Karlsruher Institut für Technologie

Master Thesis

Topological K-Theory and the image of the J-homomorphism

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Abstract

For a topological space X one defines the stable homotopy groups $\pi_r^s(X)$ as the direct limit $\varinjlim_k \pi_{r+k}(\Sigma^k X)$. It is hard to compute those groups. In fact for general spaces it is still unknown what $\pi_r^s(X)$ is. Even for $X = S^0$ the stable homotopy groups of spheres $\pi_r^s := \varinjlim_k \pi_{r+k}(S^k)$ are more or less known up to $r = 90$ and still unknown for higher r. In the paper [Ada66] from 1966, Adams characterized certain subgroups of π_r^s calculating the image of the J-homomorphism $J\colon \pi_r(SO) \to \pi_r^s$ where SO is the direct limit of the special orthogonal groups SO(n). His result can be divided into three parts.

For $r \equiv 0, 1 \bmod 8$ the image of J is a direct $\mathbb{Z}/2$ summand of π_r^s.
For $r = 4s - 1 \equiv 3 \bmod 8$ the image of J is a direct $\mathbb{Z}/m(2s)$ summand of π_r^s.
For $r = 4s - 1 \equiv 7 \bmod 8$ the image of J is either $\mathbb{Z}/m(2s)$ or $\mathbb{Z}/2m(2s)$ and in the first case it is a direct summand.
Here $m(2s)$ is the denominator of $\frac{B_s}{4s}$ and B_s is the s-th Bernoulli number.

Adams conjectured that in the third case the first option is always the case. This was known as the Adams conjecture and open for the next five years until Quillen proved the Adams conjecture to be true in 1971 ([Qui71]).

The main tool Adams used to prove his results is topological K-theory. In this thesis we will tackle the proof of the first case. Therefore we will introduce the necessary background to understand Adams proof namely vector bundles, topological K-theory and lots of tools arising from K-theory. We will then present Adams original proof of the case $r \equiv 0, 1 \bmod 8$. We will not present the other cases but their proofs use the techniques developed in this thesis (see [Ada66]).

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Contents

Conventions

By ‘space’ we mean topological space and ‘map’ between topological spaces means continuous map.
All vector bundles are either real or complex. Sometimes we just say ‘vector bundle’ instead of ‘real’ or ‘complex vector bundle’ if the statements are true in both cases.

1 Introduction

In algebraic topology one is interested in topological spaces up to homotopy. In general it is very hard to decide whether two spaces are homotopy equivalent or not. A common approach is to define invariants for a certain class of spaces which are preserved by homotopy equivalences. A naive choice would for example be the number of path components. Some more prominent examples are the Euler characteristic, singular homology and the fundamental group.
One then tries to calculate those invariants. If the results differ then the spaces cannot be homotopy equivalent. If the invariants agree one can generally not conclude, that the spaces are homotopy equivalent. Therefore one defines several invariants in the hope of being able to distinguish more spaces.
Another example for a homotopy invariant are the homotopy groups $\pi_r(X)$ for $r \in \mathbb{N}$ which are a generalization of the above mentioned fundamental group $\pi_1(X)$.
The r-th homotopy group of X is defined as all homotopy classes of basepoint preserving maps $S^r \to X$. It turns out that for $r \geq 2$ this is an abelian group.
Even though their definition is quite simple in practice these groups are very hard to determine. Even for rather easy spaces such as spheres the groups $\pi_r(S^n)$ are still unknown in general. This is very different compared to the situation in singular homology where the homology groups of spheres follow directly from the axioms.

In 1937 Freudenthal proved that for any pointed space X the homotopy groups $\pi_{r+k}(\Sigma^k X)$ stabilize which means that there is a number $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the groups $\pi_{r+n}(\Sigma^n X)$ are isomorphic. This theorem is known as the Freudenthal suspension theorem. The resulting group may also be directly defined as the direct limit

$$\pi_r^s(X) := \varinjlim_k \pi_{r+k}(\Sigma^k X).$$

and is called the r-th stable homotopy group of X.
The Freudenthal suspension theorem was one of the starting points of an area of algebraic topology known as stable homotopy theory. This area is concerned with statements about spaces that are true if one applies the suspension functor (or rather reduced suspension functor) often enough.
A very fundamental problem in this area is to calculate the stable homotopy groups for interesting spaces e.g. spheres. As for the ordinary homotopy groups this is quite challenging. It is best reflected by a quote from one of the main contributors to homotopy theory Frank Adams.

'With all due respect to anyone interested in them, the stable homotopy groups of spheres are a mess.' ([Ada95], p.202)

For the case of spheres the stable homotopy group $\pi_r^s := \varinjlim_k \pi_{r+k}(S^k)$ is called the r-stem. They have been calculated up to dimension $r \leq 90$ (see [IWX23]).
In 1966 Adams found cyclic subgroups of the r-stem which are direct summands (see

[Ada66]). These arise as the image of the so called J-homomorphism which is a map $J\colon \pi_r(SO) \to \pi_r^s$. Adams results can be divided into three parts.

For $r \equiv 0, 1 \bmod 8$ the image of J is a direct $\mathbb{Z}/2$ summand of π_r^s.
For $r = 4s-1 \equiv 3 \bmod 8$ the image of J is a direct $\mathbb{Z}/m(2s)$ summand of π_r^s.
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Here $m(2s)$ is the denominator of $\frac{B_s}{4s}$ and B_s is the s-th Bernoulli number.

Adams conjectured, that in the third case the first option is always the case. This was known as the Adams conjecture and open for the next five years until in 1971 Quillen proved the Adams conjecture to be true ([Qui71]).

