Prym covers, theta functions and Kobayashi curves in Hilbert modular surfaces

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November 13, 2012

Abstract

Algebraic curves in Hilbert modular surfaces that are totally geodesic for the Kobayashi metric have very interesting geometric and arithmetic properties, e.g. they are rigid. There are very few methods known to construct such algebraic geodesics that we call Kobayashi curves.

We give an explicit way of constructing Kobayashi curves using determinants of derivatives of theta functions. This construction also allows to calculate the Euler characteristics of the Teichmüller curves constructed by McMullen using Prym covers.

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Introduction

We call *Kobayashi curves* algebraic curves on Hilbert modular surfaces that are totally geodesic for the Kobayashi metric. They are rigid ([MV10]) and interesting both from geometric and arithmetic point of view and there are very few methods to construct Kobayashi curves. This paper can be read from two perspectives, the construction of Kobayashi curves and the calculation of invariants of Teichmüller curves.

The most obvious Kobayashi curve on a Hilbert modular surface $X_D = (\mathbb{H}\times\mathbb{H})/\mathrm{SL}(\mathfrak{o}_D \oplus \mathfrak{o}_D^\vee)$ is the image of the diagonal in $\mathbb{H}\times\mathbb{H}$. One can twist it by a matrix $M \in \mathrm{GL}_2^+(\mathbb{Q}(\sqrt{D})$, i.e. consider the image of $z \mapsto (Mz, M^\sigma z)$ where $\langle\sigma\rangle = \mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q})$. In this way one obtains further Kobayashi curves, also known as Hirzebruch-Zagier cycles or Shimura curves. These curves are more special, metrically they are even geodesic for the invariant Riemannian metric on $\mathbb{H} \times \mathbb{H}$. The first Kobayashi curves without this supplementary property were constructed implicitly in [McM03]. They were constructed as Teichmüller curves $W_D \subset \mathcal{M}_2$, the name refers to their construction using Weierstraß points. Their image under the Torelli map lies in X_D and in this sense the curves W_D generalize the modular embeddings of triangle groups in [CW90] that cover some small discriminants D. One can apply a twist by a Möbius transformation $M \in \mathrm{GL}_2^+(\mathbb{Q}(\sqrt{D}))$ also to W_D to obtain more Kobayashi curves. The geometry of the resulting curves is studied in [Wei12].

Not all the Kobayashi curves in X_D arise as twists of W_D or of the diagonal. In fact, it is shown in [Wei12] that an invariant (second Lyapunov exponent) of a Kobayashi curve is unchanged under twisting. Moreover, the image of other Teichmüller curves $W_D(6)$ constructed in [McM06a] map to curves W_D^X in X_D that are also Kobayashi curves and that have a second Lyapunov exponent different from the one of the diagonal and the second Lyapunov exponent of W_D.

The first aim of this paper is to explicitly construct modular forms whose vanishing loci are these Kobayashi curves W_D^X. This part is a continuation of [MZ11]. There, a theta function interpretation of the first series of Teichmüller curves W_D has been found. Having an explicit modular form at hand can be used to determine the period map explicitly as power series (whereas this is a great mystery from the Teichmüller curve perspective) and it can be used to retrieve all combinatorial properties of W_D (e.g. the set of cusps, etc.) from an arithmetic perspective without using the geometry of flat surfaces.

To state the first result, let $\theta_0, \theta_1, \theta_2, \theta_3$ be the classical theta constants with characteristic $(c_1, 0)$ for $c_1 \in \frac{1}{2}\mathbb{Z}^2/\mathbb{Z}^2$. The precise definition is given in Section 1. We write f' for the derivative of f in the direction z_2, where (z_1, z_2) are the coordinates in $\mathbb{H} \times \mathbb{H}$.

Theorem 0.1. *The determinant of derivatives of theta functions*

$$G_D^X(\mathbf{z}) = \begin{vmatrix} \theta_0(\mathbf{z}) & \theta_1(\mathbf{z}) & \theta_2(\mathbf{z}) & \theta_3(\mathbf{z}) \\ \theta_0'(\mathbf{z}) & \theta_1'(\mathbf{z}) & \theta_2'(\mathbf{z}) & \theta_3'(\mathbf{z}) \\ \theta_0''(\mathbf{z}) & \theta_1''(\mathbf{z}) & \theta_2''(\mathbf{z}) & \theta_3''(\mathbf{z}) \\ \theta_0'''(\mathbf{z}) & \theta_1'''(\mathbf{z}) & \theta_2'''(\mathbf{z}) & \theta_3'''(\mathbf{z}) \end{vmatrix}$$

is a modular form of weight $(2, 14)$ *for* $\mathrm{SL}(\mathfrak{o}_D \oplus \mathfrak{o}_D^\vee)$. *Its vanishing locus is the Kobayashi curve* W_D^X.

Determining invariants of Teichmüller curves is the motivation to a variant of this construction for non-principally polarized abelian varieties. From that point of view this paper is jointly with the work of [LM11] a continuation of [McM06a].

Let $W_D(6)$ be the Prym Teichmüller curves in $\mathcal{M}_4$ and let $W_D(4)$ be the Prym Teichmüller curves in $\mathcal{M}_3$. The notation refers to their construction using holomorphic one-forms with a 6-fold resp. with a 4-fold zero. (See Section 1 for the definitions and the construction of these curves via flat surfaces.)

Theorem 0.2. *For the Prym Teichmüller curves $W_D(6) \subset \mathcal{M}_4$ the Euler characteristic is given by*

$$\chi(W_D(6)) = -7\chi(X_D).$$

For genus three and $D \equiv 5(8)$ the locus $W_D(4)$ is empty. For $D \equiv 0, 4(8)$ we have

$$\chi(W_D(4)) = -\frac{5}{2}\chi(X_{D,(1,2)})$$

and for $D \equiv 1(8)$ there are two components $W_D(4)^1$ and $W_D(4)^2$ with

$$\chi(W_D(4)^j) = -\frac{5}{2}\chi(X_{D,(1,2)}) \quad \textit{for} \quad j = 1, 2.$$

Here $X_{D,(1,2)}$ is the locus in the moduli space of $(1,2)$-polarized abelian surfaces parameterizing surfaces with a choice of real multiplication, a Hilbert modular surface for some Hilbert modular group commensurable to the standard Hilbert modular groups. We give a precise definition and a way to evaluate explicitly the Euler characteristic in Section 1. The preceding theorem does not prove that $W_D(4)$ nor $W_D(4)^j$ is irreducible. This important result is shown in [LM11]. Connectedness of $W_D(6)$ is conjectured in [LM11] with evidence given by small discriminants.

Pictures of the flat surfaces generating the Teichmüller curves W_D, $W_D(6)$, $W_D(4)$ are drawn in Figure 1. Note that there is presently no algorithm known to compute the group uniformizing the curves W_D directly, i.e. using the geometry of the generating flat surfaces, if D is larger than some small explicit constant and thus W_D not a rational curve. The same statement holds for $W_D(6)$ and for $W_D(4)$.

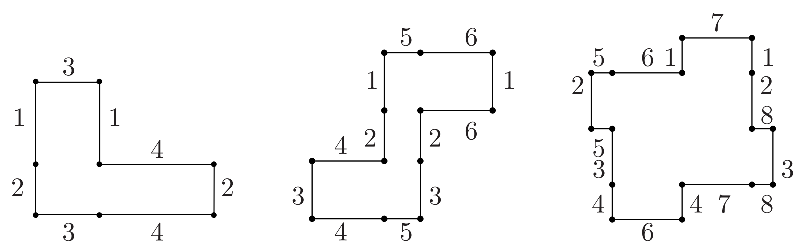

Figure 1: Flat surfaces generating respectively the Teichmüller curves W_D, $W_D(6)$ and $W_D(4)$ (from [McM06a]). Admissible side lengths are given in loc. cit. and in [LM11].

In order to calculate $\chi(W_D(4))$ we construct Hilbert modular forms for the Hilbert modular groups associated with $X_{D,(1,2)}$ whose vanishing loci are the image curves W_D^S of $W_D(4)$ in X_D. The statement that plays the role analogous to Theorem 0.2 for this case is Proposition 3.5. As for G_D^X, these Hilbert modular forms G_D^S are very 'canonical', determinants of derivatives of theta functions. Their precise from is stated in Proposition 3.5. The Euler characteristic of

$W_D(6)$ is evaluated, too, using modular forms, without ever referring to the geometry of flat surfaces. This strategy was first carried out by Bainbridge ([Bai07]) to compute the Euler characteristic of the curves W_D. There, on the contrary, the modular form cutting out W_D was first described using relative periods, i.e. flat surface geometry.

The construction raises the question to construct more, even to determine all Kobayashi curves on Hilbert modular surfaces. The construction of modular forms using determinants of derivatives of theta functions has a natural analog using higher multiples of the principal polarization and higher order derivatives. Yet, showing that these modular forms define Kobayashi curves requires techniques different from the ones used here. Some speculations in this direction are the content of Section 5.

We summarize that we are interested in two series of Teichmüller curves

$$W_D(6) \subset \mathcal{M}_g(4) \quad \text{and} \quad W_D(4) \subset \mathcal{M}_g(3).$$

Using modular forms we construct curves

$$W_D^X \subset X_D \quad \text{and} \quad W_D^L \subset X_{D,(1,2)}$$

so that under a Prym Torelli map

$$W_D(6) \to W_D^X \quad \text{and} \quad W_D(4) \to W_D^S.$$

It turns out in Section 4 that these two maps are actually bijections, hence that introducing two names for these curves served for technical purposes only. The proofs of both Theorem 0.1 and Theorem 0.2 appear at the end of Section 4.

The author thanks Olivier Debarre, Sam Grushevsky, Erwan Lanneau, Curt McMullen and Don Zagier for discussions and the referees for useful comments.

1 Background

In this section we develop the notions of real multiplication, Hilbert modular surfaces and their embedding into the moduli space of principally polarized abelian varieties in some detail. This is well-known, but the variant for $(1, 2)$-polarizations, that we also need, is treated in less detail in the literature. We end with some generalities on Kobayashi curves.

Hilbert modular surfaces. Let $\mathfrak{o} = \mathfrak{o}_D$ be the order of discriminant D in the quadratic field $K = \mathbb{Q}(\sqrt{D})$ with σ a generator of the Galois group of $K/\mathbb{Q}$. We fix once and for all two embeddings $\iota_1, \iota_2 : K \to \mathbb{R}$ and implicitly use the first embedding unless stated differently. We denote by X_D the Hilbert modular surface of discriminant D, i.e. $X_D = (\mathbb{H} \times \mathbb{H})/\mathrm{SL}(\mathfrak{o}_D \oplus \mathfrak{o}_D^\vee)$., see e.g. [vdG88] for this and other standard facts. These Hilbert modular surfaces parameterize principally polarized abelian varieties (A, Θ) with a choice of real multiplication by $\mathfrak{o}_D$, that is with the choice of a map $\mathfrak{o}_D \to \mathrm{End}(A)$. We now introduce in detail two types of coordinates on these abelian varieties, since we will soon also need the less well studied non-principally polarized version.

To a point $\mathbf{z} = (z_1, z_2) \in \mathbb{H} \times \mathbb{H}$ we associate the polarized abelian variety $A_{\mathbf{z}} = \mathbb{C}^2/\Lambda_{\mathbf{z}}$ where $\Lambda_{\mathbf{z}}$ is the lattice

$$\Lambda_{\mathbf{z}} = \{(a + bz_1, a^\sigma + b^\sigma z_2)^T \mid a \in \mathfrak{o}_D, b \in \mathfrak{o}_D^\vee\} \tag{1.1}$$

We denote the coordinates of $\mathbb{C}^2$ by $\mathbf{u} = (u_1, u_2)^T$ and we see that that real multiplication is given for $\lambda \in \mathfrak{o}_D$ by $\lambda \cdot (u_1, u_2)^T = (\lambda u_1, \lambda^\sigma u_2)^T$. Consequently, the holomorphic one-forms du_1 and du_2 on $A_{\mathbf{z}}$ are eigenforms for real multiplication and we refer to $\mathbf{u}$ as *eigenform coordinates*. We also say that du_1 is the eigenform for ι, unique up to scalar, i.e. where $\lambda \in K$ acts by $\lambda du_1 = \iota_1(\lambda) \cdot du_1$. Note that here and in the sequel we represent the universal covering $\mathbb{C}^2$ of $A_{\mathbf{z}}$ by column vectors.

