that these intersections are set-theoretically equal. A complete proof via this method would need to verify that the intersection multiplicities with the two divisors coincide.

Hyperelliptic flat surfaces in this stratum are obtained as coverings from the stratum $\mathcal{Q}(2,1,-1^7)$ and we deduce from Theorem 3.4 that $L^{\mathrm{hyp}}_{(2,1,1)} = 11/6$. Hence we may assume that C is not entirely in the hyperelliptic locus. If C intersects W

at a point $(X, 2p, q, r)$, where p is the marked point, then p is a Weierstrass point. Hence we have $2p+q+r \sim 3p+s$ for some s in X. Consequently $q+r \sim p+s$ and X must be hyperelliptic. On the other hand, suppose C intersects H at a point $(X, 2p, q, r)$. Since X is hyperelliptic, p must be a Weierstrass point.

Corollary 5.3. *A Teichmüller curve generated by a non-hyperelliptic flat surface in $\Omega\mathcal{M}_3(2,1,1)$ does intersect the hyperelliptic locus H at the boundary.*

Proof. If the statement was false for some Teichmüller curve C, we would have $C \cdot H = 0$, hence $s(C) = s(H) = 9$, contradicting $s(C) = 98/11$. □

5.5. **Varying sum in the stratum $\Omega\mathcal{M}_3(1,1,1,1)$.** We show by example that the sum of Lyapunov exponents in the principal stratum in $g = 3$ is varying, even modulo the hyperelliptic locus. In the case $\Omega\mathcal{M}_3(1,1,1,1)$, the algorithm of [EMZ03] to calculate Siegel-Veech constants for components of strata gives

$$L_{(1,1,1,1)} = 53/28, \quad s_{(1,1,1,1)} = 468/53, \quad c_{(1,1,1,1)} = 39/28.$$

Examples. *The 'eierlegende Wollmilchsau', the square-tiled surface given by the permutations ($\pi_r = (1234)(5678), \pi_u = (1836)(2745)$), generates a Teichmüller curve C with $L(C) = 1$.*

The square-tiled surface given by the permutations

$$(\pi_r = (1234)(5)(6789), \pi_u = (1)(2563)(4897))$$

generates a Teichmüller curve C with $L(C) = 2$. It attains the upper bound given by Theorem 1.1, but it is not hyperelliptic.

There exist square-tiled surfaces in this stratum whose associated Teichmüller curves C have

$$L(C) \in \{1, 3/2, 5/3, 7/4, 9/5, 11/6, 19/11, 33/19, 83/46, 544/297\}.$$

Proof of Theorem 1.1, Case $\Omega\mathcal{M}_3(1,1,1,1)$. Teichmüller curves in the locus of hyperelliptic flat surfaces in $\Omega\mathcal{M}_3(1,1,1,1)$ have $L = 2$ by [EKZ].

If the Teichmüller curve C is not contained in the hyperelliptic locus, then $C \cdot H \geq 0$, or equivalently $s(C) \leq s(H) = 9$. Using $\kappa_{(1,1,1,1)} = 1/2$ this implies $L(C) \leq 2$.

For the last statement in the theorem recall that an double cover of a genus two curve is always hyperelliptic (e.g. [Far76]). □

6. Genus four

In genus 4 we summarize the non-varying sums of Lyapunov exponents and upper bounds for varying sums in Table 2.

6.1. **The stratum $\Omega\mathcal{M}_4(6)^{\mathrm{even}}$.** In the case $\Omega\mathcal{M}_4(6)^{\mathrm{even}}$, we have

$$L_{(6)^{\mathrm{even}}} = 14/7, \quad s_{(6)^{\mathrm{even}}} = 60/7, \quad c_{(6)^{\mathrm{even}}} = 10/7.$$

Proposition 6.1. *Let C be a Teichmüller curve generated by $(X, \omega) \in \Omega\mathcal{M}_4(6)^{\mathrm{even}}$, lifted to $\overline{\mathcal{M}}_{4,1}$ using the zero of ω. Then C does not intersect the theta-null divisor Θ in $\overline{\mathcal{M}}_{4,1}$.*

Proof. Recall that the divisor $\Theta \subset \overline{\mathcal{M}}_{4,1}$ parameterizes curves that admit an odd theta characteristic whose support contains the marked point. Suppose the stable pointed curve (X, p) lies in the intersection of C and Θ. Then there exists an odd theta characteristic η on X with a section $t \in H^0(\eta)$ such that $\mathrm{div}(t) = p+q+r$ for

Degrees of zeros $(d_1, \ldots, d_n)$	Hyperelliptic or spin structure	Lyapunov exponents				
		Component		Teichmüller curves		
		$\approx$	$\sum_{j=1}^{g} \lambda_j$	$\approx$	$\sum_{j=1}^{g} \lambda_j$	Reference
(6)	*hyperelliptic*	2.28571	$\frac{16}{7}$	Non-varying		Thm. 3.4
(6)	*even*	2.00000	2	Non-varying		Sec. 6.1
(6)	*odd*	1.85714	$\frac{13}{7}$	Non-varying		Sec. 6.2
$(5,1)$	–	2.00000	2	Non-varying		Sec. 6.3
$(4,2)$	*even*	2.13333	$\frac{32}{15}$	Non-varying ?		–
$(4,2)$	*odd*	1.93333	$\frac{29}{15}$	Non-varying ?		–
$(3,3)$	*hyperelliptic*	2.50000	$\frac{5}{2}$	Non-varying		Thm. 3.4
$(3,3)$	$non-hyp$	2.00000	2	Non-varying		Sec. 6.6
$(3,2,1)$	–	2.08333	$\frac{25}{12}$	Non-varying		Sec. 6.8
$(2,2,2)$	*odd*	2.00000	2	Non-varying		Sec. 6.7
$(2,2,2)$	*even*	2.28571	$\frac{166}{75}$	2.333333	$\frac{7}{3}$	$\mathcal{Q}(3,1,-1^8)$
$(4,1,1)$	–	2.06727	$\frac{1137}{550}$	1.96792	$\frac{1043}{530}$	Eq. (16)
$(2^2,1^2)$	–	2.13952	$\frac{504}{2358}$	1.91666	$\frac{23}{12}$	$\mathcal{Q}(2,1,1,-1^7)$
$(3,1^3)$	–	2.12903	$\frac{66}{31}$	2.11523	$\frac{514}{243}$	Eq. (17)
$(2,1^4)$	–	2.18333	$\frac{131}{60}$	2.80000	$\frac{14}{5}$	$\mathcal{Q}(3,2,2,-1^{11})$
(1^6)	–	2.22546	$\frac{839}{377}$	2.50000	$\frac{5}{2}$	$\mathcal{Q}(2,2,2,-1^{10})$

FIGURE 2. Varying and non-varying sums in genus four

some q, r not both equal to p. Denote by $\mathcal{L} = \mathcal{O}_X(3p)$ the line bundle corresponding to the even theta characteristic with a section $s \in H^0(\mathcal{L})$ given by $3p$. Since $\eta^{\otimes 2} \sim \omega_X \sim \mathcal{L}^{\otimes 2}$, the function s^2t^{-2} implies that $4p \sim 2q+2r$ on X, hence p is not a base point of $|\mathcal{L}(p)| = |\mathcal{O}_X(4p)|$. Consequently we have $h^0(\mathcal{L}(p)) = 1+h^0(\mathcal{L}) \geq 3$. By $\omega_X \sim \mathcal{O}_X(6p)$ and Riemann-Roch, $h^0(\mathcal{O}_X(2p)) = h^0(\mathcal{O}_X(4p)) - 1 \geq 2$. Since X is irreducible by Corollary 4.2, this implies that X is hyperelliptic and p is a Weierstrass point. It contradicts the disjointness of the hyperelliptic locus and this component in $\overline{\mathcal{M}}_4$ by Proposition 4.4. □