The proofs by Adams use topological K-theory which is a generalized cohomology theory that arises from vector bundles. Further it provides lots of extra structure and tools.
Historically K-theory was introduced by Grothendieck studying projective algebraic varieties. Atiyah and Hirzebruch [AHAS72] then constructed a topological analogue using vector bundles. Besides the application to stable homotopy theory topological K-theory can also be used to solve various other problems. For example it can be used to solve the Hopf invariant one problem. Which then implies that the only spheres that admit a H-space structure are S^0, S^1, S^3, S^7 (see [Hat03] Section 3.2). Further topological K-theory gives a fundamentally different proof of Bott-periodicity than the original proof by Bott (see [Bot56]).

The goal of this thesis is to introduce vector bundles and topological K-theory and then present a proof of the case $r \equiv 0, 1 \bmod 8$ of Adams results on the J-homomorphism mentioned above.
We begin our journey towards this proof in Chapter 2 by defining vector bundles and prove useful properties about them. We will further see constructions and structures related to vector bundles that are not only interesting on their own, but also become important when dealing with topological K-theory in Chapter 3.
Besides defining topological K-theory, Chapter 3 further explores the tools and structures that arise from topological K-theory such as the Adams operations, the Chern character and the Thom isomorphism. We will also see the famous Bott periodicity theorem which says that the K-groups are periodic.
Finally in Chapter 4 we start tackling the proof of the theorem about the image of the J-homomorphism by first defining the stable homotopy groups of spheres and the J-homomorphism. We then use the tools from Chapter 3 to define invariants d and e. By developing techniques to calculate those invariants we will then be able to prove the theorem.

2 Vector bundles

The goal of this chapter is to first introduce vector bundles and then prove useful properties about them. We will only treat the case of complex or real vector bundles which we will later use to define complex and real K-theory. Most of this chapter is a collection of results and proofs from [Kar09] and [Hat03].

2.1 Vector bundle basics

In this section we define vector bundles and build a category where the objects are vector bundles over a fixed base space.

Definition 2.1 (Vector bundle)
Let B be a topological space. A real vector bundle over B is a topological space E together with a continuous map $p\colon E \to B$, such that the following two conditions are satisfied

1. $E_b := p^{-1}(b)$ is a finite dimensional $\mathbb{R}$-vector space for all $b \in B$.
2. p is locally trivial, that means for every point $b \in B$ there exists an open neighbourhood U and $n \in \mathbb{N}$ (n does not need to be the same for all U) and a homeomorphism $\varphi\colon p^{-1}(U) \to U \times \mathbb{R}^n$ making the following diagram commute

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\ \varphi\ } & U \times \mathbb{R}^n \\ & {\scriptstyle p|_{p^{-1}(U)}} \searrow \quad \swarrow {\scriptstyle \pi_1} & \\ & U & \end{array}$$

   where π_1 is the projection onto the first component. Furthermore for every $b \in U$ the restriction $\varphi|_{E_b}\colon E_b \to \{b\} \times \mathbb{R}^n$ of φ to the each fibre is a vector space isomorphism.

If we replace $\mathbb{R}$ by $\mathbb{C}$ we get the definition of a complex vector bundle over B.
We call B base space, E total space and E_b the fibre of b (under p). The maps φ are called local trivialisations.
We will often just write E for a vector bundle if the map $p\colon E \to B$ is clear.

Remark 2.2
Since E_x is a vector space and thus non empty, p is surjective.
As a set, E is the disjoint union $\bigsqcup_{b \in B} E_b = \bigsqcup_{b \in B} p^{-1}(b)$.

Definition 2.3 (Vector subbundle)
Let $p\colon E \to B$ be a vector bundle over B. A vector subbundle of E is a subspace $E_0 \subset E$ such that the restriction $p|_{E_0}\colon E_0 \to B$ is a vector bundle. Since the fibres of vector bundles are non-empty $(E_0)_b$ is a vector subspace of E_b.

Proposition 2.4
Let $p\colon E \to B$ be a vector bundle. The map $r\colon B \to \mathbb{N}, b \mapsto \dim_{\mathbb{K}}(E_b)$ is locally constant.

Proof. Let $b \in B$. Choose an open neighbourhood $U \subset B$ of b and a local trivialisation $\varphi\colon p^{-1}(U) \to U \times \mathbb{R}^n$. Since $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ iff $n = m$ and φ is a homeomorphism $\dim(E_{b'}) = n$ for all $b' \in U$. $\square$

Remark 2.5
Locally constant functions on a connected space are (globally) constant. Thus the map r is constant on every connected component of B. So one can view r as a function from the set of connected components of B to $\mathbb{N}$.

Definition 2.6 (Rank)
Let B be a space and $p\colon E \to B$ a vector bundle over B. We define the rank of p to be the map $r\colon B \to \mathbb{N}, b \mapsto \dim(E_b)$.
If $r(b) = n$ for all $b \in B$, we call p a vector bundle of rank n. Vector bundles of rank one are also called line bundles.

Lets see some examples of vector bundles. In the next section we will introduce a method to produce more vector bundles.

Example 2.7

(a) The trivial bundle of rank n is $E \coloneqq B \times \mathbb{R}^n$ together with the projection p to the first component. We write ε^n for the trivial bundle of rank n.

(b) The Möbius bundle is obtained as the quotient space E of $[0,1] \times \mathbb{R}$ by glueing $(0,t)$ to $(1,-t)$. The projection $[0,1] \times \mathbb{R} \to [0,1]$ induces a map $p\colon E \to S^1$ which turns E into a line bundle over S^1. The total space E is homeomorphic to a Möbius strip hence the name.

(c) Let M be a n-dimensional differentiable manifold. For every point $x \in M$ we have a tangent space T_xM. Using methods we will see later one can endow the disjoint union $TM \coloneqq \bigsqcup_{x \in M} T_xM$ with a topology such that together with the obvious map $TM \to M$ this becomes a vector bundle of rank n over M.
Manifolds which have a trivial tangent bundle are called parallelizable. It is a classical application of K-theory that S^n is parallelizable if and only if $n \in \{0,1,3,7\}$ (see [Hat03] Section 2.3).