The pairing

$$\langle (a, b), (\widetilde{a}, \widetilde{b}) \rangle = \mathrm{Tr}_{\mathbb{Q}}^K (a\widetilde{b} - \widetilde{a}b) \tag{1.2}$$

is integer valued on $\mathfrak{o}_D \oplus \mathfrak{o}_D^\vee$ (which we identify with $\Lambda_{\mathbf{z}}$) and unimodular by definition of $\mathfrak{o}_D^\vee$. It thus defines a principal polarization on $A_{\mathbf{z}}$.

Associated with any choice of $\mathbb{Z}$-basis (η_1, η_2) of $\mathfrak{o}_D$ satisfying the sign convention $\mathrm{Tr}_{\mathbb{Q}}^K(\eta_1 \eta_2^\sigma/\sqrt{D}) = 1$ there is a symplectic basis of $\mathfrak{o}_D \oplus \mathfrak{o}_D^\vee$ given by

$$a_1 = (\eta_1, 0)^T, a_2 = (\eta_2, 0)^T, b_1 = (0, \eta_2^\sigma/\sqrt{D})^T, b_2 = (0, -\eta_1^\sigma/\sqrt{D})^T.$$

In this basis of homology and in the eigenform coordinates, the period matrix of $A_{\mathbf{z}}$ is given by

$$\Pi_{\mathbf{u}} = \begin{pmatrix} \eta_1 & \eta_2 & \eta_2^\sigma z_1/\sqrt{D} & -\eta_1^\sigma z_1/\sqrt{D} \\ \eta_1^\sigma & \eta_2^\sigma & -\eta_2 z_2/\sqrt{D} & \eta_1 z_2/\sqrt{D} \end{pmatrix} = \begin{pmatrix} B & \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} A^T \end{pmatrix},$$

where

$$A = B^{-1}, \qquad B = \begin{pmatrix} \eta_1 & \eta_2 \\ \eta_1^\sigma & \eta_2^\sigma \end{pmatrix}. \tag{1.3}$$

The change of basis $\mathbf{v} = (v_1, v_2)^T = A \cdot \mathbf{u}$ results in multiplying $\Pi_{\mathbf{u}}$ from the left by A and gives

$$\Pi_{\mathbf{v}} = \begin{pmatrix} I_2 & A \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} A^T \end{pmatrix} \tag{1.4}$$

Since the period matrix of $A_{\mathbf{z}}$ is in standard form with respect to the basis $\mathbf{v} = (v_1, v_2)$ (and a $\mathbb{Z}$-basis of homology) we refer to $\mathbf{v}$ as *standard coordinates.*

Siegel modular embeddings. Any choice of a $\mathbb{Z}$-basis (η_1, η_2) of $\mathfrak{o}_D$, defines a map $\psi : \mathbb{H} \times \mathbb{H} \to \mathbb{H}_2$ that is equivariant with respect to a homomorphism $\Psi : \mathrm{SL}(\mathfrak{o}_D \oplus \mathfrak{o}_D^\vee) \to \mathrm{Sp}(4, \mathbb{Z})$ and descends to a map $X_D \to \mathcal{A}_2$. This map is gerically two to one. Its image is the locus of abelian varieties with real multiplication (without making a choice among the Galois conjugates). The image is isomorphic to X_D^τ, the quotient of X_D by the involution τ, induced by swapping the two copies of $\mathbb{H}$.

Explicitly, from (1.4) we know that

$$\psi : \quad \mathbf{z} = (z_1, z_2) \mapsto A \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} A^T \tag{1.5}$$

where A and B are defined in (1.3). If we let

$$\Psi : \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} A & 0 \\ 0 & B^T \end{pmatrix} \begin{pmatrix} \widehat{a} & \widehat{b} \\ \widehat{c} & \widehat{d} \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & A^T \end{pmatrix}, \tag{1.6}$$

where $\widehat{a}$ for $a \in K$ denotes the diagonal matrix $\operatorname{diag}(a, \sigma(a))$, then the equivariance is easily checked.

Abelian surfaces with real multiplication and a $(1,2)$-polarization. We define $X_{D,(1,2)}$ to be the moduli space of $(1,2)$-polarized abelian varieties with a choice of real multiplication by $\mathfrak{o}_D$.

Proposition 1.1. *The moduli space $X_{D,(1,2)}$ is empty for $D \equiv 5$ (8). The moduli space $X_{D,(1,2)}$ is non-empty and irreducible both for $D \equiv 0,4$ (8) and for $D \equiv 1$ (8). It is irreducible for $D \equiv 0,4$ (8), but has two irreducible components for $D \equiv 1$ (8).*

The image of $X_{D,(1,2)}$ in the moduli space of $(1,2)$-polarized abelian varieties is irreducible both for $D \equiv 0,4$ (8) and $D \equiv 1$ (8).

Proof. Suppose that $(A = \mathbb{C}^2/\Lambda, \mathcal{L})$ is an abelian variety with real multiplication and a $(1,2)$-polarization $\mathcal{L}$. Then Λ is a rank-two $\mathfrak{o}_D$-module with symplectic pairing of signature $(1,2)$. By [Bas62] such a lattice splits as a direct sum of $\mathfrak{o}_D$-modules. Moreover, although $\mathfrak{o}_D$ is not a Dedekind domain for D a non-fundamental discriminant, any rank-two $\mathfrak{o}_D$-module is isomorphic to $\mathfrak{a} \oplus \mathfrak{o}_D^\vee$ for some fractional $\mathfrak{o}_D$-ideal $\mathfrak{a}$. The isomorphism can moreover be chosen so that the symplectic form is mapped to the trace pairing (1.2). In this normalization, if $N_{\mathbb{Q}}^F(\mathfrak{a}) = h$, then the polarization has degree h. Since all polarizations of degree two are of type $(1,2)$, the locus $X_{D,(1,2)}$ is non-empty if and only if there is a fractional $\mathfrak{o}_D$ ideal $\mathfrak{a}$ with $N_{\mathbb{Q}}^F(\mathfrak{a}) = 2$, i.e. if and only if $D \not\equiv 5$ (8).

Generalizing (1.1) we define for any ideal $\mathfrak{b}$ and $\mathbf{z} \in \mathbb{H} \times \mathbb{H}$ the lattice

$$\Lambda_{\mathbf{z}}^{\mathfrak{b}} = \{(a + bz_1, a^\sigma + b^\sigma z_2)^T \mid a \in \mathfrak{b}, b \in \mathfrak{o}^\vee\}. \tag{1.7}$$

For $D \equiv 0,4$ (8) there is exactly one prime ideal $\mathfrak{a}$ of norm two, so as in the principally polarized case one shows that $X_{D,(1,2)} = (\mathbb{H} \times \mathbb{H})/\mathrm{SL}_2(\mathfrak{a} \oplus \mathfrak{o}_D^\vee)$ is connected.

For $D \equiv 1$ (8) the prime two splits $(2) = \mathfrak{a}\mathfrak{a}^\sigma$ into two prime ideals of norm two. Consequently, both $(\mathbb{H} \times \mathbb{H})/\mathrm{SL}_2(\mathfrak{a} \oplus \mathfrak{o}_D^\vee)$ and $(\mathbb{H} \times \mathbb{H})/\mathrm{SL}_2(\mathfrak{a}^\sigma \oplus \mathfrak{o}_D^\vee)$ parameterize $(1,2)$-polarized abelian varieties with real multiplication by $\mathfrak{o}_D$ given by $\mathbb{C}^2/\Lambda_{\mathbf{z}}^{\mathfrak{b}}$ and $\mathbb{C}^2/\Lambda_{\mathbf{z}}^{\mathfrak{b}^\sigma}$. Since $\mathfrak{o}_D^\vee = (\mathfrak{o}_D^\vee)^\sigma$, the map $(u_1, u_2) \mapsto (u_2, u_1)$ defines for any fractional ideal $\mathfrak{b}$ an isomorphism

$$\mathbb{C}^2/\Lambda_{(z_1,z_2)}^{\mathfrak{b}} \to \mathbb{C}^2/\Lambda_{(z_2,z_1)}^{\mathfrak{b}^\sigma} \tag{1.8}$$

of abelian varieties. Consequently, both the surfaces $(\mathbb{H}\times\mathbb{H})/\mathrm{SL}_2(\mathfrak{a}\oplus\mathfrak{o}_D^\vee)$ and $(\mathbb{H}\times\mathbb{H})/\mathrm{SL}_2(\mathfrak{a}^\sigma\oplus\mathfrak{o}_D^\vee)$ parameterize the same subset of the moduli space of $(1,2)$-polarized abelian varieties. ■

Period matrices. We choose from now on a symplectic basis of (η_1,η_2) of $\mathfrak{o}_D$, such that $(\eta_1, 2\eta_2)$ is a basis of $\mathfrak{a}$ and such that the sign convention $\mathrm{Tr}^K(\eta_1\eta_2^\sigma/\sqrt{D}) = 1$ holds. Let $P = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ be the diagonal matrix of the type of the polarization we are interested in. With this choice of generators, the basis

$$a_1 = (\eta_1, 0)^T, a_2 = (2\eta_2, 0)^T, b_1 = (0, \eta_2^\sigma/\sqrt{D})^T, b_2 = (0, -\eta_1^\sigma/\sqrt{D})^T.$$

is in standard form with respect to the trace polarization. In this basis the period matrix is

$$\Pi_{\mathbf{u}} = \begin{pmatrix} \eta_1 & 2\eta_2 & \eta_2^\sigma z_1/\sqrt{D} & -\eta_1^\sigma z_1/\sqrt{D} \\ \eta_1^\sigma & 2\eta_2^\sigma & -\eta_2 z_2/\sqrt{D} & \eta_1 z_2/\sqrt{D} \end{pmatrix} = \begin{pmatrix} B & \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} A^T \end{pmatrix},$$

where

$$A = PB^{-1}, \qquad B = \begin{pmatrix} \eta_1 & 2\eta_2 \\ \eta_1^\sigma & 2\eta_2^\sigma \end{pmatrix}. \tag{1.9}$$

The change of basis $\mathbf{v} = (v_1, v_2)^T = A\cdot\mathbf{u}$ results in multiplying $\Pi_{\mathbf{u}}$ from the left by A and gives

$$\Pi_{\mathbf{v}} = \begin{pmatrix} P & A\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} A^T \end{pmatrix}, \tag{1.10}$$

the period matrix for a $(1,2)$-polarized abelian surface with real multiplication in standard coordinates.

Euler characteristics. The notion Euler characteristic (of curves and of Hilbert modular surfaces) refers throughout to orbifold Euler characteristics. Let $D = f^2 D_0$ be the factorization of the discriminant into a fundamental discriminant D_0 and a square of $f\in\mathbb{N}$. The Euler characteristic of Hilbert modular surfaces has been computed by Siegel ([Sie36]). A reference including also the case of non-fundamental discriminants is [Bai07, Theorem 2.12]. Altogether,

$$\chi(X_D) = 2f^3\zeta_{\mathbb{Q}(\sqrt{D})}(-1)\left(\sum_{r|f}\left(\frac{D_0}{r}\right)\frac{\mu(r)}{r^2}\right), \tag{1.11}$$

where μ is the Möbius function and $\left(\frac{a}{b}\right)$ is the Jacobi symbol.