Proof of Theorem 1.2, Case $\Omega\mathcal{M}_4(6)^{\text{even}}$. Using the proposition and the class of the theta-null divisor Θ in (3), we obtain that

$$C \cdot (30\lambda + 60\omega_{\text{rel}} - 4\delta_0) = 0.$$

Using Proposition 4.5 we know

$$C \cdot \omega_{\text{rel}} = \frac{C \cdot \lambda - (C \cdot \delta)/12}{4}.$$

It now suffices to plug this in and use (12) and (13). □

6.2. **The stratum $\Omega\mathcal{M}_4(6)^{\text{odd}}$.** In the case $\Omega\mathcal{M}_4(6)^{\text{odd}}$, we have

$$L_{(6)^{\text{odd}}} = 13/7, \quad s_{(6)^{\text{odd}}} = 108/13, \quad c_{(6)^{\text{odd}}} = 9/7.$$

Proposition 6.2. *Let C be a Teichmüller curve generated by $(X, \omega) \in \Omega\mathcal{M}_4(6)^{\text{odd}}$ lifted to $\overline{\mathcal{M}}_{4,1}$ using the zero of ω. Then C does not intersect the pointed Brill-Noether divisor $BN^1_{3,(2)}$.*

Proof. Recall that $BN^1_{3,(2)} \subset \overline{\mathcal{M}}_{4,1}$ parameterizes curves that admit a linear series g^1_3 with a section containing $2p$, where p is the marked point. Suppose that C intersects $BN^1_{3,(2)}$ at (X, p). Let $\eta = \mathcal{O}_X(3p)$ denote the theta characteristic given by $3p$. Since $h^0(\eta)$ is odd, Clifford's theorem implies that $h^0(\eta) = 1$. Since (X, p) is contained in $BN^1_{3,(2)}$, we have $h^0(\mathcal{O}_X(2p + q)) = 2$ for some q different from p. By $\omega_X \sim \mathcal{O}_X(6p)$ and Riemann-Roch, we have $h^0(\mathcal{O}_X(4p - q)) = 2$, hence $h^0(\mathcal{O}_X(3p - q)) \geq 1$. Note that q is not a base point of the linear system $|\mathcal{O}_X(3p)|$ for $q \neq p$. Consequently we have $h^0(\eta) = 1 + h^0(\mathcal{O}_X(3p - q)) \geq 2$, contradicting that $h^0(\eta) = 1$. □

Proof of Theorem 1.2, Case $\Omega\mathcal{M}_4(6)^{\text{odd}}$. Recall the divisor class of $BN^1_{3,(2)}$ in (7). By $C \cdot BN^1_{3,(2)} = 0$, we have

$$C \cdot (4\omega_{\text{rel}} + 8\lambda - \delta_0) = 0.$$

It now suffices to use Proposition 4.5 and to plug the result in (12) and (13). □

6.3. **The stratum $\Omega\mathcal{M}_4(5, 1)$.** In the case $\Omega\mathcal{M}_4(5, 1)$, we have

$$L_{(5,1)} = 2, \quad s_{(5,1)} = 25/3, \quad c_{(5,1)} = 25/18.$$

Proposition 6.3. *Let C be a Teichmüller curve generated by $(X, \omega) \in \Omega\mathcal{M}_4(5, 1)$, lifted to $\overline{\mathcal{M}}_{4,1}$ using the 5-fold zero of ω. Then C does not intersect the pointed Brill-Noether divisor $BN^1_{3,(2)}$.*

Proof. Suppose that (X, ω) is contained in the intersection of C with $BN^1_{3,(2)}$, where $\mathrm{div}(\omega) = 5p + q$ with p the marked point. By Proposition 4.4, X is not hyperelliptic. For a non-hyperelliptic curve X which is either smooth, or nodal irreducible, or consisting of two components joined at three nodes, the dualizing sheaf ω_X is very ample due to [Has99, Prop. 2.3]. Hence the canonical map is an embedding. We will analyze the geometry of its image.

We have $h^0(\mathcal{O}_X(2p + r)) \geq 2$ for some smooth point r. Since $\omega_X \sim \mathcal{O}_X(5p + q)$, by Riemann-Roch, it implies that $h^0(\mathcal{O}_X(3p+q-r)) \geq 2$, hence $h^0(\mathcal{O}_X(3p-r)) \geq 1$. If $r \neq p$, then r is not a base point of $|\mathcal{O}_X(3p)|$, hence $h^0(\mathcal{O}_X(3p)) \geq 2$. If $r = p$, then we still have $h^0(\mathcal{O}_X(3p)) = h^0(\mathcal{O}_X(2p + r)) \geq 2$. In any case, $3p$ admits a g^1_3 for X. By Riemann-Roch, $2p + q$ also yields a g^1_3. These must be two different g^1_3's, for $p \neq q$.

Since X is not hyperelliptic, its canonical image is contained in a quadric surface Q in $\mathbb{P}^3$. By Geometric Riemann-Roch, a section of a g^1_3 on X corresponds to a line in $\mathbb{P}^3$ that intersects Q at three points (with multiplicity). By Bézout, this line must be a ruling of Q. Since X has two g^1_3's, Q must be smooth and its two rulings correspond to the two g^1_3's. But the two lines spanned by the sections $2p + q$ and $3p$ cannot be both tangent to X at the smooth point p, contradiction. $\square$

Proof of Theorem 1.2, Case $\Omega\mathcal{M}_4(5, 1)$. We lift the Teichmüller curve C to $\overline{\mathcal{M}}_{4,1}$ using the 5-fold zero of ω. By the proposition $C \cdot BN^1_{3,(2)} = 0$, we have

$$C \cdot (4\omega_{\mathrm{rel}} + 8\lambda - \delta_0) = 0.$$

By Proposition 4.5, we also have

$$C \cdot \omega_{\mathrm{rel}} = \frac{C.\,\lambda - (C.\,\delta)/12}{11/3}.$$

Now the result follows by combining the two equalities. $\square$

6.4. **The stratum $\Omega\mathcal{M}_4(4, 2)^{\mathrm{even}*}$.** In the case $\Omega\mathcal{M}_4(4, 2)^{\mathrm{even}}$ we have

$$L_{(4,2)^{\mathrm{even}}} = 32/15, \quad s_{(4,2)^{\mathrm{even}}} = 17/2, \quad c_{(4,2)^{\mathrm{even}}} = 68/45.$$

Based on numerical values on individual Teichmüller curves, we believe that the sum of Lyapunov exponents is non-varying in this stratum. But we have not found a moduli space and a divisor to perform the desired disjointness argument.

6.5. **The stratum $\Omega\mathcal{M}_4(4, 2)^{\mathrm{odd}*}$.** In the case $\Omega\mathcal{M}_4(4, 2)^{\mathrm{odd}}$ we have

$$L_{(4,2)^{\mathrm{odd}}} = 29/15, \quad s_{(4,2)^{\mathrm{odd}}} = 236/29, \quad c_{(4,2)^{\mathrm{odd}}} = 59/45.$$

We also believe that the sum of Lyapunov exponents is non-varying in this case. But we have not discovered a divisor that would do the job.