(d) Consider the real projective space $\mathbb{RP}^n$. We view $\mathbb{RP}^n$ as the space of lines through the origin in $\mathbb{R}^{n+1}$. The total space of the canonical line bundle is given by
$$E \coloneqq \{(l,v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} \mid v \in l\} \subset \mathbb{RP}^n \times \mathbb{R}^{n+1}.$$
The map p is the projection to the first component.
To construct local trivialisations we now view $\mathbb{RP}^n$ as the quotient space of S^n where antipodal points are identified.
For $x \in \mathbb{RP}^n$ let L be the hyperplane through the origin orthogonal to x. Choose a representative in S^n named x again. Further let U_x be the open hemisphere containing x and bounded by L.

Local trivialisations can be obtained by orthogonally projecting down to L.
If π_L denotes the orthogonal projection onto L the trivialisations are explicitly given as

$$h_x \colon p^{-1}(U_x) \to U_x \times \mathbb{R}^{n+1}, (y, v) \mapsto (y, \pi_L(v)).$$

In the same way one obtains a complex line bundle over $\mathbb{C}\mathrm{P}^n$ which we also call canonical line bundle. The canonical line bundle H over $\mathbb{C}\mathrm{P}^1 \cong S^2$ plays an important role in K-theory.

Next we want to construct a category where the objects are vector bundles over a fixed base space B. Therefore we define morphisms of vector bundles over B.

Definition 2.8 (Morphism of vector bundles)
Let B be a topological space and $p_1 \colon E_1 \to B$, $p_2 \colon E_2 \to B$ vector bundles over B.
A morphism from p_1 to p_2 is a map $f \colon E_1 \to E_2$ satisfying the following two conditions

1. f maps the fibre $p_1^{-1}(b)$ of b under p_1 to the fibre $p_2^{-1}(b)$ of b under p_2 for every $b \in B$ i.e. we have a commutative diagram of the form

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & B & \end{array}$$

2. For every $b \in B$ the map $f_b \coloneqq f|_{p_1^{-1}(b)} \colon p_1^{-1}(b) \to p_2^{-1}(b)$ is linear.

Remark 2.9
For $i = 1, 2, 3$ let $p_i \colon E_i \to B$ be vector bundles over B. Let further $f \colon E_1 \to E_2$ and $g \colon E_2 \to E_3$ be morphisms of vector bundles. We immediately get that the following maps are morphism of vector bundles

- $\mathrm{id} \colon E_1 \to E_1$
- $g \circ f \colon E_1 \to E_3$

Thus for every space B we get a category where the objects are vector bundles over B and the morphism are maps according to Definition 2.8. We denoted by $\mathrm{Vect}(B)$ (respectively $\mathrm{Vect}^n(B)$) the set of isomorphism classes of vector bundles (of rank n respectively) over B. We write $\mathrm{Vect}_{\mathbb{K}}(B)$ if we want to specify the field $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

Remark 2.10
Being locally trivial (see Definition 2.1 2.) is equivalent to being locally isomorphic to a trivial bundle, hence the name.

Let $f \colon E_1 \to E_2$ be an isomorphism in $\mathrm{Vect}(B)$. Then we get from the definition of a morphism that f is a homeomorphism and f takes each fibre $(E_1)_b$ to $(E_2)_{f(b)}$ by a vector space isomorphism. We now show that the converse is also true.

Lemma 2.11
Let $p_1 \colon E_1 \to B$, $p_2 \colon E_2 \to B$ be vector bundles over B and $h \colon E_1 \to E_2$ a morphism of vector bundles. If h_b is a vector space isomorphism for each $b \in B$, then h is an isomorphism of vector bundles.

Proof. Since h is an isomorphism on each fibre we immediately get that h is bijective by Remark 2.2.
It remains to show that h^{-1} is a morphism of vector bundles. Since h preserves fibres h^{-1} also does. So only the continuity of h^{-1} is left to show.
Let $b \in B$. Since continuity is a local property we may choose an open set $U \subset B$ such that both p_1 and p_2 are trivial over U. Thus we may assume that h is given by

$$h \colon U \times \mathbb{R}^n \to U \times \mathbb{R}^n, (b, v) \mapsto (b, g_b(v)),$$

where $g_b \in \mathrm{GL}_n(\mathbb{R})$, since h_b is a linear isomorphism. Therefore h^{-1} is given by $(b, v) \mapsto (b, g_b^{-1}(v))$. The continuity of h implies that g_b depends continuously on b. Viewing g_b and g_b^{-1} as matrices we get that every entry of g_b^{-1} depends polynomially on the entries of g_b (one sees this by using the formula $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$, where $\operatorname{adj}(A)$ is the adjunct matrix of A). So g_b^{-1} also depends continuously on b and thus h is continuous. □

2.2 Operations on vector bundles

This section is dedicated to the construction of new bundles out of old ones. We will also generalize constructions known for vector spaces such as direct sums and tensor products to the case of vector bundles.

Definition 2.12 (Restriction of vector bundles)
Let $p \colon E \to B$ be a vector bundle over B and $U \subset B$. We call $p|_{p^{-1}(U)} \colon p^{-1}(U) \to U$ the restriction of E over U and denote it by E_U.

Remark 2.13
E_U is again a vector bundle. Indeed the fibres of E_U are fibres of E. To obtain local trivialisations for E_U one may restrict local trivialisations of E to $p^{-1}(U)$.
Mapping E to E_U induces a functor from the category of vector bundles over B to the category of vector bundles over U.

Definition 2.14 (Pullback bundle)
Let $p \colon E \to B$ be a vector bundle over B and $f \colon A \to B$ a map of spaces. We define the pullback bundle (of E along f) as

$$f^*(E) := \{(a, e) \in A \times E \mid f(a) = p(e)\} \subset A \times E$$

together with the map $f^*(E) \to A, (a, e) \mapsto a$.
For a map $h \colon E_1 \to E_2$ between vector bundles E_1 and E_2 over the same base B we further define $f^*(h) \colon f^*(E_1) \to f^*(E_2), (a, e_1) \mapsto (a, h(e_1))$.