The groups $\mathrm{SL}(\mathfrak{o}_D\oplus\mathfrak{o}_D^\vee)$ and $\mathrm{SL}_2(\mathfrak{a}\oplus\mathfrak{o}_D^\vee)$ are commensurable. To determine the indices in their intersection, we conjugate both groups by $\begin{pmatrix} \sqrt{D} & 0 \\ 0 & 1 \end{pmatrix}$. This takes $\mathrm{SL}(\mathfrak{o}_D\oplus\mathfrak{o}_D^\vee)$ into $\mathrm{SL}(\mathfrak{o}_D\oplus\mathfrak{o}_D)$, and $\mathrm{SL}_2(\mathfrak{a}\oplus\mathfrak{o}_D^\vee)$ into $\mathrm{SL}_2(\mathfrak{a}\oplus\mathfrak{o}_D)$. The two images under conjugation contain

$$\Gamma_{\mathfrak{a}} = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\mathrm{SL}_2(K)\,:\, a,d\in\mathfrak{o}_D,\, b\in\mathfrak{a},\, c\in\mathfrak{o}_D\right\}. \tag{1.12}$$

with a finite index that we now calculate. We reduce mod $\mathfrak{a}$. Since $\mathfrak{o}_D/\mathfrak{a} \cong \mathbb{F}_2$, the group $\mathrm{SL}(\mathfrak{o}_D \oplus \mathfrak{o}_D)$ reduces to the full group $\mathrm{SL}_2(\mathbb{F}_2)$ and $\Gamma_\mathfrak{a}$ reduces to the group of lower triangular matrices. Thus $[\mathrm{SL}(\mathfrak{o}_D \oplus \mathfrak{o}_D^\vee) : \Gamma_\mathfrak{a}] = 3$.

Suppose the fractional ideal $\mathfrak{a}$ is not invertible, i.e. $2|f$, hence $D/4$ is also the discriminant of a ring. We claim that $\mathfrak{a}^{-1} = \mathfrak{o}_{D/4}$. In fact, $\mathfrak{a} = \langle 2, x\rangle_\mathbb{Z}$ for some x of norm 4. Then $\mathfrak{a}^{-1} = \langle 1, x^\sigma/2\rangle_\mathbb{Z} = \mathfrak{o}_{D/4}$. Using the claim we consider $\Gamma_\mathfrak{a}$ and $\mathrm{SL}_2(\mathfrak{a} \oplus \mathfrak{o}_D)$ as subgroups of $\mathrm{SL}(\mathfrak{o}_{D/4} \oplus \mathfrak{o}_{D/4})$. Both contain the kernel of the reduction mod $\mathfrak{a}$ to $\mathrm{SL}_2(\mathfrak{o}_{D/4}/\mathfrak{a})$. The images are groups of lower triangular matrices of size 2 and 4 respectively. We conclude that $[\mathrm{SL}(\mathfrak{a} \oplus \mathfrak{o}_D) : \Gamma_\mathfrak{a}] = 2$ in this case.

Suppose that $\mathfrak{a}$ is invertible. If $\mathfrak{a}$ is a principal ideal, generated by λ, then conjugation by $\mathrm{diag}(\lambda, 1)$ takes $\mathrm{SL}_2(\mathfrak{a} \oplus \mathfrak{o}_D)$ into $\mathrm{SL}(\mathfrak{o}_D \oplus \mathfrak{o}_D)$ and $\Gamma_\mathfrak{a}$ into the transposed group. Hence $[\mathrm{SL}(\mathfrak{a} \oplus \mathfrak{o}_D) : \Gamma_\mathfrak{a}] = 3$ in this case. The general case with $\mathfrak{a}$ invertible behaves similarly. Outside the prime(s) lying over the ideal (2) the groups $\mathrm{SL}(\mathfrak{a} \oplus \mathfrak{o}_D)$ and $\Gamma_\mathfrak{a}$ agree. We localize at $\mathfrak{a}$ and take the completion. Since $2 \not| f$, the ideal $\mathfrak{a}$ now becomes principal and by the preceding argument $[\mathrm{SL}(\mathfrak{a} \oplus \mathfrak{o}_D) : \Gamma_\mathfrak{a}]$ divides 3. Since the two groups are not the same, equality holds. Altogether we have shown the following proposition.

Proposition 1.2. *The Euler characteristic of $X_{D,(1,2)}$ and of X_D are related as follows.*

$$\frac{\chi(X_{D,(1,2)})}{\chi(X_D)} = \begin{cases} 1 & \text{if } 2 \not| f \\ 3/2 & \text{if } 2 \mid f. \end{cases} \tag{1.13}$$

Theta functions. Let $(A, \mathcal{L})$ be a P-polarized abelian surface, where P is a diagonal matrix, the type of the polarization. If we fix a basis of homology so that the polarization is in standard form $\begin{pmatrix} 0 & P \\ -P & 0 \end{pmatrix}$, then $A = \mathbb{C}^2/\Lambda$, where $\Lambda = (P \quad Z) \cdot \mathbb{Z}^4$ and $Z \in \mathbb{H}_2$. The classical theta functions with characteristic $(c_1, c_2) \in (\frac{1}{2}\mathbb{Z}^2)^2$ are given (using standard coordinates $\mathbf{v} = (v_1, v_2)^T$) by

$$\theta_{c_1,c_2}(Z, \mathbf{v}) = \sum_{\mathbf{x} \in \mathbb{Z}^2} \mathbf{e}(\pi i(\mathbf{x} + c_1)^T Z(\mathbf{x} + c_1) + 2\pi i(\mathbf{x} + c_1)^T(\mathbf{v} + c_2)). \tag{1.14}$$

The main result we need is that for (c_1, c_2) fixed, the set

$$\left\{\theta_{c_1+m_1,c_2}(Z, \mathbf{v}), \quad m_1 \in P^{-1}\mathbb{Z}^2/\mathbb{Z}^2\right\} \tag{1.15}$$

forms a basis of global section of a translate of the line bundle $\mathcal{L}$ by the point $c = Zc_1 + c_2$, see [BL04, Section 3]).

If one wants to work out explicitly the modular forms G_D^X defining W_D^X (see Theorem 0.1) and the corresponding modular form for the genus three construction (see Proposition 3.5) one has to restrict these modular forms via a Siegel modular embedding and translate by an appropriate theta characteristic. We will determine this characteristic in Proposition 2.2 resp. at the end of Section 3 explicitly.

Good compactifications. Let $\overline{X_D}$ (resp. $\overline{X_{D,(1,2)}}$) denote a good compactification of X_D (resp. of $X_{D,(1,2)}$) in the sense of [Mum77] with boundary divisor B. Hirzebruch's minimal smooth compactification is good, the one constructed by Bainbridge ([Bai07]) to study the curves W_D is good, too. Let ω_i for $i = 1, 2$ denote the line bundles of the natural foliations $\mathcal{F}_1 = \{\mathbb{H} \times \{pt\}\}$ and $\mathcal{F}_2 = \{\{pt\} \times \mathbb{H}\}$ of the Hilbert modular surface X_D. They extend on the good compactification to line bundles with the property ([Mum77])

$$\Omega^1_{\overline{X_D}}(\log B) = \omega_1 \oplus \omega_2. \tag{1.16}$$

These compactifications will be used to perform intersection theory. We denote the class of line bundles ω_i (or divisors like W_D^X) in the intersection ring of $\overline{X_D}$ (i.e. up to numerical equivalence) by $[\omega_i]$ (resp. by $[W_D^X]$).

Teichmüller curves and Kobayashi curves. A *Teichmüller curve* is an algebraic curve C with a generically injective map $C \to \mathcal{M}_g$ to the moduli stack of curves that is a totally geodesic subvariety for the Teichmüller metric. On $\mathcal{M}_g$ the Teichmüller metric agrees with the Kobayashi metric and thus Teichmüller curves are also Kobayashi curves in the following sense.

For any algebraic varity Y we define a *Kobayashi curve C in Y* to be an algebraic curve C together with a generically injective map $C \to Y$ that is totally geodesic for the Kobayashi metric. In this paper we will apply this notion (besides for $\mathcal{M}_g$) only for Hilbert modular surfaces X_D. The universal covering of X_D contains uncountably many Kobayashi geodesics, but only few of them descend to algebraic curves, i.e. to Kobayashi curves on X_D.

We recall the following characterization of Kobayashi curves from [MV10] to illustrate the various ways to interpret these curves. We will only need the implication i) to iv) in the sequel. Note that we changed terminology from [MV10], where the notion Kobayashi geodesic was used for what we now call Kobayashi curve.

Proposition 1.3. *Let $C \to X_{D,P}$ be an algebraic curve in a Hilbert modular surface for some polarization P with completion $\overline{C} \to \overline{X_{D,P}}$. Then the following conditions are equivalent.*

i) The curve C is a Kobayashi curve.

ii) The variation of Hodge structures (VHS) over C has a rank two subsystem that is maximal Higgs (see [MV10] for the definitions).

iii) For (at least) one of the two foliation classes ω_i we have $[\omega_i][\overline{C}] = \chi(C)$.

iv) There exists (at least) one (of the two) natural foliations $\mathcal{F}$ of $X_{D,P}$ such that the curve C is everywhere transversal to $\mathcal{F}$.

v) The inclusion $T_C(\log \overline{C} \setminus C) \to T_{X_{D,P}}(\log B)|_C$ splits.

Proof. The proofs are simplifications of [MV10, Theorem 1.2], since for Hilbert modular surfaces the variation of Hodge structures already decomposes into rank two summands. Only these two summands are candidates for the maximal Higgs sub-VHS. ■

Teichmüller curves are generated as $\mathrm{SL}_2(\mathbb{R})$-orbits of *flat surfaces* (X, ω), i.e. of pairs of a Riemann surface X and a non-zero holomorphic one-form ω. The moduli space of flat surfaces is a vector bundle minus the zero section over $\mathcal{M}_g$, denoted by $\Omega\mathcal{M}_g$. Its quotient by the action of $\mathbb{C}^*$ is denoted by $\mathbb{P}\Omega\mathcal{M}_g$.

The Teichmüller curve is isomorphic to $\mathbb{H}/\mathrm{SL}(X, \omega)$, where $\mathrm{SL}(X, \omega)$ is the *affine group* of the flat surface, i.e. the group of matrix parts of orientation-preserving homeomorphisms of X that are affine maps in the ω charts. At the time of writing there is (still) no effective algorithm known to determine $\mathrm{SL}(X, \omega)$ in general or to detect for a given (X, ω) if $\mathrm{SL}(X, \omega)$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$. Examples of flat surfaces generating Teichmüller curves are given in Figure 1. We explain more conceptually how these flat surfaces were constructed as Prym covers in the next section. See e.g. [MT02] or [Möl11b] for more background on flat surfaces, the $\mathrm{SL}_2(\mathbb{R})$-action and Teichmüller curves.

2 Prym varieties

A Prym curve is a (connected smooth algebraic) curve X together with an involution ρ and a supplementary condition. This supplementary condition is different in two of our primary references. In [BL04] it is required that the Prym variety $\mathrm{Prym}(X) = \mathrm{Prym}(X, \rho)$ introduced below naturally acquires a multiple of a principal polarization. In [McM06a] the author focusses on the case that $\mathrm{Prym}(X)$ is two-dimensional, i.e. an abelian surface. Although the first terminology seems to be more widely used, we stick to the second since we are interested in Hilbert modular *surfaces* and applications to curves on these surfaces.