6.6. **The stratum $\Omega\mathcal{M}_4(3, 3)^{\mathrm{non-hyp}}$.** In the case $\Omega\mathcal{M}_4(3, 3)^{\mathrm{non-hyp}}$ we have

$$L_{(3,3)^{\mathrm{non-hyp}}} = 2, \quad s_{(3,3)^{\mathrm{non-hyp}}} = 33/4, \quad c_{(3,3)^{\mathrm{non-hyp}}} = 11/8.$$

Proposition 6.4. *Let C be a Teichmüller curve generated by a flat surface $(X, \omega) \in \Omega\mathcal{M}_4(3, 3)^{\mathrm{non-hyp}}$, lifted to $\overline{\mathcal{M}}_{4,2}$ (after a degree two base change). Then C does not intersect the divisor* Lin^1_3.

Proof. Recall that $\mathrm{Lin}_3^1 \subset \overline{\mathcal{M}}_{4,2}$ parameterizes pointed curves (X, p, q) that admit a g_3^1 with a section vanishing at p, q, r for some $r \in X$. Suppose (X, p, q) is contained in the intersection of C with Lin_3^1. Since $\omega_X \sim \mathcal{O}_X(3p+3q)$ and $h^0(\mathcal{O}_X(p+q+r)) \geq 2$, by Riemann-Roch we know that $h^0(\mathcal{O}_X(2p+2q-r)) \geq 2$. If $r \neq p, q$, then $h^0(\mathcal{O}_C(2p+2q)) \geq 3$, hence $2p+q$ and $2q+p$ both admit g_3^1. If C is not hyperelliptic, using the canonical image of C contained in a quadric in $\mathbb{P}^3$ and the preceding argument of rulings, one concludes that $2p+q$ and $2q+p$ span the same line (connecting p, q) on the quadric, contradiction. If $r = p$ or q, again, $2p+q$ and $2q+p$ both admit g_3^1 and consequently C is hyperelliptic. But this stratum is non-hyperelliptic, and Proposition 4.4 yields the desired contradiction. □

Proof of Theorem 1.2, Case $\Omega\mathcal{M}_4(3,3)$. It suffices to carry out the calculation for the second test curve in the proof of Proposition 2.6 backwards. By $C \cdot \mathrm{Lin}_3^1 = 0$ together with Proposition 4.5 and $\kappa_{(3,3)} = 5/8$, the result follows immediately. □

6.7. **The stratum $\Omega\mathcal{M}_4(2,2,2)^{\mathrm{odd}}$.** In the case $\Omega\mathcal{M}_4(2,2,2)^{\mathrm{odd}}$ we have

$$L_{(2,2,2)^{\mathrm{odd}}} = 2, \quad s_{(2,2,2)^{\mathrm{odd}}} = 8, \quad c_{(2,2,2)^{\mathrm{odd}}} = 4/3.$$

Note that by [KZ03, Prop. 7] this stratum contains the hyperelliptic curves where all the zeros are fixed, but not those, where a pair of zeros are exchanged.

Proof of Theorem 1.2, Case $\Omega\mathcal{M}_4(2,2,2)^{\mathrm{odd}}$. We just need to apply the argument of Section 5.3 to obtain the result. It does apply to this case, because an odd theta characteristic on a genus four curve cannot have three or more sections by Clifford's theorem (Theorem 2.5). □

6.8. **The stratum $\Omega\mathcal{M}_4(3,2,1)$.** In the case $\Omega\mathcal{M}_4(3,2,1)$, we have

$$L_{(3,2,1)} = 25/12, \quad s_{(3,2,1)} = 41/5, \quad c_{(3,2,1)} = 205/144.$$

Proposition 6.5. *Let C be a Teichmüller curve generated by a flat surface $(X, \omega) \in \Omega\mathcal{M}_4(3,2,1)$, lifted to $\overline{\mathcal{M}}_{4,3}$. Then C does not intersect the divisor $BN^1_{4,(1,1,2)}$.*

Proof. Recall that the Brill-Noether divisor $BN^1_{4,(1,1,2)} \subset \overline{\mathcal{M}}_{4,3}$ parameterizes curves with a g_4^1 given by $p+q+2r$. Suppose that (X, ω) is contained in the intersection of C with $BN^1_{4,(1,1,2)}$, where $\mathrm{div}(\omega) = 3p+2q+r$ and p, q, r are the (ordered) marked points. By $h^0(\mathcal{O}_X(p+q+2r)) \geq 2$ and Riemann-Roch, we have $h^0(\mathcal{O}_X(2p+q-r)) \geq 1$. Consequently $h^0(\mathcal{O}_X(2p+q)) \geq 2$ and by Riemann-Roch again, we have $h^0(\mathcal{O}_X(p+q+r)) \geq 2$. Note that if $2p+q \sim p+q+r$, then $p \sim r$, which is impossible. If these are two different g_3^1's and X is smooth, or stable but non-hyperelliptic and at least three-connected, then its canonical map is an embedding. Consequently both p and q lie on two different rulings of the quadric containing the canonical image of X. This is impossible.

Since hyperelliptic stable fibers cannot occur by Proposition 4.4, the last case to be excluded consists of a stable curve which is only two-connected. Given the constraints in Corollary 4.2, there are two possibilities for such a stable curve. First, there are two irreducible nodal curves X_1 and X_2 of arithmetic genus one and two, respectively, joined at two nodes $\{x, y\}$ with q lying on X_1 and p and r lying on X_2. Second, there are irreducible nodal curves X_1, X_0 and Y_1, the index specifying the arithmetic genus with p on Y_1, q on X_1 and r on X_0, whose intersection is given by $Y_1 \cdot X_1 = \{x\}$, $Y_1 \cdot X_0 = \{z_1, z_2\}$, $X_1 \cdot X_0 = \{y\}$.

For the first type, consider the linear system $|\mathcal{O}_X(2p+q)|$. Since in both cases p and q lie on different components of the stable curve, q has to be a base point of this linear system. Hence in the first case $\omega_{X_2}(x+y) \sim \mathcal{O}_{X_2}(4p)$, i.e. p is a Weierstrass point for the line bundle $\omega_{X_2}(x+y)$. By the same argument q is a base point of $|\mathcal{O}_X(p+q+r)|$, hence $\omega_{X_2}(x+y) \sim \mathcal{O}_{X_2}(2p+2r)$. Since $p \neq r$, this implies that p is not a Weierstrass point and we obtain the desired contradiction.

For the second type of the stable curve X, the condition $h^0(\mathcal{O}_X(p+q+r)) \geq 2$ provides an immediate contradiction, since all the three points lie on different components of the stable curve. $\square$

Proof of Theorem 1.2, Case $\Omega\mathcal{M}_4(3,2,1)$. Recall the divisor class of $BN^1_{(1,1,2)}$ in (6). By $C \cdot BN^1_{4,(1,1,2)} = 0$ together with Proposition 4.5, the result follows directly. $\square$

6.9. **Varying sum in the stratum $\Omega\mathcal{M}_4(2,2,2)^{\mathrm{even}}$.** In the case $\Omega\mathcal{M}_4(2,2,2)^{\mathrm{even}}$, we have

$$L_{(2,2,2)^{\mathrm{even}}} = 166/75, \quad s_{(2,2,2)^{\mathrm{even}}} = 696/83, \quad c_{(2,2,2)^{\mathrm{even}}} = 116/75.$$

Note that by [KZ03, Prop. 7] this stratum contains the hyperelliptic curves where a pair of zeros are exchanged, but not those, where all the zeros are fixed. For Teichmüller curves C^{hyp} contained in the locus of hyperelliptic flat surfaces within this stratum, we have

$$L(C^{\mathrm{hyp}}) = 7/3.$$

Examples. *The square-tiled surface* $(X : y^6 = x(x-1)(x-t), \omega = dx/y)$ *found by Forni and Matheus* [FM] *has maximally degenerate Lyapunov spectrum, i.e.* $L(C) = 1$.