Proposition 2.15 (Properties of the pullback)
Let $p\colon E \to B$ be a vector bundle over B and $f\colon A \to B$, $g\colon C \to A$ maps of spaces. then we have

(i) $f^*(E)$ is a vector bundle over A

(ii) $\mathrm{id}_B^*(E) \cong E$

(iii) $(f \circ g)^*(E) \cong g^*(f^*(E))$

(iv) $\mathrm{Vect}(B)$ depends contravariantly on B

(v) $f^*(\mathrm{id}) = \mathrm{id}$

(vi) $f^*(h_1 \circ h_2) = f^*(h_1) \circ f^*(h_2)$

(vii) f induces a contravariant functor from the category of vector bundles over B to the category of vector bundles over A

Proof. (i) By definition we have $f^*(E)_a = E_{f(a)}$. Thus the fibres of $f^*(E)$ are finite dimensional vector spaces. To see that $f^*(E)$ is locally trivial let $a \in A$ and $U \subset B$ be an open neighbourhood of $f(a)$ such that $E_U \cong \varepsilon^n$. Then $U' := f^{-1}(U)$ is an open neighbourhood of a. Since the pullback of a trivial bundle is again trivial we get $f^*(E)_{U'} \cong (f|_{U'})^*(E_U) \cong (f|_{U'})^*(\varepsilon^n) \cong \varepsilon^n$.

(ii) Clear from the definition.

(iii) Let $p'\colon f^*(A) \to A, (a,e) \mapsto a$ be the map turning $f^*(E)$ into a vector bundle. We have

$$\begin{aligned} g^*(f^*(E)) &= g^*(\{(a,e) \in A \times E \mid f(a) = p(e)\}) \\ &= \{(c,(a,e)) \in C \times (A \times E) \mid g(c) = p'(a,e) = a, f(a) = p(e)\} \\ &= \{(c, g(c), e) \in C \times A \times E \mid f(g(c)) = p(e)\} \\ &\cong \{(c,e) \in C \times E \mid (f \circ g)(c) = p(e)\} \\ &= (f \circ g)^*(E). \end{aligned}$$

(iv) This is a combination of (ii) and (iii)

(v) Clear from the definition.

(vi) $f^*(h_1 \circ h_2)(a, e_1) = (a, (h_1(h_2(e_1)) = f^*(h_1)(a, h_2(e_1)) = (f^*(h_1) \circ f^*(h_2))(a, e_1)$

(vii) This is a combination of (i), (v) and (vi)

□

Remark 2.16
The restriction E_U is the pullback of $E \to B$ along the inclusion $U \hookrightarrow B$.

Let $\mathrm{Vect}_{\mathbb{R}}$ denote the category of finite dimensional $\mathbb{R}$-vector spaces. On the set of vector spaces over $\mathbb{R}$ we have operations like the direct sum or the tensor product. We want to generalize those to vector bundles. For example given two vector bundles E and E' we want to construct a vector bundle $E \oplus E'$ with fibres $(E \oplus E')_b = E_b \oplus E'_b$. In particular if the base space is a single point we recover the original operation on vector spaces.

Definition 2.17
A functor $\varphi\colon \mathrm{Vect}_{\mathbb{R}} \to \mathrm{Vect}_{\mathbb{R}}$ is called continuous if for all vector spaces M, N the induced map $\varphi_{M,N}\colon \mathrm{Hom}(M,N) \to \mathrm{Hom}(\varphi(M),\varphi(N))$ is continuous.

Functors F that preserve limits are sometimes called continuous due to the equation $\lim(F(D)) = F(\lim(D))$, but this notion is different from the one we just defined.

Construction 2.18
Let $\varphi\colon \mathrm{Vect}_{\mathbb{R}} \to \mathrm{Vect}_{\mathbb{R}}$ be a continuous functor and E a vector bundle over B. We define

$$\varphi'(E) \coloneqq \bigsqcup_{b \in B} \varphi(E_b).$$

$\varphi'(E)$ comes with a canonical projection $\varphi'(E) \to B$ induced by the projections $\varphi(E_b) \to B$ for each $b \in B$ and we have $\varphi'(E)_b = \varphi(E_b)$ for each $b \in B$.
All we now need to do is equip $\varphi'(E)$ with a suitable topology. We will start by choosing a cover $\{U_i\}$ of B and local trivialisations $\alpha_i\colon E_{U_i} \to U_i \times \mathbb{R}^n$. These induce bijections

$$\beta_i\colon \varphi'(E)_{U_i} \to U_i \times \varphi(\mathbb{R}^n).$$

The space on the right hand side carries a natural topology and we equip $\varphi'(E)_{U_i}$ with the topology induced this bijection. Finally we equip $\varphi'(E)$ with the largest topology making the inclusions $\varphi'(E)_{U_i} \to \varphi'(E)$ continuous.
There are some technical details to be discussed, but let us believe for the moment that this is well defined.
Then the resulting map $\varphi'(E) \to B$ is locally trivial because $\varphi'(E)_{U_i} \cong U_i \times \varphi(\mathbb{R}^n)$.
We can further make this construction functorial in vector bundles.
For a map $f\colon E \to F$ between vector bundles E and F over B let $f'\colon \varphi'(E) \to \varphi'(F)$ be the map which is defined by setting $f'_b \coloneqq \varphi(f_b)\colon \varphi(E_b) \to \varphi(F_b)$ on each fibre $\varphi(E_b)$ of $\varphi'(E)$. It follows from the continuity of φ that f' is continuous and therefore a morphism of vector bundles (see [Kar09] I.4.6 for a detailed argument).

As mentioned before we still need to check two technical details verifying the well definedness of $\varphi'(E)$.