Prym varieties. Suppose that X is a curve with an involution ρ and we let $Y = X/\langle\rho\rangle$. The quotient map $\pi : X \to Y$ induces a map

$$q : \mathrm{Jac}(X) \to \mathrm{Jac}(Y)$$

and we let $\mathrm{Prym}(X)$ be the connected component of the identity of the kernel of q. This abelian variety acquires a polarization $\mathcal{O}_{\mathrm{Jac}(X)}(\Theta)|_{\mathrm{Prym}(X)}$ by restriction of the principal polarization on $\mathrm{Jac}(X)$. We call this polarized abelian variety the *Prym variety* of (X, ρ).

Originally, Prym varieties were invented to construct more principally polarized abelian varieties using curve (and covering) theory than just Jacobians. Thus, it was required that the polarization on $\mathrm{Prym}(X)$ has a multiple of a principal polarization. This holds for étale double covers and genus two double covers ramified at two points.

Double covers with two-dimensional Prym variety. In the sequel we use the modified terminology of [McM06a] and say that (X, ρ) is a Prym pair if $\dim(\mathrm{Prym}(X)) = 2$. This happens for $g(X) = 3$ with 4 ramification points, for $g(X) = 4$ with two ramification points and also for $g(X) = 5$ with ρ fixed point free. The last case is less suitable for the construction of Teichmüller curves, so we disregard it here.

If $g(X) = 4$, the principal polarization of $\mathrm{Jac}(X)$ restricts to a polarization of type $(2, 2)$ on $\mathrm{Prym}(X)$ by [BL04, Corollary 12.1.5]. That is, $\mathcal{O}_{\mathrm{Jac}(X)}(\Theta)|_{\mathrm{Prym}(X)}$ is a positive line bundle on $\mathrm{Prym}(X)$ of type $(2, 2)$. Consequently, there is a line bundle $\mathcal{L}$ on $\mathrm{Prym}(X)$ that defines a principal polarization and such that $\mathcal{L}^{\otimes 2} = \mathcal{O}_{\mathrm{Jac}(X)}(\Theta)|_{\mathrm{Prym}(X)}$.

Since $\mathcal{L}$ is a polarization, the Euler characteristic is the product of the d_i appearing in the type and there is no higher cohomology, i.e. we have $\chi(\mathcal{L}^{\otimes 2}) = 2 \cdot 2 = H^0(\mathrm{Prym}(X), \mathcal{L}^{\otimes 2})$. By Riemann-Roch, the self-intersection number of $\mathcal{L}^{\otimes 2}$ equals 8 and hence by adjunction the vanishing locus of any section of $\mathcal{L}^{\otimes 2}$ is a curve of arithmetic genus five (hence possibly a curve of genus 4 with 2 points identified to form a node).

The Prym Teichmüller curves $W_D(6)$ and $W_D(4)$. For $g(X) = 4$ the *Prym eigenform locus* ΩE_D is defined in [McM06a] to be the subvariety in $\Omega\mathcal{M}_4$ of flat surfaces (X, ω) such that X admits a Prym involution ρ such that $\mathrm{Prym}(X)$ admits real multiplication by $\mathfrak{o}_D$ and, finally, such that ω is an eigenform for real multiplication by $\mathfrak{o}_D$. The eigenform condition includes in particular, that ω is in the -1-eigenspace of the action of ρ on $H^0(X, \Omega^1_X)$. The intersection of ΩE_D with the minimal stratum $\Omega\mathcal{M}_4(6)$ is shown in [McM06a] to be (complex) two-dimensional. Its image in $\mathbb{P}\Omega\mathcal{M}_4$ is an algebraic curve $W_D(6)$ that projects isomorphically to a Teichmüller curve $\mathcal{M}_4$ that will also be denoted by $W_D(6)$. We do not assume here that $W_D(6)$ is irreducible, but by [LM11] this is conjecturally indeed the case. These curves can be generated by X-shaped flat surfaces (see Figure 1). We use this to label their images in Hilbert modular surfaces by an upper index X, see below.

Fully similarly, for $g(X) = 3$ the *Prym eigenform locus* ΩE_D is defined in [McM06a] to be the subvariety in $\Omega\mathcal{M}_3$ of flat surfaces (X, ω) such that X admits a Prym involution ρ such that $\mathrm{Prym}(X)$ admits real multiplication by $\mathfrak{o}_D$ and, finally, such that ω is an eigenform for real multiplication by $\mathfrak{o}_D$. The intersection of ΩE_D with the minimal stratum $\Omega\mathcal{M}_3(4)$ is shown in [McM06a] to be two-dimensional. Its image in $\mathbb{P}\Omega\mathcal{M}_3$ is a curve $W_D(4)$, that projects to a Teichmüller curve in $\mathcal{M}_3$ the will also be denoted by $W_D(4)$. These curves can be generated by S-shaped flat surfaces (see Figure 1) and we use this to label their images in Hilbert modular surfaces by an upper index S, see below.

The Abel-Jacobi image In the following lemma the non-hyperellipticity is still parallel for $W_D(6)$ and $W_D(4)$, but in all further steps the two cases differ. We thus concentrate on $W_D(6)$ for the rest of this section.

If we fix a point P on X, then the composition of the Abel-Jacobi map

$X \to \mathrm{Jac}(X)$ based at P with the dual of the inclusion $\mathrm{Prym}(X) \to \mathrm{Jac}(X)$ defines a map $\varphi : X \to \mathrm{Prym}(X)$, called the *Abel-Prym map* (based at P). Note for comparison with the W_D^S case that in this (standard) definition we have used the principal polarizations of $\mathrm{Prym}(X)$ and $\mathrm{Jac}(X)$ to obtain isomorphisms $\mathrm{Prym}(X) \to \mathrm{Prym}(X)^\vee$ and $\mathrm{Jac}(X) \to \mathrm{Jac}(X)^\vee$ that, composed with the canoical dual of the inclusion $\mathrm{Jac}(X)^\vee \to \mathrm{Prym}(X)^\vee$ give the natural map $\mathrm{Jac}(X) \to \mathrm{Prym}(X)$. Explicitly, identifying $\mathrm{Prym}(X) \subset \mathrm{Jac}(X) = \mathrm{Pic}^0(X)$ we obtain $\varphi(x) = x - \rho(x)$.

Lemma 2.1. *The curves $W_D(6)$ and $W_D(4)$ are disjoint from the hyperelliptic locus.*

Moreover, for each curve $[X] \in W_D(6)$ the Abel-Prym map is an immersion outside the fixed points of ρ and maps the two fixed points P and Q of ρ to a single point in $\mathrm{Prym}(X)$.

Proof. Suppose (X, ρ) was hyperelliptic with hyperelliptic involution h. By the uniqueness of the hyperelliptic involution $\langle h, \rho\rangle \cong (\mathbb{Z}/2)^2$. In particular $\tau = h \circ \rho$ is another involution. Moreover, by [Mum74] $\mathrm{Prym}(X) = \mathrm{Jac}(X/\tau)$ and hence the one-form ω is a pullback from X/τ. This contradicts that ω has a single zero of order 6 resp. of order 4.

The disjointness from the hyperelliptic locus is the hypothesis needed in [BL04, Section 12.5] to deduce the remaining claims. ■

For the construction of modular forms using theta functions below we need a more precise description of the Abel-Prym image in terms of the theta divisor on $\mathrm{Jac}(X)$ and thus the theta divisor on $\mathrm{Prym}(X)$. Let $W^3 \subset \mathrm{Pic}^3(X)$ denote the canonical theta-divisor in the Picard group of X, i.e. the image of the 3-fold symmetric product of X in $\mathrm{Pic}^3(X)$.

Proposition 2.2. *Let (X, ω) be a surface in $W_D(6)$ and $\mathrm{div}(\omega) = 6P$. Then the spin structure determined by $\mathcal{O}_X(3P)$ is even, i.e. $h^0(X, \mathcal{O}_X(3P)) = 2$.*

Suppose that the Abel-Prym map is based at P. Then the two line bundles $\mathcal{O}_{\mathrm{Prym}(X)}(\varphi(X))$ and $\mathcal{O}_{\mathrm{Jac}(X)}(\Theta)|_{\mathrm{Prym}(X)} = \mathcal{L}^2$ are linearly equivalent, where Θ is the translate of $W^3 \subset \mathrm{Pic}^3(X)$ by $-3P$.

Proof. The possible configurations of cylinder decompositions of such a flat surface are listed in the appendix of [LM11]. There are only two of them (one is also visible in Figure 1). In both cases, the parity of the spin structure can be calculated using the winding number of a homology basis (as e.g. explained in [KZ03]) to be even. Clifford's theorem implies that for an even spin structure $h^0(X, \mathcal{O}_X(3P)) = 2$.

The algebraic equivalence of $\mathcal{O}_{\mathrm{Prym}(X)}(\varphi(X))$ and $\mathcal{O}_{\mathrm{Jac}(X)}(\Theta)|_{\mathrm{Prym}(X)} = \mathcal{L}^2$ is a consequence of intersection theory, known as Welters' criterion (see [BL04, Theorem 12.2.2]). The translation image of a linear equivalence class on an abelian variety is precisely the algebraic equivalence class. So we have to show that this translation is zero for the two bundles in question. Equivalently, we have to show that $\varphi(X) \subset \Theta$, where we consider $\varphi(X) \subset A \subset \mathrm{Jac}(X) =$

$\mathrm{Pic}^0(X)$. Since $\varphi(X) \subset \mathrm{Pic}^0(X)$ is the set of classes $x - \rho(x)$, we have to show that $3P+x-\rho(x)$ is an effective divisor for all $x \in X$. Since $h^0(X, \mathcal{O}_X(3P)) = 2$, this is obviously true. ■

The curve W_D^X in X_D. For any curve X representing a point $[X] \in W_D(6)$ the Prym variety has a principal polarization by $\mathcal{L}$ defined above and real multiplication by $\mathfrak{o}_D$ by definition. We thus obtain a map $W_D(6) \to X_D$, whose image W_D^X we now describe. Recall that du_1 and du_2 are the eigenforms for real multiplication on the abelian variety $A_{(z_1,z_2)}$ in the eigenform coordinates $\mathbf{u} = (u_1, u_2)$ introduced in Section 1. Since the abelian varieties $A_{(z_1,z_2)}$ and $A_{(z_2,z_1)}$ are isomorphic with an isomorphism interchanging du_1 and du_2, we may assume in the sequel that du_1 is the eigenform with the 6-fold zero. The other choice describes the flipped curve $\tau(W_D^X)$, where $\tau(z_1, z_2) = (z_2, z_1)$.

Description of W_D^X using theta functions. By Lemma 2.1 the Abel-Jacobi image of X is cut out in $\mathrm{Prym}(X)$ by some section in $\mathcal{L}^{\otimes 2}$. If we fix a basis $\theta_0, \theta_1, \theta_2, \theta_3$ of global sections of $\mathcal{L}^{\otimes 2}$, it is cut out by the vanishing of

$$\theta_X(\mathbf{z}, \mathbf{u}) = \sum_{j=0}^{3} a_j(\mathbf{z})\theta_j(\mathbf{z}, \mathbf{u})$$

for some choice of coefficients $a_j(\mathbf{z})$. We now determine the coefficient tuple $(a_0 : \ldots : a_3)$ as a projective tuple such that $\varphi(X) = \{\theta_X = 0\}$.

First, by the choice of the base point of the Abel-Prym map, the divisor θ_X has to contain zero, hence

$$\sum_{j=0}^{3} a_j(\mathbf{z})\theta_j(\mathbf{z}, 0) = 0 \tag{2.1}$$

From Lemma 2.1 we deduce that $\varphi(X)$ has an ordinary double point at $\mathbf{u} = (0, 0)$.