Proposition 6.6. *A Teichmüller curve C generated by a non-hyperelliptic flat surface* $(X,\omega) \in \Omega\overline{\mathcal{M}}_4(2,2,2)^{\mathrm{even}}$ *has*

$$L(C) \leq 16/7.$$

In particular the sum of Lyapunov exponents of any Teichmüller curve generated by a non-hyperelliptic flat surface in this stratum is strictly smaller than the sum of Lyapunov exponents of any Teichmüller curve generated by a hyperelliptic flat surface in this stratum.

Proof. Recall the divisor class in (11) of the Gieseker-Petri divisor $GP^1_{4,3}$ on $\overline{\mathcal{M}}_4$. It has slope equal to $17/2$. Hence if C is not entirely contained in this divisor, we have $s(C) \leq 17/2$ which translates into $L(C) \leq 16/7$.

If C is contained in $GP^1_{4,3}$, we try to intersect C with the Brill-Noether divisor $BN^1_{4,(3,1)}$. If C is not contained in $BN^1_{4,(3,1)}$, using Proposition 4.5 with $\kappa = 2/3$ we obtain a better bound $L(C) \leq 2$.

Suppose that a generic surface (X,ω) parameterized by C lies in $BN^1_{4,(3,1)}$. By definition, we have $h^0(\mathcal{O}_X(3p+q)) \geq 2$, which implies $h^0(\mathcal{O}_X(q+2r-p)) \geq 1$ by Riemann-Roch. Consequently we have $h^0(\mathcal{O}_X(2r+q)) \geq 2$ and $h^0(\mathcal{O}_X(2p+q)) \geq 2$. Since we excluded hyperelliptic X, the assumption that X is parameterized in $GP^1_{4,3}$ implies that $2r+q \sim 2p+q$. Hence $2p \sim 2r$, which contradicts the assumption that X is not hyperelliptic. $\square$

6.10. **Varying sum in the stratum $\Omega\mathcal{M}_4(4,1,1)$.** In this stratum we have

$$L_{(4,1,1)} = 1137/550 \approx 2.06727, \quad s_{(4,1,1)} = 3118/379, \quad c_{(4,1,1)} = 1559/1100.$$

The stratum contains the locus of hyperelliptic flat surfaces coming from $\mathcal{Q}(3,2,-1^9)$. Hence for a Teichmüller curve C^{hyp} in this locus, the sum of Lyapunov exponents is

$$L(C^{\mathrm{hyp}}) = 23/10.$$

Examples. A Teichmüller curve C generated by the square-tiled surface with

$$(\pi_r = (12)(3)(4)(5)(6\ 7)(8)(9\ 10), \pi_u = (132456879)(10)) \tag{16}$$

has $L(C) = 1043/530 \approx 1.96792$.

A Teichmüller curve C generated by the square-tiled surface with

$$(\pi_r = (12)(3)(4)(5)(6)(7,8)(9)(10\ 11), \pi_u = (1\,3)(2,4,5,6,7,9)(8\,10)(11))$$

has $L(C) = 267163/129510 \approx 2.06287$.

This stratum also contains Teichmüller curves C generated by square-tiled surfaces with

$$L(C) \in \{1043/530 \approx 1.96792, 579/290, 4101/1990, 1799/870 \approx 2.06782, 23/10\}.$$

Proposition 6.7. *A Teichmüller curve C generated by a non-hyperelliptic flat surface $(X,\omega) \in \Omega\mathcal{M}_4(4,1,1)$ has*

$$L(C) \leq 21/10.$$

In particular the sum of Lyapunov exponents of any Teichmüller curve generated by a non-hyperelliptic flat surface in this stratum is strictly smaller than the sum of Lyapunov exponents of any Teichmüller curve generated by a hyperelliptic flat surface in this stratum.

Proof. By the same argument as in the proof for the stratum $\Omega\mathcal{M}_3(2,1,1)$ we may pass to an unramified covering and label the zeros of ω such that $\omega_X \sim \mathcal{O}_X(4p+q+r)$ along the whole family. Using p and q we lift this covering to a curve in $\overline{\mathcal{M}}_{4,2}$ that we continue to call C. We will show that C is not entirely contained in the divisor $BN^1_{4,(2,2)}$, provided that X is not hyperelliptic. Then $C \cdot BN^1_{4,(2,2)} \geq 0$ together with Proposition 4.5 using $\kappa_{(4,1,1)} = 39/60$ implies the claim.

Suppose that the flat surface (X,ω) on C is contained in $BN^1_{4,(2,2)}$. Then by definition $h^0(\mathcal{O}_X(2p+2q)) \geq 2$ and by Riemann-Roch $h^0(\mathcal{O}_X(2p-q+r)) \geq 1$. Hence $h^0(\mathcal{O}_X(2p+r)) \geq 2$ and by Riemann-Roch again $h^0(\mathcal{O}_X(2p+q)) \geq 2$. Since X is not hyperelliptic, we consider the quadric surface containing its canonical image. The ruling that is tangent to X at p intersects X at a third point. This point has to be both q and r due to the g^1_3's given by $2p+q$ and $2p+r$, which is absurd for $q \neq r$. $\square$

6.11. **Varying sum in the stratum $\Omega\mathcal{M}_4(3,1,1,1)$.** In this stratum we have

$$L_{(3,1,1,1)} = 66/31 \approx 2.12903, \quad s_{(3,1,1,1)} = 65/8, \quad c_{(3,1,1,1)} = 715/496.$$

This stratum does not contain any submanifolds obtained by double covering constructions.

Examples. The Teichmüller curve C generated by the square-tiled surface with

$$(\pi_r = (123456789\,10), \pi_u = (145836\,10)(279)) \tag{17}$$

has $L(C) = 514/243 \approx 2.11523$.

The Teichmüller curve C generated by the square-tiled surface with

$$(\pi_r = (123456789\,10\,11), \pi_u = (1458\,10\,27\,11)(369))$$

has $L(C) = 1531/720 \approx 2.12639$.

There exist Teichmüller curves C generated by square-tiled surfaces in this stratum with

$$L(C) \in \{241/114 \approx 2.114035, 72167/33984, 1531/720 \approx 2.1263\}.$$

Proposition 6.8. *A Teichmüller curve C generated by a flat surface $(X, \omega) \in \Omega\overline{\mathcal{M}}_4(3, 1, 1, 1)$ has*

$$L(C) \leq 7/3.$$

Proof. The proof is identical to the one given below for the stratum $\Omega\mathcal{M}_4(2, 1, 1, 1, 1)$ using two different lifts to $\overline{\mathcal{M}}_{4,3}$ and the divisor $BN^1_{4,(1,1,2)}$. See Section 6.13. □

6.12. **Varying sum in the stratum $\Omega\mathcal{M}_4(2, 2, 1, 1)$.** In this stratum we have

$$L_{(2,2,1,1)} = 5045/2358 \approx 2.13952, \quad s_{(2,2,1,1)} = 8178/1009, \quad c_{(2,2,1,1)} = 6815/4716.$$

The stratum contains two loci of hyperelliptic flat surfaces. One of them corresponds to the orientation double covers of $\mathcal{Q}(4, 2, -1^{10})$, hence for a Teichmüller curve C in this locus, the sum of Lyapunov exponents is $L(C) = 5/2$. In this locus, the zeros are permuted in pairs by the hyperelliptic involution.