1. The resulting bundle is independent of the choice of the cover and local trivialisations.

2. The last step where we equipped $\varphi'(E)$ with a topology is possible.
   For example the following problem might occur. Lets take two inclusions
   $\iota_i\colon \varphi'(E)_{U_i} \to \varphi'(E)$ and $\iota_j\colon \varphi'(E)_{U_j} \to \varphi'(E)$ such that $U_i \cap U_j \neq \emptyset$. There might exist some $W \subset E_{U_i} \cap E_{U_j} \neq \emptyset$ such that $\iota_i^{-1}(W)$ is open but $\iota_j^{-1}(W)$ is not.
   So ι_i being continuous requires W to be open in $\varphi'(E)$ (since then ι_i is a homeomorphism onto its image) but ι_j being continuous forbids that.

Both points are resolved by the following lemma.

Lemma 2.19
Let $\alpha_1 \colon E_{U_1} \to U_1 \times M$ and $\alpha_2 \colon E_{U_2} \to U_2 \times N$ be local trivialisations of B, such that $U_1 \cap U_2 \neq \emptyset$. Furthermore let $\beta_1 \colon \varphi'(E)_{U_1} \to U_1 \times \varphi(M)$ and $\beta_2 \colon \varphi'(E)_{U_2} \to U_2 \times \varphi(N)$ be the induced bijections. We equip $\varphi'(E)_{U_1}$, respectively $\varphi'(E)_{U_2}$ with the topology induced by β_1 respectively β_2.
Then these topologies agree on $\varphi'(E)_{U_1 \cap U_2} = \varphi'(E)_{U_1} \cap \varphi'(E)_{U_2}$ and $\varphi'(E)_{U_1 \cap U_2}$ is open in both $\varphi'(E)_{U_1}$ and $\varphi'(E)_{U_2}$.

Proof. Define

$$s = \alpha_2|_{U_1 \cap U_2} \circ \alpha_1|_{U_1 \cap U_2}^{-1} \colon U_1 \cap V_1 \to \mathrm{Hom}(M, N)$$

and

$$\delta \coloneqq \varphi_{M,N} \circ s \colon U_1 \cap U_2 \to \mathrm{Hom}(\varphi(M), \varphi(N)).$$

The map $\varphi_{M,N}$ is continuous because φ is a continuous functor. Therefore δ is continuous. Interchanging M, N shows that δ^{-1} is also continuous. δ induces a continuous map

$$\hat{\delta} \colon (U_1 \cap U_2) \times \varphi(M) \to (U_1 \cap U_2) \times \varphi(N)$$

which is actually a homeomorphism since δ^{-1} induces $\hat{\delta}^{-1}$. Furthermore we have a commutative diagram of the form

$$\begin{array}{ccc} & & (U_1 \cap U_2) \times \varphi(M) \\ & \overset{\beta_1|_{U_1 \cap U_2}}{\nearrow} & \\ \varphi'(E)_{U_1 \cap U_2} & & \Big\downarrow \hat{\delta} \\ & \underset{\beta_2|_{U_1 \cap U_2}}{\searrow} & \\ & & (U_1 \cap U_2) \times \varphi(N) \end{array} .$$

Thus the topologies on $\varphi'(E)_{U_1}$ and $\varphi'(E)_{U_2}$ agree. The projection $p_1 \colon \varphi'(E)_{U_1} \to U_1$ is continuous so we further get that $\varphi'(E)_{U_1 \cap U_2} = p_1^{-1}(U_1 \cap U_2)$ is open in $\varphi'(E)_{U_1}$. Similarly $\varphi'(E)_{U_1 \cap U_2}$ is open in $\varphi'(E)_{U_2}$. ☐

For Construction 2.18 we have so far only considered functors taking a single real vector space and also returning only a single real vector space in a covariantly manner. The construction works the same for several generalisations. The first is that we can allow complex vector spaces. These may be mixed, meaning that the input is a real vector space and the output is a complex vector space. Resulting in a construction that has as input a real vector bundle and as output a complex vector bundle.
Furthermore we can input more than one vector bundle (one might also allow more than one output (See [Kar09] I.4.7), but we do not need that here).
The last generalisation is that we allow the output to depend contravariantly on the input.
The final construction takes the following form. Define $\mathcal{C}$ to be the category

$$\mathcal{C} := \prod_{i_1 \in I_1} \mathrm{Vect}_{\mathbb{R}} \times \prod_{i_2 \in I_2} \mathrm{Vect}_{\mathbb{C}} \times \prod_{i_3 \in I_3} \mathrm{Vect}_{\mathbb{R}}^{op} \times \prod_{i_4 \in I_4} \mathrm{Vect}_{\mathbb{C}}^{op}$$

where I_j, $j = 1, \ldots, 4$ are finite sets. We consider functors

$$\varphi \colon \mathcal{C} \to \mathrm{Vect}_{\mathbb{K}},$$

where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $(-)^{op}$ denotes the opposite category. We also require that for all objects $M, N \in \mathcal{C}$ the map

$$\varphi_{M,N} \colon \mathrm{Hom}(M, N) \to \mathrm{Hom}(\varphi(M), \varphi(N))$$

is continuous. Running through the construction we end up with a functor

$$\widetilde{\varphi} \colon \widetilde{\mathcal{C}} \to \mathrm{Vect}_{\mathbb{K}}(B)$$

where $\widetilde{\mathcal{C}} := \prod_{i_1 \in I_1} \mathrm{Vect}_{\mathbb{R}}(B) \times \prod_{i_2 \in I_2} \mathrm{Vect}_{\mathbb{C}}(B) \times \prod_{i_3 \in I_3} \mathrm{Vect}_{\mathbb{R}}^{op}(B) \times \prod_{i_4 \in I_4} \mathrm{Vect}_{\mathbb{C}}^{op}(B)$.
We can now apply the construction to produce various examples of vector bundles.