Lemma 2.3. *Fix the point $\mathbf{z} \in \mathbb{H} \times \mathbb{H}$ and consider $\theta_X(\mathbf{z}, \mathbf{u})$ for a tuple of $a_j(\mathbf{z})$ such that (2.1) holds. Then the differential du_1 restricted to X has a zero of order 6 at $\mathbf{u} = (0, 0)$ if and only if $\frac{\partial^k}{\partial u_2^k}\theta_X(\mathbf{z}, \mathbf{u})|_{\mathbf{u}=(0,0)}$ vanishes for $k = 1, \ldots, 6$.*

Proof. If $\varphi(X)$ had a single branch through the origin $(0, 0)$ in $A_{\mathbf{z}}$ the vanishing of these partial derivatives would be exactly a reformulation of an eigenform having a 6-fold zero at that point $(0, 0)$. But since by Lemma 2.1 there are two branches through $(0, 0)$ the vanishing of these partial derivatives only implies that $\mathrm{ord}_P(du_1) + \mathrm{ord}_Q(du_1) \geq 5$. But since P and Q are fixed points of ρ and du_1 in the (-1)-eigenspace of ρ, the vanishing orders $\mathrm{ord}_P(du_1)$ and $\mathrm{ord}_Q(du_1)$ are both even. Hence the possible orders are $(6, 0)$ and $(4, 2)$. The latter case is impossible since then ω^2 would descend to the quotient curve X/ρ to a quadratic differential with zero divisor of type $(3, 1)$. Such a differential does not exist on a genus two curve, thus proving the claim. ■

We use the shorthand notation $\theta_j(\mathbf{z}) = \theta_j(\mathbf{z}, (0,0))$ and the classical terminology *theta constants* for these restrictions. Indices of theta constants are to be read modulo 4 in the sequel. For a function $f(\mathbf{z}, \mathbf{u})$ the prime denotes the derivative $f'(\mathbf{z}) = \frac{\partial}{\partial z_2} f(\mathbf{z}, \mathbf{u})|_{\mathbf{u}=(0,0)}$.

Proposition 2.4. *The function*

$$G_D^X(\mathbf{z}) = \sum_{j=0}^{3} a_j(\mathbf{z})\theta_j'''(\mathbf{z}), \text{ where } a_j(\mathbf{z}) = \begin{vmatrix} \theta_{j+1}(\mathbf{z}) & \theta_{j+2}(\mathbf{z}) & \theta_{j+3}(\mathbf{z}) \\ \theta'_{j+1}(\mathbf{z}) & \theta'_{j+2}(\mathbf{z}) & \theta'_{j+3}(\mathbf{z}) \\ \theta''_{j+1}(\mathbf{z}) & \theta''_{j+2}(\mathbf{z}) & \theta''_{j+3}(\mathbf{z}) \end{vmatrix} \tag{2.2}$$

is a modular form of weight $(2, 14)$ *for* $\mathrm{SL}(\mathfrak{o}_D \oplus \mathfrak{o}_D^\vee)$ *and for some character. Its vanishing locus contains* W_D^X.

Equivalently, we can write

$$G_D^X(\mathbf{z}) = \begin{vmatrix} \theta_0(\mathbf{z}) & \theta_1(\mathbf{z}) & \theta_2(\mathbf{z}) & \theta_3(\mathbf{z}) \\ \theta'_0(\mathbf{z}) & \theta'_1(\mathbf{z}) & \theta'_2(\mathbf{z}) & \theta'_3(\mathbf{z}) \\ \theta''_0(\mathbf{z}) & \theta''_1(\mathbf{z}) & \theta''_2(\mathbf{z}) & \theta''_3(\mathbf{z}) \\ \theta'''_0(\mathbf{z}) & \theta'''_1(\mathbf{z}) & \theta'''_2(\mathbf{z}) & \theta'''_3(\mathbf{z}) \end{vmatrix} \tag{2.3}$$

Hence (up to a scalar) $G_D^X(\mathbf{z})$ does not depend on the choice of the basis of $\mathcal{L}^{\otimes 2}$.

Proof. The functions $\theta_j(\mathbf{z})$ are modular forms of weight $(1/2, 1/2)$ for a subgroup of Γ finite index of $\mathrm{SL}(\mathfrak{o}_D \oplus \mathfrak{o}_D^\vee)$. (For Siegel theta functions of second order this group is the congruence group $\Gamma(4,8)$, so $\Gamma \supset \psi^{-1}(\psi(\mathrm{SL}(\mathfrak{o}_D \oplus \mathfrak{o}_D^\vee)) \cap \Gamma(4,8))$, where ψ is a Siegel modular embedding. The precise form of Γ will play no role.) First, we claim that the $a_j(\mathbf{z})$ are modular forms of weight $(3/2, 15/2)$ for the same subgroup Γ. This is a general principle, extending Rankin-Cohen brackets (see e.g. [Zag08]) to three-by-three determinants. Roughly, the derivative of a modular form f of weight (k_1, k_2) in the second variable is a modular form of weight $(k_1, k_2 + 2)$ plus a non-modular contribution. These non-modular contributions cancel when taking the linear combinations that appear in a determinant. All the summands in the determinant are of weight $(1/2, 1/2) + (1/2, 5/2) + (1/2, 9/2)$, thus proving the claim. More precisely, for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(\mathfrak{o}_D \oplus \mathfrak{o}_D^\vee)$ we have

$$\begin{aligned} \frac{\partial}{\partial z_2} f\left(\frac{az_1 + b}{cz_1 + d}, \frac{a^\sigma z_2 + b^\sigma}{c^\sigma z_2 + d^\sigma}\right) &= (cz_1 + d)^{k_1} (c^\sigma z_2 + d^\sigma)^{k_2+2} \frac{\partial}{\partial z_2} f(z_1, z_2) \\ &\quad + k_2 c^\sigma (cz_1 + d)^{k_1} (c^\sigma z_2 + d^\sigma)^{k_2+1} f(z_1, z_2). \end{aligned} \tag{2.4}$$

Hence $\left(\frac{\partial f}{\partial z_2}\right) g$ differs from a modular form in a summand

$$k_2 c^\sigma (cz_1 + d)^{k_1} (c^\sigma z_2 + d^\sigma)^{k_2+1} f \cdot g.$$

Since this summand is symmetric in f and g, if f and g have the same weight and thus the same k_2, we deduce the modularity of $fg' - f'g$. Differentiating

(2.4) once more gives the result for second derivatives and three-by-three determinants.

By Lemma 2.3 on the locus W_D^X the section $\frac{\partial^k}{\partial u_2^k}\theta_X(\mathbf{z},\mathbf{u})|_{\mathbf{u}=(0,0)}$ vanishes for $k = 1,\ldots,5$. Vanishing for odd k is automatic, since all the θ_i and hence θ_X is even. By the heat equation (see e.g. [BL04, Section 8.5]) $\frac{\partial^{2k}}{\partial u_2^{2k}}\theta_i(\mathbf{z},\mathbf{u})|_{\mathbf{u}=(0,0)} = \frac{\partial^k}{\partial z_2^k}\theta_i(\mathbf{z})$. Thus the $a_j(\mathbf{z})$ have to satisfy

$$\frac{\partial^{2k}}{\partial u_2^{2k}}\left(\sum_{j=0}^{3} a_j(\mathbf{z})\theta_j(\mathbf{z},\mathbf{u})|_{\mathbf{u}=(0,0)}\right) = \sum_{j=0}^{3} a_j(\mathbf{z})\frac{\partial^k}{\partial z_2^k}\theta_j(\mathbf{z}) = 0$$

for $k = 0,1,2$. The $a_j(\mathbf{z})$ given in (2.2) are, up to a common scalar factor, the unique solution to these conditions.

We now use the same argument derived from differentiating (2.4) again. Since $\theta_i(\mathbf{z},\mathbf{u})$ are modular forms of weight $(1/2,1/2)$ and the $a_j(\mathbf{z})$ are modular forms of weight $(1/2,1/2)$, the sum G_D^X is a modular form of weight $(3/2,15/2)+(1/2,13/2) = (2,14)$ plus a multiple of $\sum a_j(\mathbf{z})\theta_j''(\mathbf{z})$, which is known to vanish by the choice of the $a_j(\mathbf{z})$.

Finally, we consider the action of $\mathrm{SL}(\mathfrak{o}_D \oplus \mathfrak{o}_D^\vee)/\Gamma$. It is known for Siegel theta functions that $\Gamma(2)/\Gamma(4,8)$ acts by characters and that $\mathrm{Sp}(2g,\mathbb{Z})/\Gamma(2)$ acts by a linear representation on the basis of $\mathcal{L}^{\otimes 2}$. These statements obviously also hold for $\psi(\mathrm{SL}(\mathfrak{o}_D \oplus \mathfrak{o}_D^\vee))$. ¿From the determinantal form of G_D^X it is obvious that the change of basis leaves G_D unchanged. Consequently, G_D^X is a modular form for the full Hilbert modular group $\mathrm{SL}(\mathfrak{o}_D \oplus \mathfrak{o}_D^\vee)$ for some character. ■

Explicit construction. We may take the classical theta functions (1.14) and restrict them via a Siegel modular embedding. By Proposition 2.2 no translation by a characteristic is required. Consequently, by (1.15) applied to $P = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$ and $c_1 = c_2 = 0$ we obtain the desired basis $\theta_0,\ldots,\theta_3$ used to construct G_D^X.

3 A Prym variety with $(1,2)$-polarization.

The aim of this section is a characterization of W_D^S in terms of derivatives of theta functions, parallel to Proposition 2.4. The corresponding modular form is constructed in Proposition 3.5. Consequently, in this section we restrict to $g(X) = 3$ and ρ is an involution on X with 4 fixed points.

The Abel-Jacobi map revisited. Let $\iota : \mathrm{Prym}(X) \to \mathrm{Jac}(X)$ be the inclusion of the Prym variety, defined in the preceding section as the connected component of the identity of the kernel of $q : \mathrm{Jac}(X) \to \mathrm{Jac}(Y)$. The restriction of the principal polarization of $\mathrm{Jac}(X)$ to $\mathrm{Prym}(X)$ is a polarization of type $(1,2)$ that we denote by $\mathcal{L} = \mathcal{O}_{\mathrm{Jac}(X)}(\Theta)|_{\mathrm{Prym}(X)}$. Now, the canonical map $\phi_{\mathcal{L}} : \mathrm{Prym}(X) \to \mathrm{Prym}^\vee(X)$ associated with $\mathcal{L}$ is no longer an isomorphism, but of degree 4. Consequently, there is a dual isogeny $(\phi_{\mathcal{L}})^\vee : \mathrm{Prym}(X) \to$

$\mathrm{Prym}^\vee(X)$ with the property that $(\phi_{\mathcal{L}})^\vee \circ \phi_{\mathcal{L}} = [2]$ is the multiplication by two map. The map $(\phi_{\mathcal{L}})^\vee$ is induced by a line bundle $\check{\mathcal{L}}$ on $\mathrm{Prym}^\vee(X)$ which is also a polarization of type $(1,2)$ (see [BL99]). The map $(\phi_{\mathcal{L}})^\vee = \phi_{\check{\mathcal{L}}}$ depends only on the image of $\check{\mathcal{L}}$ in the Néron-Severi group, i.e. for the moment $\check{\mathcal{L}}$ is well-defined only up to translations.

Still identifying $\mathrm{Jac}(X)$ with its dual we have the dual inclusion map $\check{\imath} : \mathrm{Jac}(X) \to \mathrm{Prym}^\vee(X)$ and the Abel-Jacobi map φ defined above generalizes as the composition of the map $X \to \mathrm{Jac}(X)$ (still depending on the choice of a base point) of with $\phi_{\check{\mathcal{L}}} \circ \check{\imath}$. We let φ_0 be the composition of $X \to \mathrm{Jac}(X)$ and $\check{\imath}$, so that $\varphi = \phi_{\check{\mathcal{L}}} \circ \varphi_0$. We call φ_0 the *pre-Abel-Jacobi map* and $\varphi_0(X)$ the *pre-Abel-Jacobi image* of X. In the sequel we will always take φ_0 to be based at a fixed point of ρ. Most of the following lemma is also covered by results in [Bar87].