The second one corresponds to $\mathcal{Q}(2, 1, 1, -1^8)$, hence for a Teichmüller curve C in this locus, the sum of Lyapunov exponents is $L(C) = 13/6 \approx 2.16$.

Examples. The Teichmüller curve C generated by the square-tiled surface with

$$(\pi_r = (12)(3)(4)(5)(67)(8)(9)(10\,11)(12), \pi_u = (134)(256789\,10\,11\,12))$$

has $L(C) = 3313/1590 \approx 2.083$.

The Teichmüller curve C generated by the square-tiled surface with

$$(\pi_r = (12)(3)(4)(56)(7)(89)(10\,11), \pi_u = (134578)(269\,10)(11))$$

has $L(C) = 4919/2312 \approx 2.1275$.

There exist Teichmüller curves C generated by square-tiled surfaces in this stratum with

$$L(C) \in \{3313/1590 \approx 2.083, 157/75, 273529/128580, 4919/2312 \approx 2.1275\}.$$

Proposition 6.9. *A Teichmüller curve C generated by a non-hyperelliptic flat surface $(X, \omega) \in \Omega\overline{\mathcal{M}}_4(2, 2, 1, 1)$ has*

$$L(C) \leq 13/6.$$

In particular the sum of Lyapunov exponents of any Teichmüller curve generated by a non-hyperelliptic flat surface in this stratum is strictly smaller than the sum of Lyapunov exponents of any Teichmüller curve generated by a hyperelliptic flat surface where the four zeros are permuted in pairs.

Proof. By the same argument as in the proof for the stratum $\Omega\mathcal{M}_3(2,1,1)$ we may pass to an unramified covering and label the zeros of ω such that $\omega_X \sim \mathcal{O}_X(2p+2q+r+s)$ along the whole family. We lift this covering to a curve in $\overline{\mathcal{M}}_{4,3}$ by marking p,q,r and continue to call it C. We will show that C is not entirely contained in the divisor $BN^1_{4,(1,1,2)}$. Then $C \cdot BN^1_{4,(1,1,2)} \geq 0$ together with Proposition 4.5 implies this proposition.

Suppose that the flat surface (X,ω) parameterized by C is contained in $BN^1_{4,(1,1,2)}$. Then by definition $h^0(\mathcal{O}_X(p+q+2r)) \geq 2$ and by Riemann-Roch $h^0(\mathcal{O}_X(p+q-r+s)) \geq 1$. Hence $h^0(\mathcal{O}_X(p+q+s)) \geq 2$ and by Riemann-Roch again $h^0(\mathcal{O}_X(p+q+r)) \geq 2$. For X non-hyperelliptic this means that on the quadric containing the canonical image of X in $\mathbb{P}^3$, there are two rulings passing through p and q (hence they are the same ruling), one intersecting the curve moreover at r and the other intersecting the curve moreover at s. This is absurd for $r \neq s$. $\square$

6.13. **Varying sum in the stratum $\Omega\mathcal{M}_4(2,1,1,1,1)$.** In this stratum we have

$$L_{(2,1,1,1,1)} = 131/60 \approx 2.18333, \quad s_{(2,1,1,1,1)} = 1052/131, \quad c_{(2,1,1,1,1)} = 263/180.$$

This stratum contains the locus of hyperelliptic flat surfaces corresponding to $\mathcal{Q}(3,2,2,-1^{11})$. Hence for a Teichmüller curve C in this locus, the sum of Lyapunov exponents is $L(C) = 14/5 = 2.8$.

Examples. The Teichmüller curve C generated by the square-tiled surface with

$$(\pi_r = (12)(3)(4)(5,6)(7)(8)(9\,10)(11\,12)(13), \pi_u = (132457689\,11\,10\,12\,13))$$

has $L(C) = 268/129 \approx 2.0775$.

The Teichmüller curve C generated by the square-tiled surface with

$$(\pi_r = (12)(3)(4)(5)(6)(78)(9\,10)(11)(12\,13), \pi_u = (13)(245679\,11\,8\,12\,10)(13))$$

has $L(C) = 207826/95511 \approx 2.1759$.

There exist Teichmüller curves C generated by square-tiled surfaces in this stratum with

$$L(C) \in \{268/129 \approx 2.0775, 239/114, 4031/1923, 207826/95511 \approx 2.175\}.$$

Proposition 6.10. *A Teichmüller curve C generated by a non-hyperelliptic flat surface $(X,\omega) \in \Omega\overline{\mathcal{M}}_4(2,1,1,1,1)$ has*

$$L(C) \leq 7/3.$$

In particular the sum of Lyapunov exponents of any Teichmüller curve generated by a non-hyperelliptic flat surface in this stratum is strictly smaller than the sum of Lyapunov exponents of any Teichmüller curve generated by a hyperelliptic flat surface in this stratum.

Proof. By the same argument as in the proof for the stratum $\Omega\mathcal{M}_3(2,1,1)$, we may pass to an unramified covering and label the zeros of ω such that $\omega_X \sim \mathcal{O}_X(2p+q+r+s+u)$ along the whole family. First, using p, q and r we lift this covering to a curve in $\overline{\mathcal{M}}_{4,3}$ that we continue to call C.

If C is not entirely contained in the divisor $BN^1_{4,(1,2,1)}$, then $C \cdot BN^1_{4,(1,2,1)} \geq 0$ together with Proposition 4.5 using $\kappa_{(2,1,1,1,1)} = 13/18$ implies the claim. If C is entirely contained in the divisor $BN^1_{4,(1,2,1)}$, we can lift C to $\overline{\mathcal{M}}_{4,3}$ alternatively by marking p, q and r. Again, if C is not contained in $BN^1_{4,(1,2,1)}$, the claim holds.

Suppose that C is contained in the Brill-Noether divisor for both lifts. Then for (X,ω) parameterized in C, by definition we have $h^0(\mathcal{O}_X(p+2q+r)) \geq 2$, consequently we obtain $h^0(\mathcal{O}_X(s+u+p-q)) \geq 1$ and $h^0(\mathcal{O}_X(s+u+p)) \geq 2$. For the second lift we deduce from $h^0(\mathcal{O}_X(p+2q+u)) \geq 2$ that $h^0(\mathcal{O}_X(s+r+p)) \geq 2$. Since X is not hyperelliptic, the canonical map is an embedding and its image lies on a quadric surface in $\mathbb{P}^3$. Then the unique line on the quadric passing through s and p cannot have a third intersection point with C at both r and u for $r \neq u$. $\square$

6.14. **Varying sum in the stratum $\Omega\mathcal{M}_4(1,1,1,1,1,1)$.** In this stratum we have

$$L_{(1,1,1,1,1,1)} = \frac{839}{377} \approx 2.22546, \quad s_{(1,1,1,1,1,1)} = \frac{6675}{839}, \quad c_{(1,1,1,1,1,1)} = \frac{2225}{1508}.$$

The stratum contains the locus of hyperelliptic flat surfaces corresponding to the stratum $\mathcal{Q}(2,2,2,-1^{10})$. Hence for a Teichmüller curve C in this locus we have $L(C) = 5/2$.

Examples. The Teichmüller curve C generated by the square-tiled surface with

$$(\pi_r = (12)(3)(4)(5)(6)(78)(9)(10)(11\,12)(13)(14), \pi_u = (132456798\,10\,11\,13)(12\,14))$$

has $L(C) = 125/58 \approx 2.15517$.

The Teichmüller curve C generated by the square-tiled surface with

$$(\pi_r = (12)(3)(4)(56)(7)(8)(9\,10)(11)(12), \pi_u = (132457689\,11\,10\,12))$$

has $L(C) = 9/4 = 2.25$.