Example 2.20

(a) Let $\varphi \colon \mathrm{Vect}_{\mathbb{K}} \times \mathrm{Vect}_{\mathbb{K}} \to \mathrm{Vect}_{\mathbb{K}}$ be the functor given by $(M, N) \mapsto M \oplus N$. We obtain a functor $\widetilde{\varphi} \colon \mathrm{Vect}_{\mathbb{K}}(B) \times \mathrm{Vect}_{\mathbb{K}}(B) \to \mathrm{Vect}_{\mathbb{K}}(B)$. For vector bundles E, F we call $\widetilde{\varphi}(E, F)$ Whitney sum (or direct sum) of E and F and denote it by $E \oplus F$.

(b) Taking $\varphi \colon \mathrm{Vect}_{\mathbb{K}} \times \mathrm{Vect}_{\mathbb{K}} \to \mathrm{Vect}_{\mathbb{K}}$ to be the tensor product functor given by $(M, N) \mapsto M \otimes_{\mathbb{K}} N$ we obtain the tensor product $\widetilde{\varphi}(E, F) = E \otimes F$ of E and F.

(c) We also have the i-th exterior power $\lambda^i(E)$ induced by the functor given by $M \mapsto \Lambda^i(M)$.

(d) Let $\varphi \colon \mathrm{Vect}_{\mathbb{R}} \to \mathrm{Vect}_{\mathbb{C}}, M \mapsto M \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification functor. We get a functor $\widetilde{\varphi} \colon \mathrm{Vect}_{\mathbb{R}}(B) \to \mathrm{Vect}_{\mathbb{C}}(B)$ turning real vector bundles into complex vector bundles.
Since every complex vector space is also a real vector space we get a functor turning a complex vector bundle into a real vector bundle induced by the forgetful functor $\mathrm{Vect}_{\mathbb{C}} \to \mathrm{Vect}_{\mathbb{R}}$.

(e) If M is a complex vector space then its conjugate vector space $\bar{M}$ has the same underlying set but the scalar multiplication is defined by $\lambda * v := \bar{\lambda} v$, where on the right hand side we use the scalarmultiplication in M.
The functor $\varphi \colon \mathrm{Vect}_{\mathbb{C}} \to \mathrm{Vect}_{\mathbb{C}}, M \mapsto \bar{M}$ induces a functor $\mathrm{Vect}_{\mathbb{C}}(B) \to \mathrm{Vect}_{\mathbb{C}}(B)$ called the conjugation of vector bundles.

(f) From the functor $\varphi \colon \mathrm{Vect}_{\mathbb{K}}^{op} \times \mathrm{Vect}_{\mathbb{K}} \to \mathrm{Vect}_{\mathbb{K}}$ induced by $(M, N) \mapsto \mathrm{Hom}(M, N)$ we get the Hom bundle $\mathrm{Hom}(E, F)$

(g) The dual space functor $\mathrm{Vect}_{\mathbb{K}}^{op} \to \mathrm{Vect}_{\mathbb{K}}, M \mapsto M^*$ induces a functor $\mathrm{Vect}_{\mathbb{K}}(B) \to \mathrm{Vect}_{\mathbb{K}}(B)$. We call E^* the dual bundle of E.

The Whitney sum and the tensor product are of particular importance in K-theory since, as we will see later, they induce a ring structure on the K-groups. The exterior power is used to construct the Adams operation.

Remark 2.21 (Alternative definitions of the direct sum)
There are two alternative definitions of the Whitney sum which are more explicit.

1. $E_1 \oplus E_2 = \{(e_1, e_2) \in E_1 \times E_2 \mid p_1(e_1) = p_2(e_2)\}$
2. $E_1 \oplus E_2 = \Delta^*(E_1 \times E_2)$. Where we view $E_1 \times E_2$ as a vector bundle over $B \times B$ via the mapping $(e_1, e_2) \mapsto (p(e_1), p(e_2))$ and Δ is the diagonal map $\Delta\colon B \to B \times B, b \mapsto (b, b)$.

The next propositions record some useful properties for later use. The proof strategy will be the same for all of these. Namely one writes down a fibre preserving continuous map that is a vector space isomorphism on each fibre and then uses Lemma 2.11 to conclude that this map is an isomorphism of vector bundles. Most of the properties are obtained by using the corresponding isomorphism of vector spaces on each fibre. We will illustrate this procedure once and omit the other proofs.

Proposition 2.22 (Properties of the direct sum)
Let $E_i \to B$ be vector bundles over B for $i = 1, 2, 3$ and $f\colon A \to B$ be a map of spaces. We have the following properties

(i) $E_1 \oplus (E_2 \oplus E_3) \cong (E_1 \oplus E_2) \oplus E_3$

(ii) $E_1 \oplus E_2 \cong E_2 \oplus E_1$

(iii) $f^*(E_1 \oplus E_2) \cong f^*(E_1) \oplus f^*(E_2)$

Proof. We illustrate the procedure for (iii).
For $(a, e_1, e_2) \in f^*(E_1 \oplus E_2)$ we define $h((a, e_1, e_2)) := ((a, e_1), (a, e_2)) \in f^*(E_1) \oplus f^*(E_2)$. h is continuous, fibre preserving and linear on each fibre thus a morphism of vector bundles. It is clear that the induced map on the fibres is an vector space isomorphism. □

Proposition 2.23 (Properties of the tensor product)
Let $E_i \to B$ be vector bundles over B for $i = 1, 2, 3$ and $f\colon A \to B$ be a map of spaces. We have the following properties

(i) $E_1 \otimes (E_2 \otimes E_3) \cong (E_1 \otimes E_2) \otimes E_3$

(ii) $E_1 \otimes E_2 \cong E_2 \otimes E_1$

(iii) $E_1 \otimes \varepsilon^1 \cong E_1$

(iv) $f^*(E_1 \otimes E_2) \cong f^*(E_1) \otimes f^*(E_2)$

Proposition 2.24 (Tensor product and direct sum)
Let $E_i \to B$ be vector bundles over B for $i = 1, 2, 3$. Then we have

$$(E_1 \oplus E_2) \otimes E_3 \cong (E_1 \otimes E_3) \oplus (E_2 \otimes E_3).$$