Lemma 3.1. *The pre-Abel-Jacobi image of X is embedded into* $\mathrm{Prym}^\vee(X)$*. The image of the fixed points are two-torsion points in* $\mathrm{Prym}^\vee(X)$*.*

The Abel-Jacobi image of X is embedded into $\mathrm{Prym}(X)$ *outside the four fixed points of ρ. These four fixed points are mapped to* $0 \in \mathrm{Prym}(X)$*.*

The map (-1) *on* $\mathrm{Prym}^\vee(X)$ *induces the involution ρ on X.*

Proof. Recall that the map $q^\vee : \mathrm{Jac}(Y) \to \mathrm{Jac}(X)$ is given for any degree zero divisor D by $D \mapsto D + \rho(D)$. The map $\check{\imath}$ is the quotient map by the image of $q^\vee$. Hence two points D_1 and D_2 are the same in $\mathrm{Prym}^\vee(X)$, if and only if they differ by an element of the form $D + \rho(D)$. This implies that a fixed point of ρ maps to a point of order two in $\mathrm{Prym}^\vee(X)$ and the last statement.

The Prym variety $\mathrm{Prym}(X)$ is the complementary subvariety to $\mathrm{Jac}(Y)$ inside $\mathrm{Jac}(X)$ in the sense of [BL04, Section 12.1]. Consequently, $\mathrm{Prym}(X)$ is the image of $(1+\rho) : \mathrm{Jac}(X) \to \mathrm{Jac}(X)$. Suppose that $\varphi(x) = \varphi(y)$. Then $(x-P) + (\rho(x) - P) \sim (y - P) + (\rho(y) - P)$, hence $x - \rho(x) \sim y - \rho(y)$. Since X is not hyperelliptic, this can only happen if x and y are both fixed points of ρ. This implies the pointwise injectivity for both φ_0 and φ outside the fixed points.

Let P_i denote the images in Y of the fixed points of ρ. The projectivised differential of the Abel-Prym map is the composition $X \to Y \to \mathbb{P}(H^0(\mathcal{O}_Y \otimes \eta))$ where η is a line bundle defining the double covering, i.e. $\eta^{\otimes 2} = \mathcal{O}_Y(P_1 + P_2 + P_3 + P_4)$. The Abel-Prym map is not an embedding at a point x, if and only if x is a base point of $\mathcal{O}_Y \otimes \eta$, i.e. if $h^0(\mathcal{O}_Y \otimes \eta) = h^0(\mathcal{O}_Y \otimes \eta(-x))$. (Details on both statements can be found in [BL04, Proposition 12.5.3 and Corollary 12.5.5], the principal polarization hypothesis is not used.) On a curve of genus one the bundle $\mathcal{O}_Y$ is trivial and since $\deg(\eta) = 2$, Riemann-Roch implies the claim.

It remains to show that the images of the fixed points of ρ are actually distinct in $\mathrm{Prym}^\vee(X)$. If not, then for some fixed point Q we have $Q - P \sim D + \rho(D)$ for some degree zero divisor D on X. Since Y is an elliptic curve, hence equal to its Jacobian, we may moreover suppose that $D \sim (R - P)$ for some point $R \in X$. Together we obtain $Q + P \sim R + \rho(R)$, which contradicts that X is not hyperelliptic. ∎

Lemma 3.2. *For a flat surface (X, ω) parameterized by $W_D(4)$ the parity of the spin structure is odd.*

Proof. This can be checked on any flat surface representing (X, ω) using the winding number definition given in [KZ03]. Alternatively, we can use the classification of strata in [KZ03] together with Lemma 2.1 stating that we are not in the hyperelliptic stratum. ■

We denote by $K(\check{\mathcal{L}}) \subset \mathrm{Prym}^\vee(X)$ the kernel of $\phi_{\check{\mathcal{L}}}$. It is shown in [BL04, Lemma 10.1.2] that a polarization $\check{\mathcal{L}}$ of type $(1, 2)$ on an abelian variety has exactly 4 base points of the linear system $|\check{\mathcal{L}}|$, i.e. there are exactly 4 points that all the vanishing loci of sections in $H^0(\check{\mathcal{L}})$ have in common. Moreover, the 4 base points form one orbit under the translation by $K(\check{\mathcal{L}})$.

Lemma 3.3. *There is a unique choice of $\check{\mathcal{L}}$ within its algebraic equivalence class such that zero is a base point of $\check{\mathcal{L}}$.*

With this choice of $\check{\mathcal{L}}$, if the flat surface (X, ω) lies in $W_D(4)$, then the pre-Abel-Prym image $\varphi_0(X)$ is the vanishing locus of some section in $H^0(\check{\mathcal{L}})$). Moreover $(-1)^*\check{\mathcal{L}} = \check{\mathcal{L}}$ with this choice of $\check{\mathcal{L}}$, and all the global sections of $\check{\mathcal{L}}$ are odd.

A similar statement does not seem to hold for the Abel-Prym image. By Lemma 3.1 it has arithmetic genus 6 and the obvious polarizations on $\mathrm{Prym}(X)$ are of type $(1, 2)$ or maybe $(1, 4)$. By adjunction the vanishing loci of their global sections are curves of arithmetic genus 3 or 9 respectively.

In the sequel we use the endomorphism $\delta(C, D)$ associated with a curve C and a divisor D of an abelian variety A. It is defined by mapping $a \in A$ to the sum of intersection points of the curve C translated by a and the divisor D.

Proof. Recall that the algebraic equivalence class of a line bundles consists exactly of its translations by any point in $\mathrm{Prym}(X)^\vee$. Two translations of $\check{\mathcal{L}}$ that have zero as base point differ by an element in $K(\check{\mathcal{L}})$. But for $c \in K(\check{\mathcal{L}})$ we have $t_c^*\check{\mathcal{L}} \sim \check{\mathcal{L}}$ by definition of $\phi_{\check{\mathcal{L}}}$. This proves the first statement.

For the second statement we first show that $\varphi_0(X)$ and $\check{\mathcal{L}}$ are algebraically equivalent. Following the strategy in [BL04, Lemma 12.2.3] we need to show by [BL04, Theorem 11.6.4] that $\delta(\varphi_0(X), \check{\mathcal{L}}) = \delta(\check{\mathcal{L}}, \check{\mathcal{L}})$. By [BL04, Proposition 5.4.7] we have $\delta(\check{\mathcal{L}}, \check{\mathcal{L}}) = -2\mathrm{id}_{\mathrm{Prym}^\vee(X)}$. On the other hand, still identifying $\mathrm{Jac}(X)$ with its dual using the polarizations Θ we have

$$\delta(\varphi_0(X), \check{\mathcal{L}}) = -\check{\iota} \circ \iota \circ \phi_{\check{\mathcal{L}}} = -\phi_{\mathcal{L}} \circ \phi_{\check{\mathcal{L}}} = -2\mathrm{id}_{\mathrm{Prym}^\vee(X)}$$

by [BL04, Proposition 11.6.1], by definition of $\mathcal{L}$ as restriction of $\Theta_{\mathrm{Jac}(X)}$ and by definition of $\check{\mathcal{L}}$.

Next, we check that we correctly normalized $\check{\mathcal{L}}$ within its algebraic equivalence class. If two points in $\varphi_0(X)$ differ by an element in $K(\check{\mathcal{L}})$, they are mapped to the same point in $\varphi(X)$. By Lemma 3.1 this happens for any two points among the fixed points of ρ and for no other pair of points. Since the base points of $\check{\mathcal{L}}$ differ by elements in $K(\check{\mathcal{L}})$, this implies the claim.

The last statement is a special case of the results in [BL04, Sections 4.6 and 4.7]. ■

The curve W_D^S in $X_{D,(1,2)}$. We keep the normalization of $\check{\mathcal{L}}$ within its algebraic equivalence class from now on. For any curve representing a point $[X] \in W_D(4)$ the Prym variety has a polarization by $\mathcal{L}$ of type $(1,2)$ defined above and real multiplication by $\mathfrak{o}_D$ by definition. We thus get a map $W_D(4) \to X_{D,(1,2)}$, whose image W_D^S we now describe. Recall that du_1 and du_2 are the eigenforms for real multiplication on the abelian variety $A_{(z_1,z_2)}$ in the eigenform coordinates $\mathbf{u}$ introduced in Section 1.

For $D \equiv 0\,(4)$ we may suppose that du_1 is the eigenform and that the abelian varieties in $X_{D,(1,2)}$ are $A_{\mathbf{z}} = \mathbb{C}^2/\Lambda_{\mathbf{z}}^{\mathfrak{a}}$, compare (1.8). For $D \equiv 1\,(4)$, however, we have to replace $\mathfrak{a}$ by $\mathfrak{a}^\sigma$, when insisting on this normalization. This will play a role in the next section.

We can now strengthen the previous lemma using the real multiplication and the 4-fold zero condition.

Lemma 3.4. *If the image of the flat surface (X,ω) lies in W_D^S, then there exists some section θ_X of $H^0(A_{\mathbf{z}}, \check{\mathcal{L}})$ such that the partial derivatives*

$$\frac{\partial^k}{\partial u_2^k}\theta_X(\mathbf{z},\mathbf{u})$$

for $k = 0,\dots,4$ vanish at the point $\mathbf{u} = (0,0)$.

Since c was chosen in the base locus of $\check{\mathcal{L}}$ and since the sections of $\check{\mathcal{L}}$ are odd, the vanishing of the derivatives for $k = 0,2,4$ is automatic. We now express the vanishing for $k = 1$ and $k = 3$ in terms of theta functions.

We now fix a basis $\{\theta_0(\mathbf{z},\mathbf{u}), \theta_1(\mathbf{z},\mathbf{u})\}$ of sections of $\check{\mathcal{L}}$. We define

$$D_2\theta_j(\mathbf{z}) = \frac{\partial}{\partial u_2}\theta_j(\mathbf{z},\mathbf{u})|_{\mathbf{u}=(0,0)} \quad \text{for} \quad j = 0,1$$

and for analogous purposes as in Section 2 we let

$$a_j(\mathbf{z}) = (-1)^j D_2\theta_{j+1}(\mathbf{z}).$$

Recall from the previous section the definition $f'(\mathbf{z}) = \frac{\partial}{\partial z_2} f(\mathbf{z},\mathbf{u})|_{\mathbf{u}=(0,0)}$.

Proposition 3.5. *Let $\mathfrak{a}$ be a fractional $\mathfrak{o}_D$-ideal of norm two. Then the function*

$$\begin{aligned} G_D^S &= a_0(\mathbf{z})\frac{\partial^3}{\partial u_2^3}\theta_0(\mathbf{z},\mathbf{u})|_{\mathbf{u}=(0,0)} + a_1(\mathbf{z})\frac{\partial^3}{\partial u_2^3}\theta_1(\mathbf{z},\mathbf{u})|_{\mathbf{u}=(0,0)} \\ &= \begin{vmatrix} D_2\theta_0(\mathbf{z}) & D_2\theta_1(\mathbf{z}) \\ D_2\theta_0'(\mathbf{z}) & D_2\theta_1'(\mathbf{z}) \end{vmatrix} \end{aligned} \tag{3.1}$$

is a modular form of weight $(1,5)$ for the Hilbert modular group $\mathrm{SL}(\mathfrak{a}\oplus\mathfrak{o}^\vee)$. The union of the vanishing loci of G_D^S for the one or two choices of $\mathfrak{a}$ (depending on $D \equiv 0$ (4) or $D \equiv 1$ (8)) contains the image W_D^S of the Teichmüller curve $W_D(4)$ in the real multiplication locus $X_{D,(1,2)}$.