There exist Teichmüller curves C generated by square-tiled surfaces in this stratum with

$$L(C) \in \{125/58 \approx 2.15517, 419/194, 1019/470, 8498/3867 \approx 2.1975, 9/4\}.$$

Proposition 6.11. *A Teichmüller curve C generated by a flat surface $(X,\omega) \in \Omega\overline{\mathcal{M}}_4(1,1,1,1,1,1)$ has*

$$L(C) \leq 5/2.$$

Proof. The argument is completely analogous to the stratum $\Omega\overline{\mathcal{M}}_4(2,1,1,1,1)$. $\square$

7. Genus five

In genus 5 only few strata have a non-varying sum of Lyapunov exponents. We summarize the behavior in Tables 3 and 4. Contrary to genus 4 we do not give an upper bound for the sum in all the (components of) strata where the sum is varying but provide only one example which often comes from the locus of hyperelliptic flat surfaces.

7.1. **The stratum $\Omega\mathcal{M}_5(8)^{\mathrm{even}}$.** In the case $\Omega\mathcal{M}_5(8)^{\mathrm{even}}$ we have

$$L_{(8)^{\mathrm{even}}} = 20/9, \quad s_{(8)^{\mathrm{even}}} = 8, \quad c_{(8)^{\mathrm{even}}} = 50/27.$$

Proposition 7.1. *Let C be a Teichmüller curve generated by a flat surface in $\Omega\mathcal{M}_5(8)^{\mathrm{even}}$. Then C does not intersect the Brill-Noether divisor BN_3^1 on $\overline{\mathcal{M}}_5$.*

Proof. Teichmüller curves in this stratum are disjoint from the hyperelliptic locus even at the boundary of $\overline{\mathcal{M}}_5$, since the hyperelliptic component is a different component and by Proposition 4.4. Suppose that (X,ω) is a flat surface contained in the intersection of C and BN_3^1. Since X is trigonal (possibly nodal but irreducible)

Degrees of zeros $(d_1,\dots,d_n)$	Hyperelliptic or spin structure	Lyapunov exponents				
		Component		Teichmüller curves		
		$\approx$	$\sum_{j=1}^{g}\lambda_j$	$\approx$	$\sum_{j=1}^{g}\lambda_j$	Reference
(8)	*hyperelliptic*	2.777778	$\frac{25}{9}$	Non-varying		Thm. 3.4
(8)	*even*	2.222222	$\frac{20}{9}$	Non-varying		Sec. 7.1
(8)	*odd*	2.111111	$\frac{19}{9}$	Non-varying		Sec. 7.2
$(7,1)$	–	2.227022	$\frac{2423}{1088}$	2.229062	$\frac{7133}{3200}$	Eq. (18)
$(6,2)$	*even*	2.301983	$\frac{178429}{77511}$	2.619047	$\frac{55}{21}$	$\mathcal{Q}(5,1,-1^{10})$
$(6,2)$	*odd*	2.190476	$\frac{46}{21}$	Non-varying ?		–
$(6,1,1)$	–	2.285384	$\frac{59332837}{25961866}$	2.785714	$\frac{39}{14}$	$\mathcal{Q}(5,2,-1^{11})$
$(5,3)$	–	2.250000	$\frac{9}{4}$	Non-varying		Sec. 7.3
$(5,2,1)$	–	2.300563	$\frac{4493}{1953}$	2.302594	$\frac{48541}{21081}$	Eq. (19)
$(5,1,1,1)$	–	2.340909	$\frac{103}{44}$	2.337802	$\frac{12381}{5296}$	Eq. (20)
$(4,4)$	*hyperelliptic*	3.000000	3	Non-varying		Thm. 3.4
$(4,4)$	*even*	2.311111	$\frac{104}{45}$	2.400000	$\frac{12}{5}$	Eq. (21)
$(4,4)$	*odd*	2.191613	$\frac{228605}{104309}$	2.600000	$\frac{13}{5}$	$\mathcal{Q}(3,3,-1^{10})$
$(4,3,1)$	–	2.306255	$\frac{438419}{190100}$	2.302715	$\frac{777627}{337700}$	Eq. (23)
$(4,2,2)$	*even*	2.374007	$\frac{34981}{14735}$	2.800000	$\frac{14}{5}$	$\mathcal{Q}(4,3,-1^{11})$
$(4,2,2)$	*odd*	2.260315	$\frac{538102}{238065}$	2.466666	$\frac{37}{15}$	$\mathcal{Q}(3,1,1,-1^{9})$
$(4,2,1,1)$	–	2.354799	$\frac{646039}{274350}$	2.633333	$\frac{79}{30}$	$\mathcal{Q}(3,2,1,-1^{10})$

FIGURE 3. Varying and non-varying sums in genus five, part I

Degrees of zeros $(d_1,\ldots,d_n)$	Hyperelliptic or spin structure	Lyapunov exponents				
		Component		Teichmüller curves		
		$\approx$	$\sum_{j=1}^{g}\lambda_j$	$\approx$	$\sum_{j=1}^{g}\lambda_j$	Reference
$(4,1^4)$	–	2.393586	$\frac{640763}{267700}$	2.800000	$\frac{14}{5}$	$\mathcal{Q}(3,2,2,-1^{11})$
$(3,3,2)$	–	2.318020	$\frac{61307}{26448}$	2.833333	$\frac{17}{6}$	$\mathcal{Q}(6,1,-1^{11})$
$(3,3,1,1)$	–	2.358542	$\frac{47435}{20112}$	3.000000	3	$\mathcal{Q}(6,2,-1^{12})$
$(3,2,2,1)$	–	2.366588	$\frac{6049}{2556}$	2.362268	$\frac{2041}{864}$	Eq. (24)
$(3,2,1^3)$	–	2.405498	$\frac{700}{291}$	2.398764	$\frac{3495}{1457}$	Eq. (25)
$(3,1^5)$	–	2.443023	$\frac{2101}{860}$	2.431085	$\frac{77785}{31996}$	Eq. (26)
$(2,2,2,2)$	*even*	2.434379	$\frac{2096}{861}$	2.666666	$\frac{8}{3}$	$\mathcal{Q}(4,1,1,-1^{10})$
$(2,2,2,2)$	*odd*	2.319961	$\frac{355309}{153153}$	2.333333	$\frac{7}{3}$	$\mathcal{Q}(1,1,1,1,-1^{8})$
$(2^3,1^2)$	–	2.413574	$\frac{79981}{33138}$	2.833333	$\frac{17}{6}$	$\mathcal{Q}(4,2,1,-1^{11})$
$(2,2,1^4)$	–	2.451217	$\frac{266761}{108828}$	2.666666	$\frac{8}{3}$	$\mathcal{Q}(2,2,1,1,-1^{10})$
$(2,1^6)$	–	2.487756	$\frac{35861}{14415}$	2.833333	$\frac{17}{6}$	$\mathcal{Q}(2,2,2,1,-1^{11})$
(1^8)	–	2.523451	$\frac{235761}{93428}$	3.000000	3	$\mathcal{Q}(2,2,2,2,-1^{12})$

FIGURE 4. Varying and non-varying sums in genus five, part II

and is not hyperelliptic, its canonical model lies on a cubic scroll in $\mathbb{P}^4$ whose ruling is spanned by the sections of the g^1_3, cf. e.g. [ACGH85, Chap. III].