Proof. We again use the idea from the proof of Proposition 2.22. The continuous fibre preserving map is given by $(e_1, e_2) \otimes e_3 \mapsto (e_1 \otimes e_3, e_2 \otimes e_3)$. □

Proposition 2.25 (Properties of the exterior power)
Let $E, F \to B$ be vector bundles over B and $f\colon A \to B$ be a map of spaces. We have the following properties

(i) $\lambda^0(E) = \varepsilon^1$

(ii) $\lambda^1(E) = E$

(iii) $\lambda^k(E) = 0$ for k larger then the highest dimension of a fibre of E

(iv) $\lambda^k(E \oplus F) \cong \bigoplus_{i+j=k} (\lambda^i(E) \otimes \lambda^j(F))$

(v) $f^*(\lambda^k(E)) = \lambda^k(f^*(E))$

To finish off this section, we are going to introduce two more operations on vector bundles. At first they might look like Example 2.20 (a) and (b) we have already seen but they differ in that they allow as input vector bundles over different base spaces B and B' and produce a vector bundle over the cartesian product $B \times B'$.

Definition 2.26
Let $p_1\colon E \to B$ and $p_2\colon E' \to B'$ be vector bundles. Further let $\pi_1\colon B \times B' \to B$ and $\pi_2\colon B \times B' \to B'$ be the projection to the first respectively second component. We define the external Whitney sum as

$$E \boxplus E' \coloneqq \pi_1^*(B) \oplus \pi_2^*(B').$$

Further we define the external tensor product as

$$E \boxtimes E' \coloneqq \pi_1^*(B) \otimes \pi_2^*(B').$$

By definition those are vector bundles over $B \times B'$.

Remark 2.27

1. As a set we have $E \boxplus E' = E \times E'$.
2. The fibres are given by $(E \boxplus E')_{(b,b')} = E_b \oplus E'_{b'}$ and $(E \boxtimes E')_{(b,b')} = E_b \otimes E'_{b'}$ for $(b, b') \in B \times B'$.

2.3 Homotopical aspects of vector bundles

The goal of this short section is to show that for a map $f\colon A \to B$ the induced map $f^*\colon \mathrm{Vect}(B) \to \mathrm{Vect}(A)$ on vector bundles only depends on the homotopy class of f. From this we will later deduce the homotopy invariance of K-theory.

Theorem 2.28
Let X be compact Hausdorff and $p\colon E \to X \times I$ a vector bundle where $I = [0,1]$. Then the restrictions $E_{X\times\{0\}}$ and $E_{X\times\{1\}}$ are isomorphic.

Proof. For the vector bundle p there exists an open cover of $X\times I$ such that the restriction of p to each open subset is trivial. We start by showing that the cover may be chosen of a particular type. Namely of the form $\{U_\alpha \times I\}$ where $\{U_\alpha\}$ is an open cover of X.
We fix $x \in X$. For every point $(x,t) \in X \times I$ there is an open neighbourhood such that the restriction of p is trivial. Each of these neighbourhoods contains a subset of the form $U \times (t,t')$. Those sets form an open cover of $\{x\} \times I$ which has a finite subcover by compactness of I. By choosing a number in each intersection of (t,t') with $(\widetilde{t},\widetilde{t'})$ we see that there exists a partition $0 = t_0 < \cdots < t_k = 1$ of $[0,1]$ and open neighbourhoods $U_{x,1},\dots,U_{x,k} \subset X$ of x such that p restricted to $U_{x,i} \times [t_{i-1},t_i]$ is trivial.
Define $U_\alpha \coloneqq U_{x,1}\cap\cdots\cap U_{x,k}$. We now want to show that p is trivial over $U_\alpha \times I$. For this let $h_0\colon p^{-1}(X\times[t_0,t_1]) \to X\times[t_0,t_1]\times\mathbb{R}^n$ and $h_1\colon p^{-1}(X\times[t_1,t_2]) \to X\times[t_1,t_2]\times\mathbb{R}^n$ be local trivialisations.
On the slice $X\times\{t_1\}\times\mathbb{R}^n$ we have the isomorphism

$$\phi \coloneqq h_0 \circ h_1^{-1}\colon X\times\{t_1\}\times\mathbb{R}^n \to X\times\{t_1\}\times\mathbb{R}^n,$$

which is the identity in the first two component. Therefore we view ϕ as an automorphism of $X\times\mathbb{R}^n$. We obtain an induced automorphism again called ϕ

$$\phi\colon X\times[t_1,t_2]\times\mathbb{R}^n \to X\times[t_1,t_2]\times\mathbb{R}^n$$

by using ϕ from above in every slice $X\times\{t\}\times\mathbb{R}^n \subset X\times[t_1,t_2]\times\mathbb{R}^n$.
Now $\phi\circ h_2$ is a local trivialisation of $p^{-1}(X\times[t_1,t_2])$ which coincides with h_1 on $p^{-1}(X\times[t_0,t_1])\cap p^{-1}(X\times[t_1,t_2])$. We can therefore put them together to get a local trivialisation of $p^{-1}(X\times[t_0,t_2])$. By induction we get a local trivialisation of $U_\alpha\times I$.
Since X is compact finitely many of the U_α's cover $X\times I$ and we call them U_i, $i=1,\dots,m$. Furthermore we choose a partition of unity $\{\varphi_i\}_{i=1,\dots,m}$ for the cover $\{U_i\}_{i=1,\dots,m}$. In particular φ_i has support in U_i.
Let $\psi_i \coloneqq \varphi_1+\cdots+\varphi_i$ for $i=1,\dots,m$ and let $X_i \coloneqq \{(x,\psi_i(x)) \mid x\in X\}$ be the graph of ψ_i. Since $\{\varphi_i\}_{i=1,\dots,m}$ is a partition of unity we have $X_0 = X\times\{0\}$ and $X_m = X\times\{1\}$.
Finally let $p_i\colon E_i \to X_i$ be the restriction of p to X_i.
We have a map $\widetilde{h}_i\colon X_i \to X_{i-1}, (x,\psi_i(x)) \mapsto (x,\psi_{i-1}(x))$. The difference between ψ_i and ψ_{i-1} is φ_i and $\widetilde{h}_i$ is the identity outside of U_i because the support of φ_i is contained in U_i.
We can lift $\widetilde{h}_i$ to a morphism of vector bundles $h_i\colon E_i \to E_{i-1}$ by mapping the point $(x,\psi_i(x),v) \in p^{-1}(U_i\times I) = U_i\times I\times\mathbb{R}^n$ to $(x,\psi_{i-1}(x),v)$. This is an isomorphism because p is trivial over $U_i\times I$.
Consequently the composition $h_1\circ\cdots\circ h_m$ is an isomorphism from $E_{X\times\{0\}}$ to $E_{X\times\{1\}}$ which finishes the proof. $\square$