Proof. By the definition of c both θ_0 and θ_1 vanish at $\mathbf{u} = (0,0)$. Since the θ_j are modular forms of weight $(1/2, 1/2)$ it is an immediate consequence of the transformation formula for theta functions and this vanishing property that $D_2\theta_j$ is a modular forms of weight $(1/2, 3/2)$ for some subgroup of the Hilbert modular group. By the principle of the construction of Rankin-Cohen brackets, G_D^S is a modular form of weight $(1/2, 3/2) + (1/2, 7/2) = (1, 5)$, as claimed.

For the second statement, note that $\theta_X(\mathbf{z}, \mathbf{u}) = a_0(\mathbf{z})\theta_0(\mathbf{z}, \mathbf{u}) + a_1(\mathbf{z})\theta_1(\mathbf{z}, \mathbf{u})$ is a section of $\check{\mathcal{L}}$ whose first two partial derivatives in the u_2-direction vanish at $(0,0)$. The vanishing of G_D^S implies the vanishing of the third partial derivative and also the forth derivative vanishes, since θ_X is odd. Lemma 3.4 now implies the claim. ∎

Explicit construction. The classical theta functions (1.14) with characteristic $c = 0$ are even functions in $\mathbf{v}$ (and $\mathbf{u}$). More generally, a shift by a half-integral characteristic $c = (c_1, c_2) \in (\frac{1}{2}\mathbb{Z}^2)^2$ is an even function, if $4c_1^T c_2$ is even and odd otherwise. Consequently, for a $(1,2)$-polarized abelian variety, both $\theta_0 = \theta_{(\frac{1}{2},0),(\frac{1}{2},0)}$ and $\theta_1 = \theta_{(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},0)}$ are odd functions, and sections of the same line bundle by (1.15). Since these functions are odd, zero is a base point of the global sections of the theta line bundle shifted by the characteristic $c = Z\begin{pmatrix}1/2\\0\end{pmatrix} + \begin{pmatrix}1/2\\0\end{pmatrix}$. Consequently, θ_0 and θ_1 are an explicit form of the basis needed to construct G_D^S.

4 Euler characteristic of W_D^X and W_D^S

The preceding theta-function interpretation gives a way to calculate the Euler characteristic of W_D^X and W_D^S, and of the Teichmüller curves $W_D(6)$ and $W_D(4)$. In this section we prove all the results announced in the introduction and cross-check our calculations of Euler characteristics with the examples in [McM06a]. We invite the reader to compare this result to the work of Bainbridge [Bai07]. He computed the Euler characteristics of the Teichmüller curves W_D using a modular form that he defined using flat geometry. His formula is

$$\chi(W_D) = -\frac{9}{2}\chi(X_D). \tag{4.1}$$

Subsequently, a theta function construction of his modular form was found in [MZ11].

Theorem 4.1. *For the Kobayashi curves W_D^X and for the Prym Teichmüller curves $W_D(6) \subset \mathcal{M}_4$ the Euler characteristic is given by*

$$\chi(W_D^X) = \chi(W_D(6)) = -7\chi(X_D).$$

The explicit formula for $\chi(X_D)$ is stated in (1.11).

Example 4.2. For $D = 8$ we have $\chi(X_8) = 1/6$ and thus $\chi(W_8(6)) = -7/6$. This corresponds to the curve $V(X_1)$ calculated in [McM06a] with genus zero, two cusps and two elliptic points of order two and three respectively.

For $D = 12$ we obtain $\chi(X_{12}) = 1/3$ and thus $\chi(W_{12}(6)) = -7/3$. This corresponds to the curve $V(X_3)$ calculated in [McM06a] with genus zero, three cusps, one elliptic point of order two and one elliptic point of order six.

Theorem 4.3. *For genus three and $D \equiv 5\,(8)$ the locus $W_D(4)$ is empty (and W_D^S is not defined). For $D \equiv 0, 4\,(8)$ we have*

$$\chi(W_D^S) = \chi(W_D(4)) = -\frac{5}{2}\chi(X_{D,(1,2)}).$$

For $D \equiv 1\,(8)$ there are two components $W_D(4)^1$ and $W_D(4)^2$, each mapping to a Kobayashi curve $W_D^{S,j}$ in $X_{D,(1,2)}$ and the Euler characteristic is given by

$$\chi(W_D^{S,j}) = \chi(W_D(4)^j) = -\frac{5}{2}\chi(X_{D,(1,2)}) \quad \textit{for} \quad j = 1, 2.$$

Example 4.4. For $D = 12$ we have $\chi(X_{12}) = \chi(X_{12,(1,2)})$ by Proposition 1.2. Hence $\chi(X_{12,(1,2)}) = 1/3$ as in the preceding example and thus $\chi(W_{12}^S) = -5/6$. By [LM11] the locus $W_{12}(4)$ for this discriminant has one component only. It corresponds to the curve $V(S_1)$ calculated in [McM06a] with genus zero, two cusps and one elliptic points of order six.

For $D = 20$ we have $\chi(X_{20,(1,2)}) = 3/2\chi(X_{20})$ by Proposition 1.2. Consequently, $\chi(X_{20,(1,2)}) = 1$ and thus $\chi(W_{20}^S) = -5/2$. By [LM11] the locus $W_{20}(4)$ for this discriminant has one component only. It corresponds to the curve $V(S_2)$ calculated in [McM06a] with genus zero, four cusps and one elliptic point of order two.

The proof of both theorems will be completed at the end of this section.

A Torelli-type theorem. The Prym-Torelli map associates with any Prym pair (X, ρ) (or equivalently with a (quotient) curve Y and the covering datum) its Prym variety. For $g(X) = 4$ we have $g(Y) = 2$ with two fixed points, thus a moduli space of dimension 5. Since $\dim \mathcal{A}_2 = 3$, there cannot exist a Torelli theorem retrieving the curve from its Prym variety. Nevertheless, a corresponding statement holds when restricted to real multiplication and eigenforms with a zero of high multiplicity.

Proposition 4.5. *Let $(A_{\mathbf{z}}, \mathcal{L})$ be a principally polarized abelian surface with real multiplication by $\mathfrak{o}_D$ and suppose that the corresponding point $[A_{\mathbf{z}}] \in X_D$ is in the vanishing locus of G_D^X. Then there is one and only one curve X of genus four with $[X] \in W_D(6)$ whose Prym variety is $A_{\mathbf{z}}$ and such that the eigenform with a 6-fold zero is du_1.*

In particular the vanishing locus of G_D^X is equal to W_D^X.

Proof. We reverse the reasoning in the construction of G_D^X. Take a basis $\theta_0(\mathbf{z},\mathbf{u}),\ldots,\theta_3(\mathbf{z},\mathbf{u})$ of sections of $\mathcal{L}^{\otimes 2}$ and choose $a_j(\mathbf{z})$ as in (2.2) (where we now work with partial derivatives in the u_2-direction of order 0,2 and 4, to avoid z_2-derivatives in this pointwise argument). Now let

$$\theta_X(\mathbf{z},\mathbf{u}) = \sum_{j=0}^{3} a_j(\mathbf{z})\theta_j(\mathbf{z},\mathbf{u})$$

and define $X'_{\mathbf{z}}$ to be the vanishing locus of θ_X. One easily checks that the function θ_X does not depend on the choice of the basis for $\mathcal{L}^{\otimes 2}$.

Suppose that θ_X is not zero. By Riemann-Roch and adjunction, $X'_{\mathbf{z}}$ is a curve of arithmetic genus 5. Since all these sections of $\mathcal{L}^{\otimes 2}$ are even and since $X'_{\mathbf{z}}$ passes through the origin of $A_{\mathbf{z}}$, it has a singularity there. Since $\mathcal{L}^{\otimes 2}$ is of type $(2,2)$ there are at most (hence precisely) two branches through the origin. Let P and Q denote the preimages of the origin in the normalization of $X'_{\mathbf{z}}$. The vanishing of the derivatives implies that $\mathrm{ord}_P(du_1)+\mathrm{ord}_Q(du_1) \geq 5$. Since θ_X is even, the vanishing orders are even, at least $(6,0)$ or $(4,2)$. Consequently, the normalization of $X_{\mathbf{z}}$ of $X'_{\mathbf{z}}$ is a curve of genus four. The curve $X_{\mathbf{z}}$ has an involution ρ induced by $\mathbf{u} \mapsto -\mathbf{u}$ on $A_{\mathbf{z}}$ and by Welters' criterion for a curve to generate a Prym variety ([BL04, Theorem 12.2.2]) $A_{\mathbf{z}}$ is the Prym variety of (X,ρ). As in the proof of Lemma 2.3 we argue that du_1 has in fact a zero of order 6 at one of the branches through the origin. We conclude that $X_{\mathbf{z}} \in W_D(6)$. This shows that there is a curve $W_D(6)$ whose Prym image is $A_{\mathbf{z}}$ and by the argument leading to Proposition 2.4 the curve X just constructed is the only choice with du_1 restricting to a zero of order 6, if we can rule out that all the $a_j(\mathbf{z})$ are zero, which is equivalent to the assumption $\theta_X \neq 0$. Together with the inclusion stated in Proposition 2.4 this concludes the proof of the proposition under the assumption on θ_X.

Suppose that θ_X was zero for some $\mathbf{z} \in \mathbb{H}\times\mathbb{H}$. This implies that all the $a_j(\mathbf{z})$ vanish. Consequently, if we let

$$M = \begin{pmatrix} \theta_0(\mathbf{z}) & \theta_1(\mathbf{z}) & \theta_2(\mathbf{z}) & \theta_3(\mathbf{z}) \\ \theta'_0(\mathbf{z}) & \theta'_1(\mathbf{z}) & \theta'_2(\mathbf{z}) & \theta'_3(\mathbf{z}) \\ \theta''_0(\mathbf{z}) & \theta''_1(\mathbf{z}) & \theta''_2(\mathbf{z}) & \theta''_3(\mathbf{z}) \end{pmatrix}, \quad \text{then} \quad \mathrm{rank}(M) \leq 2. \tag{4.2}$$

The image of $A_{\mathbf{z}}$ under the projective embedding $A_{\mathbf{z}} \to \mathbb{P}^3$ defined by the sections of $\mathcal{L}^{\otimes 2}$ is known to be a Kummer surface, the quotient of $A_{\mathbf{z}}$ by the involution (-1) (see [BL04, Section 10]). Such a Kummer surface has 16 nodes at the images of two-torsion points, i.e. singular points with local equation $x^2+y^2+z^2=0$. Since the Hessian of such a singularity has non-zero determinant, this contradicts the above hypothesis $\mathrm{rank}(M) \leq 2$. ∎

The same line of arguments works for $g=3$, with a different geometric argument to rule out $\theta_X = 0$ and with an extra twist due to the decomposition behavior of the prime two.

Proposition 4.6. *Fix a fractional $\mathfrak{o}_D$-ideal $\mathfrak{a}$ of norm two and a realization $X_{D,(1,2)} \cong (\mathbb{H}\times\mathbb{H})/\mathrm{SL}(\mathfrak{a}\oplus\mathfrak{o}_D^\vee)$. Let $(A_{\mathbf{z}},\mathcal{L})$ be a $(1,2)$-polarized abelian surface with real multiplication by $\mathfrak{o}_D$ and suppose that the corresponding point $[A_{\mathbf{z}}] \in X_{D,(1,2)}$ is in the vanishing locus of G_D^S. Then there is one and only one curve X of genus four with $[X] \in W_D^X$ whose Prym variety is $[A_{\mathbf{z}}]$ and such that the eigenform with a 4-fold zero is du_1.*

Moreover, if $D \equiv 1\,(8)$, the preimages $W_D(4)^1$ and $W_D(4)^2$ in $\mathcal{M}_4$ of the vanishing locus of G_D^S for the two choices $\mathfrak{a}$ and $\mathfrak{a}^\sigma$ of a prime ideal of norm two are generically different.