Suppose that the scroll is smooth as the embedding of the Hirzebruch surface F_1 by the linear system $|e+2f|$, where

$$e^2=-1,\ e\cdot f=1,\ f^2=0.$$

Then X has class $3e+5f$. Note that $4p$ admits a g^1_4, which comes from the projection of X from a plane Λ to a line in $\mathbb{P}^4$. This plane Λ intersects X at ≥ 4 points (with multiplicity) and the intersection contains the residual $4p$. But F_1 has degree three, so the intersection $F_1\cap\Lambda$ consists of a curve B with possibly finitely many points outside B. If B is a ruling, then $B\cdot X=3$ and Λ also intersects X at a point outside B, such that Λ is spanned by B and that point. Then we cannot have $4p \subset \Lambda \cap X$, contradiction. If B has higher degree, it can only be a conic (or its degeneration) of class $e+f$. Then $B \cdot X = 5$, so $\Lambda \cap X = 4p + q$ admits a g^2_5 by Geometric Riemann-Roch. For $q \neq p$ we obtain $h^0(\mathcal{O}_X(4p-q)) = 2$ and hence $h^0(\mathcal{O}_X(4p)) = 3$, contradiction. For $q = p$, the residual $3p$ admits a g^1_3, so it gives rise to a ruling L on the cubic scroll. Then L and B are both tangent to X at p. But $L \cdot B = f \cdot (e+f) = 1$, contradiction.

If the scroll is singular, it is isomorphic to F_3 by the linear system $|e+3f|$, where

$$e^2 = -3, \; e \cdot f = 1, \; f^2 = 0.$$

Since $X \cdot f = 3$ and $X \cdot (e+3f) = 8$, it has class $3e + 8f$. Then $X \cdot e = -1$, which implies that X consists of e union a curve of class $2e + 8f$, contradicting the irreducibility of X. □

Proof of Theorem 1.3, Case $\Omega\mathcal{M}_5(8)^{\mathrm{even}}$. By the proposition we have $C \cdot BN^1_3 = 0$. Since this divisor has slope equal to 8, cf. (4), the Teichmüller curve C has the same slope $s(C) = 8$. □

7.2. **The stratum $\Omega\mathcal{M}_5(8)^{\mathrm{odd}}$.** In the case $\Omega\mathcal{M}_5(8)^{\mathrm{odd}}$ we have

$$L_{(8)^{\mathrm{odd}}} = 19/9, \quad s_{(8)^{\mathrm{odd}}} = 148/19, \quad c_{(8)^{\mathrm{odd}}} = 37/27.$$

Proposition 7.2. *Let C be a Teichmüller curve generated by a flat surface $(X,\omega) \in \Omega\mathcal{M}_5(8)^{\mathrm{odd}}$ lifted to $\overline{\mathcal{M}}_{5,1}$ using the zero of ω. Then C does not intersect the divisor* $\mathrm{Nfold}^1_{5,4}(1)$.

Proof. Suppose that (X,ω) is contained in the intersection of C with $\mathrm{Nfold}^1_{5,4}(1)$. Note that X is not hyperelliptic, as this component and the hyperelliptic component are disjoint and by Proposition 4.4. Recall that $\mathrm{Nfold}^1_{5,4}(1) \subset \overline{\mathcal{M}}_{5,1}$ parameterizes curves that admit a g^1_4 given by $3p+q$, where p is the marked point and q is a random point. Then it implies that $h^0(\mathcal{O}_X(3p+q)) = 2$ and $h^0(\mathcal{O}_X(4p)) = 1$ by Clifford's theorem, hence we have $q \neq p$. By Riemann-Roch, we have $h^0(\mathcal{O}_X(5p-q)) = 2$. Since $h^0(\mathcal{O}_X(5p)) = 2 = h^0(\mathcal{O}_X(5p-q))$, it implies that q is a base point of $|\mathcal{O}_X(5p)|$. This is impossible. □

Proof of Theorem 1.3, Case $\Omega\mathcal{M}_5(8)^{\mathrm{odd}}$. By the proposition we have

$$C \cdot \mathrm{Nfold}^1_{5,4}(1) = 0.$$

It now suffices to plug the result of Proposition 4.5 with $m_1 = 8$ into the divisor class of $\mathrm{Nfold}^1_{5,4}(1)$ in (9) to obtain the desired numbers. □

7.3. **The stratum $\Omega\mathcal{M}_5(5,3)$.** In the case $\Omega\mathcal{M}_g(5,3)$ we have

$$L_{(5,3)} = 9/4, \quad s_{(5,3)} = 209/27, \quad c_{(5,3)} = 209/144.$$

Proposition 7.3. *Let C be a Teichmüller curve generated by a flat surface $(X,\omega) \in \Omega\mathcal{M}_5(5,3)$, lifted to $\overline{\mathcal{M}}_{5,2}$ by the zeros of ω. Then C does not intersect the divisor* $BN^1_{4,(1,2)}$.

Proof. Note that the degenerate fibers of the family over C are either irreducible or consist of two components connected by an odd number (≥ 3) of nodes by Proposition 4.3. Moreover by Proposition 4.4 the degenerate fibers are not hyperelliptic. Consequently ω_X is very ample by [Has99, Prop. 2.3].

Suppose that contrary to the claim, (X,p,q) is contained in the intersection of C and $BN^1_{4,(1,2)}$, i.e. $h^0(\mathcal{O}_X(2p+q+r)) = 2$ for some $r \in X$. Since $\omega_X \sim \mathcal{O}_X(5p+3q)$, by Riemann-Roch we have $h^0(\mathcal{O}_X(3p+2q-r)) = 2$. For $r = p$ or q, these equalities reduce to $h^0(\mathcal{O}_X(2p+2q)) = 2$ and $h^0(\mathcal{O}_X(3p+q)) = 2$. If $r \neq p, q$, then r is not a base point of $|\mathcal{O}_X(3p+2q)|$, hence $h^0(\mathcal{O}_X(3p+2q)) = 3$ and $h^0(\mathcal{O}_X(2p+q)) = 2$. Then we still have $h^0(\mathcal{O}_X(2p+2q)) = h^0(3p+q) \geq 2$, hence they are equal to 2 by Clifford's theorem. In any case, $3p+q$ and $2p+2q$ span two different planes with the corresponding contact orders at p and q to the canonical image of X in $\mathbb{P}^4$. The two planes contain a common line spanned by p, q whose intersection with X is $2p+q$. By Geometric Riemann-Roch, $2p+q$ gives rise to a g^1_3, hence X is trigonal.

For a trigonal genus 5 curve X with ω_X very ample, its canonical image is contained in a cubic scroll in $\mathbb{P}^4$. The residual g^2_5 given by $3p+2q$ maps X to a plane quintic Y, whose image differs from X at a double point u (like a node or cusp), for the arithmetic genus of Y is 6. The unique g^1_3 on X is given by intersections of lines passing through u with Y (subtracting $2u$ from the base locus). But $2p+q$ is contained in the g^1_3, and the line spanned by p, q has contact order 3 at p and 2 at q to Y, hence u must be p or q by Bézout. For $u = q$, subtracting $2u$ from $3p+2q$, we know that $3p$ is also in the g^1_3, hence $3p \sim 2p+q$, $p \sim q$, impossible. For $u = p$, we have $p+2q$ is in the g^1_3. Hence $p+2q \sim 2p+q$, which implies $p \sim q$ and this is also impossible. $\square$

Proof of Theorem 1.3, Case $\Omega\mathcal{M}_5(5,3)$. Proposition 7.3 says that $C \cdot BN^1_{4,(1,2)} = 0$ for a Teichmüller curve C in this stratum. Using the divisor class of $BN^1_{4,(1,2)}$ in (10) together with Proposition 4.5, the result follows immediately. $\square$

7.4. **The stratum $\Omega\mathcal{M}_5(6,2)^{\mathrm{odd}*}$.** In the case $\Omega\mathcal{M}_5(6,2)^{\mathrm{odd}}$ we have

$$L_{(6,2)^{\mathrm{odd}}} = 46/21, \quad s_{(6,2)^{\mathrm{odd}}} = 176/23, \quad c_{(6,2)^{\mathrm{odd}}} = 209/144.$$

Based on numerical values on individual Teichmüller curves, we believe that the sum of Lyapunov exponents is non-varying in this stratum. But we have not discovered a divisor to carry out the desired disjointness argument.

7.5. **Examples of square-tiled surfaces in $g = 5$ and $g = 6$.** In this section we list examples of square-tiled surfaces in $g = 5$ to justify that the sum of Lyapunov exponents in the remaining strata is indeed varying.

In the stratum $\Omega\mathcal{M}_5(7,1)$ varying sum can be checked using the square-tiled surface

$$(\pi_r = (123456789\,10), \pi_u = (1596)(247\,10)). \tag{18}$$

In the stratum $\Omega\mathcal{M}_5(5,2,1)$ varying sum can be checked using the square-tiled surface

$$(\pi_r = (123456789\,10\,11), \pi_u = (1\,11)(23)(46)(79)). \tag{19}$$

In the stratum $\Omega\mathcal{M}_5(5,1,1,1)$ varying sum can be checked using the square-tiled surface

$$(\pi_r = (123456789\,10\,11\,12), \pi_u = (1\,12)(23)(46)(810)). \tag{20}$$

In the stratum $\Omega\mathcal{M}_5(4,4)^{\mathrm{even}}$ varying sum can be checked using the square-tiled surface

$$(\pi_r = (123456789\,10), \pi_u = (1\,10)(29)(3568)). \tag{21}$$

In the stratum $\Omega\mathcal{M}_5(4,4)^{\rm odd}$ varying sum can be cross-checked using, besides the hyperelliptic locus, the square-tiled surface

$$(\pi_r = (123456789\,10), \pi_u = (1\,10)(23)(56)(78)). \tag{22}$$

In the stratum $\Omega\mathcal{M}_5(4,3,1)$ varying sum can be checked using the square-tiled surface

$$(\pi_r = (123456789\,10\,11\,12), \pi_u = (11\,194)(2\,10\,356)(7\,12)). \tag{23}$$

In the stratum $\Omega\mathcal{M}_5(3,2,2,1)$ varying sum can be checked using the square-tiled surface

$$(\pi_r = (123)(456789\,10\,11\,12), \pi_u = (1\,11)(10\,5\,13)(27)). \tag{24}$$

In the stratum $\Omega\mathcal{M}_5(3,2,1,1,1)$ varying sum can be checked using the square-tiled surface

$$(\pi_r = (123456789\,10\,11\,12), \pi_u = (1\,12)(24)(57)(8\,10)). \tag{25}$$

In the stratum $\Omega\mathcal{M}_5(3,1,1,1,1,1)$ varying sum can be checked using the square-tiled surface

$$(\pi_r = (123456789\,10\,11\,12\,13), \pi_u = (1\,14)(24)(68)(10\,12)). \tag{26}$$

To indicate that the phenomenon of non-varying sum of Lyapunov exponents is restricted to low genus and special loci, such as e.g. the hyperelliptic locus, we show that already in $g = 6$ the best candidates fail.

Proposition 7.4. *For $g = 6$ the sum of Lyapunov exponents is varying in the strata $\Omega\mathcal{M}_6(10)^{\rm odd}$ and $\Omega\mathcal{M}_6(10)^{\rm even}$.*

Proof. For $\Omega\mathcal{M}_6(10)^{\rm odd}$ the sum for the measure supported on the whole stratum is $L_{(10)^{\rm odd}} = \frac{82680540070}{35169130909}$ using [EMZ03], but the square-tiled surface

$$(\pi_r = (123456789\,10\,11), \pi_u = (13579\,11)) \tag{27}$$

provides an example with $L(C) = \frac{3166}{1375}$. In the even case $L_{(10)^{\rm even}} = \frac{9085753953118}{3770001658049}$ but the square-tiled surface

$$(\pi_r = (123456789\,10\,11), \pi_u = (1579\,11)(24)) \tag{28}$$

gives an example with $L(C) = \frac{244729}{101893}$. □

8. The hyperelliptic strata and moduli spaces of pointed curves

Using Teichmüller curves in the hyperelliptic strata we reverse our engine to present an application for the geometry of moduli spaces of pointed curves.

In the study of the geometry of a moduli space, a central question is to ask about the extremity of a divisor class, e.g. if it has non-negative intersection numbers with various curve classes on the moduli space. Now consider the moduli space $\overline{\mathcal{M}}_{g,1}$ of genus g curves with one marked point. Define a divisor class

$$D_1 = 4g(g-1)\omega_{\rm rel} - 12\lambda + \delta,$$

where δ is the total boundary class. Let $\mathcal{X} \to B$ be a one-dimensional family of stable one-pointed curves with smooth generic fibers. Harris and Morrison [HM98, (6.31)] showed that $D_1 \cdot B$ is always non-negative. They asked further [HM98, Prob. (6.34)] if this is optimal, i.e. if there exists such a family B satisfying $D_1 \cdot B = 0$.

Define another divisor class on the moduli space $\overline{\mathcal{M}}_{g,2}$ of genus g curves with two marked points:

$$D_2 = (g^2-1)(\psi_1+\psi_2) - 12\lambda + \delta,$$

where ψ_i is the first Chern class of the cotangent line bundle associated to the i-th marked point. By a completely analogous argument as in [HM98, (6.31)], one easily checks that D_2 has non-negative intersection with any one-dimensional family of stable two-pointed curves with smooth generic fibers. Similarly we can ask if this is optimal. Below we show that in both cases the zero-intersection can be attained.

Theorem 8.1. *Let C_1, C_2 be Teichmüller curves in $\Omega\mathcal{M}_g(2g-2)^{\mathrm{hyp}}$ and $\Omega\mathcal{M}_g(g-1,g-1)^{\mathrm{hyp}}$, lifted to $\overline{\mathcal{M}}_{g,1}$ and $\overline{\mathcal{M}}_{g,2}$ using the zeros of Abelian differentials, respectively. Then we have*

$$C_1\cdot D_1 = 0,$$
$$C_2\cdot D_2 = 0.$$

Proof. By the value of $s(C_1)$ given in Corollary 3.5 and Proposition 4.5, we obtain that

$$\frac{C_1\cdot\lambda}{C_1\cdot\omega_{\mathrm{rel}}} = g^2,$$
$$\frac{C_1\cdot\delta}{C_1\cdot\omega_{\mathrm{rel}}} = 4g(2g+1).$$

Plugging them into the intersection $C_1\cdot D_1$, an elementary calculation shows that $C_1\cdot D_1 = 0$.

Similarly we have

$$\frac{C_2\cdot\lambda}{C_2\cdot(\psi_1+\psi_2)} = \frac{g(g+1)}{4},$$
$$\frac{C_2\cdot\delta}{C_2\cdot(\psi_1+\psi_2)} = (g+1)(2g+1).$$

One easily checks that $C_2\cdot D_2 = 0$. □

Since square-tiled surfaces in a stratum correspond to 'lattice points' under the period map coordinates, the union of all such Teichmüller curves C_1, C_2 forms a Zariski dense subset in the hyperelliptic locus. Hence they provide infinitely many solutions to the above question.

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