Proof. To prove the first statement, we reverse the argument of Proposition 3.5 and use the notations introduced there. Fix a basis $\theta_0(\mathbf{z},\mathbf{u}),\theta_1(\mathbf{z},\mathbf{u})$ of sections of $\mathcal{L}$ and consider $\theta_X = a_0(\mathbf{z})\theta_0 + a_1(\mathbf{z})\theta_0$. Suppose that θ_X is not zero. Then the vanishing locus $X = \{\theta_X = 0\}$ is a curve of arithmetic genus three and by construction du_1 is a holomorphic one-form on X with a zero of order (at least) 4 at $0 \in X$. This implies that X is smooth. Since by Lemma 3.3 the map (-1) on $A_{\mathbf{z}}$ induces an involution on X with 4 fixed points, we conclude that $[X] \in W_D(4)$. This shows that there is a curve $W_D(4)$ whose Prym image is $A_{\mathbf{z}}$ and by the argument leading to Proposition 3.5 the curve X just constructed is the only choice with du_1 restricting to a zero of order 4, if we can rule out that both $D_2\theta_0(\mathbf{z},0) = -a_1(\mathbf{z}) = 0$ and $D_2\theta_1(\mathbf{z},0) = a_0(\mathbf{z}) = 0$. Together with the inclusion stated in Proposition 3.5 this implies the first statement of the proposition under the assumption on θ_X.

Suppose that θ_X was zero for some $\mathbf{z} \in \mathbb{H}\times\mathbb{H}$, i.e. $a_0(\mathbf{z}) = a_1(\mathbf{z}) = 0$. Consider the family of arithmetic genus three curves given by the vanishing locus of $a\theta_0 + b\theta_1$ parameterized by $(a:b) \in \mathbb{P}^1$. If we blow up the four base points of $\mathcal{L}$ in $A_{\mathbf{z}}$ we obtain a fibered surface with Euler number -4. If all the fibers were smooth, the formula for a genus three fiber bundle over a projective line gives Euler number -8, a contradiction. The possible singular fibers of a section of $\mathcal{L}$ are determined in [Bar87], see also [BL04, Exercise 10.8.(1)]. The first possibility is a genus two curve with one node, necessarily disjoint from the base points, the other two possibilities consist of configurations of elliptic curves. (They can occur only on some modular curves in $X_{D,(1,2)}$, but we will not use this.) Since $a_0 = a_1 = 0$, the holomorphic one-form du_1 restricted to any global section of $\mathcal{L}$ has a zero of order at least two at zero. This already rules out all the configurations of elliptic curves.

Still assuming that $a_0 = a_1 = 0$, we consider the fibered surface $f : \mathcal{X} = \mathrm{Blowup}_{4\ \mathrm{points}}(A_{\mathbf{z}}) \to \mathbb{P}^1$ all whose singular fibers are of geometric genus two. By Lemma 3.3 all the fibers admit an involution ρ induced by (-1) with (generically) 4 fixed points. The quotient is thus a curve of arithmetic genus one. Since for the singular fibers the 4 base points are disjoint from the node, the arithmetic genus one curve is smooth if and only if the corresponding fiber of f is smooth. We claim that this implies that f is a pullback of a Teichmüller curve generated by a square-tiled surface, whose family of Jacobians has a two-dimensional fixed part, the abelian surface $A_{\mathbf{z}}$. In fact, consider the image of the moduli map $\mathbb{P}^1 \to \mathcal{M}_3$. The image is embedded in $\mathcal{M}_3$, so its tangent map, the Kodaira-Spencer map, vanishes nowhere. Since the 2-dimensional abelian subvariety $A_{\mathbf{z}}$ of the family of Jacobians is constant, this implies that the Kodaira-Spencer map of the quotient family of elliptic curves $\mathcal{X}/\langle\rho\rangle$ never vanishes. Together with the statement on singularities the hypothesis for the characterization of Teichmüller curves [BM10, Theorem 1.2] are met.

The claim implies that this fibered surface also defines a Shimura curve and by [Möl11a, Lemma 4.5], the singular fibers of such a family cannot be of geometric genus two, more precisely, the fibered surface f has to be the unique such curve in $\mathcal{M}_3$, described in detail in [Möl11a, Section 3] or in [HS08]. This contradiction concludes the proof that θ_X is nowhere zero.

For the second statement we look at the periods of the eigenform ω with a 4-fold zero using our conventions (1.1). The periods of the first eigenform are, by definition, $\mathfrak{a} \oplus \mathfrak{o}_D^\vee z$ for some $z \in \mathbb{H}$ (as opposed to $\mathfrak{a}^\sigma \oplus \mathfrak{o}_D^\vee z$ for the second eigenform). If the two components $W_D(4)^1$ and $W_D(4)^2$ coincided at some point the two eigenforms would lie on the same abelian surface with real multiplication and $(1,2)$-polarization. We thus would obtain an $\mathfrak{o}_D$-linear isomorphism

$$\mathfrak{a} \oplus \mathfrak{o}_D^\vee \cong \mathfrak{a}^\sigma \oplus \mathfrak{o}_D^\vee.$$

Taking determinants of both sides we obtain $\mathfrak{a} \cong \mathfrak{a}^\sigma$, contradicting $D \equiv 1\,(8)$. ∎

The last argument is given in coordinates on flat surfaces explicitly in [LM11, Lemma 6.2], and serves for the same purpose of distinguishing the two components.

We can now collect all the information and prove all the theorems stated in the introduction as well as at the beginning of this section.

Proof of Theorem 0.1. Proposition 2.4 and 4.5 prove the first statement of the theorem. For the second statement, note that the proof that $W_D(6)$ is a Teichmüller curve uses the fact that universal covering of $\mathbb{H} \to \mathcal{T}_4$ of $C \to \mathcal{M}_4$ composed with the Torelli map $\mathcal{T} \to \mathbb{H}_4$ to the Siegel upper half space can be composed with a projection $\mathbb{H}_4 \to \mathbb{H}$ so that the composition is a Möbius transformation, hence a Kobayashi isometry. The non-expansion property of the Kobayashi metric implies that $\mathbb{H} \to \mathcal{T}_4$ is a Kobayashi curve. Since $\mathbb{H}_4 \to \mathbb{H}$ was constructed using the periods of the eigenform with a 6-fold zero, the composition

$$\mathbb{H} \to \mathcal{T}_4 \to \mathbb{H}_4 \to \mathbb{H}$$

factors through the universal covering map $\mathbb{H} \to \mathbb{H} \times \mathbb{H}$ of $W_D^X \to X_D$. By the same argument, this is a Kobayashi curve. ∎

Proof of Theorem 4.1. First we show that the vanishing locus of G_D^X is simple. In fact its restriction to the diagonal $(z,z) \subset \mathbb{H} \times \mathbb{H}$ is a non-zero modular form G_Δ of weight $2 + 14 = 16$ for $\mathrm{SL}_2(\mathbb{Z})$. Expanding the determinant defining G_Δ we check that G_Δ vanishes at the cusp. Consequently, $G_\Delta = \Delta \cdot E_4$ and hence the vanishing locus of G_Δ consists of a single point in the interior of Δ and the cusp. This implies the claim.

Using the definition of the line bundles ω_i along with (1.16), the class of the vanishing locus of a modular form is

$$[W_D^X] = [\omega_1] + 7[\omega_2]. \tag{4.3}$$

Since $[W_D^X]$ is a Kobayashi curve with the isometric embedding given by the first variable, we have $-[W_D^X][\omega_1] = \chi(W_D^X)$. Thus, pairing (4.3) with $-[\omega_1]$ and using that $[\omega_1][\omega_2] = \chi(X_D)$ on a Hilbert modular surface gives the desired result. ∎

The *proof of Theorem 4.3* follows from the same intersection argument, using that the restriction of G_D^s to the diagonal is a non-zero modular form of weight $1 + 5 = 6$, i.e. the Eisenstein series E_6, whose vanishing locus is simple.

The *proof of Theorem 0.2* is now an immediate consequence of Theorem 4.1 together with Proposition 4.5 and Theorem 4.3 together with Proposition 4.6.

5 An invariant and possible generalizations

Aiming to construct more, essentially different Kobayashi curves one can use the procedure involving theta functions to construct Hilbert modular forms of non-parallel weight generalizing the preceding construction. Of course we thus leave the world of Teichmüller curves. We propose replace the integer 2 (type of the polarization) in the genus four discussion by an arbitrary $N \in \mathbb{N}$. First, we start with the definition of an invariant.

Suppose that $C \to X_D$ is a Kobayashi curve and let $\overline{C} \to \overline{X_D}$ the closure in a good compactification of X_D. We define

$$\lambda_2(C) = \frac{[\omega_1] \cdot [\overline{C}]}{[\omega_2] \cdot [\overline{C}]}$$

and call this ratio the *second Lyapunov exponent* of the Kobayashi curve. Although we defined the intersection number on a compactification, the value of λ_2 is independent of the choice of a compactification since $[\omega_i] \cdot [B_j]$ for $i = 1, 2$ and all components B_j of the boundary divisor B. Justification for the terminology, i.e. the relation to a Lyapunov exponent for the $\mathrm{SL}_2(\mathbb{R})$-action is given in [Wei12], implicitly also in the last section of [Möl11b].

The Propositions 2.4, 3.5, 4.5 and 4.6 can in this language be summarized as follows.

Proposition 5.1. *Each of the Hilbert modular surfaces X_D contains Kobayashi curves of second Lyapunov exponent 1, $1/3$ and $1/7$. The Hilbert modular surface X_5 moreover contains Kobayashi curves of second Lyapunov exponent $1/2$.*

Proof. Hirzebruch-Zagier cycles gives $\lambda_2 = 1$, the curve W_D give $\lambda_2 = 1/3$, the W_D^X are curves with $\lambda_2 = 1/7$. The decagon generates a Teichmüller curve with $\lambda_2 = 1/2$ in X_5 ([McM06b]). ∎

Problem 5.2. *What is the set of second Lyapunov exponents for Kobayashi curves in X_D?*

The same question can be formulated for Hilbert modular surfaces of other genera, e.g. in $X_{D,(1,2)}$ the curves W_D^S have $\lambda_2 = 1/5$.

There is an obvious generalization of the construction of G_D^X. The (Hilbert) theta functions $\theta(c_1, 0)(\mathbf{z}, \mathbf{u})$ for $c_1 \in \frac{1}{N}\mathbb{Z}^2/\mathbb{Z}^2$ are a basis of $\mathcal{L}^{\otimes N}$ on the abelian variety $A_{\mathbf{z}}$. Numbering elements in $\frac{1}{N}\mathbb{Z}^2/\mathbb{Z}^2$ by $0, \ldots, N^2 - 1$ we obtain theta functions $\theta_0, \ldots, \theta_{N^2-1}$. We let

$$G_D^{[N]}(\mathbf{z}) = \begin{vmatrix} \theta_0(\mathbf{z}) & \theta_1(\mathbf{z}) & \cdots & \theta_{N^2-1}(\mathbf{z}) \\ \theta_0'(\mathbf{z}) & \theta_1'(\mathbf{z}) & \cdots & \theta_{N^2-1}'(\mathbf{z}) \\ \vdots & \vdots & & \vdots \\ \theta_0^{(N^2-1)}(\mathbf{z}) & \theta_1^{(N^2-1)}(\mathbf{z}) & \cdots & \theta_{N^2-1}^{(N^2-1)}(\mathbf{z}) \end{vmatrix}.$$

As in the proof of Proposition 2.4 we conclude that $G_D^{[N]}$ is a modular form of weight $(\frac{N^2}{2}, \frac{N^2}{2}(2N^2 - 1))$.

Problem 5.3. *Is the zero locus of $G_D^{[N]}$ irreducible? Are its components Kobayashi curves in X_D?*

A positive answer to this problem would at least show that the set of second Lyapunov exponents for Kobayashi curves in X_D is infinite, containing the values $\lambda_2 = \frac{1}{2N^2-1}$.

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