Theorem 2.29
Let X be compact Hausdorff and let $f_0, f_1 \colon X \to Y$ be two homotopic maps. Further let $p \colon E \to Y$ be a vector bundle. Then $f_0^*(E)$ and $f_1^*(E)$ are isomorphic.

Proof. Let $F \colon X \times I \to Y$ be a homotopy from f_0 to f_1. Then $\widetilde{E} \coloneqq F^*(E)$ is a vector bundle over $X \times I$. We further have $\widetilde{E}_{X\times\{0\}} = f_0^*(E)$ and $\widetilde{E}_{X\times\{1\}} = f_1^*(E)$. So $f_0^*(E) \cong f_1^*(E)$ by the previous theorem. □

Corollary 2.30 (Homotopy invariance of $\mathrm{Vect}(X)$)
Let X be compact Hausdorff. Then $\mathrm{Vect}(X)$ only depends on the homotopy type of X. More explicitly if $f \colon X \to Y$ is a homotopy equivalence between compact spaces X and Y, then $f^* \colon \mathrm{Vect}(Y) \to \mathrm{Vect}(X)$ is a bijection.
In particular every vector bundle over a contractible space is trivial.

Proof. Let $g \colon Y \to X$ be a homotopy inverse of f. Using the previous Theorem and Proposition 2.15 *(ii)*, *(iii)* we get $f^* \circ g^* = (g \circ f)^* = \mathrm{id}^* = \mathrm{id}$ and $g^* \circ f^* = (f \circ g)^* = \mathrm{id}^* = \mathrm{id}$. So g^* is the inverse of f^*. □

Remark 2.31
Since the pullback preserves the rank (or rather the rank of every connected component has the rank of its image) the corollary is also true for $\mathrm{Vect}^n(X)$ instead of $\mathrm{Vect}(X)$.

2.4 Clutching construction and vector bundles on spheres

In this section we want to introduce the clutching construction. This is will allow us to classify vector bundles over spheres.

The clutching construction allows us to clutch bundles on subspaces together to get a bundle on the whole space. To motivate the construction we take a vector bundle $p \colon E \to B$ and choose an open cover $\{U_\alpha\}$ with local trivialisations $h_\alpha \colon p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$. Using the h_α we can reconstruct p in the following way.
Start with the disjoint union $\bigsqcup_\alpha (U_\alpha \times \mathbb{R}^n)$ and let $x \in U_\alpha \cap U_\beta$ for $U_\alpha \cap U_\beta \neq \emptyset$.
Every point $y \in p^{-1}(x)$ can be identified with both $h_\alpha(y) = (x, v_1) \in U_\alpha \times \mathbb{R}^n$ and $h_\beta(y) = (x, v_2) \in U_\beta \times \mathbb{R}^n$. To reconstruct E we need to glue (x, v_1) and (x, v_2) together. Therefore to obtain the bundle E from $\bigsqcup_\alpha (U_\alpha \times \mathbb{R}^n)$ we repeat this procedure for every point in E.
This may be rephrased as glueing $(x, v) \in U_\alpha \times \mathbb{R}^n$ to $h_\beta h_\alpha^{-1}(x, v) \in U_\beta \times \mathbb{R}^n$ whenever $x \in U_\alpha \cap U_\beta \neq \emptyset$. The resulting quotient space of $\bigsqcup_\alpha (U_\alpha \times \mathbb{R}^n)$ is the bundle E we started with.
The functions $h_\beta h_\alpha^{-1}$ are called transition functions and can be seen as maps

$$(U_\alpha \cap U_\beta) \times \mathbb{R}^n \to (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

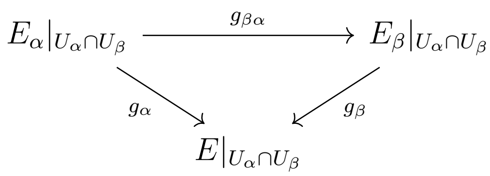
which are the identity on the first component. Since h_α and h_β are local trivialisations, they induce isomorphism on each fibre. So one might view the $h_\beta h_\alpha^{-1}$ as continuous maps

$$g_{\beta\alpha}\colon U_\alpha \cap U_\beta \to \mathrm{GL}_n(\mathbb{R})$$

further satisfying the so called cocycle condition $g_{\gamma\beta} \circ g_{\beta\alpha} = g_{\gamma\alpha}$ on $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$.

The following theorem says that the opposite is also true. This means given vector bundles p_α on each U_α of the cover $\{U_\alpha\}$ together with functions $g_{\beta\alpha}\colon U_\alpha \cap U_\beta \to \mathrm{GL}_n(\mathbb{R})$ satisfying the cocycle condition one can construct a bundle $p\colon E \to B$ such that $p|_{U_\alpha} \cong p_\alpha$.

Theorem 2.32 (Clutching of bundles)
Let $\{U_\alpha\}$ be an open cover of B and $p_\alpha\colon E_\alpha \to U_\alpha$ be vector bundles. Furthermore let $g_{\beta\alpha}\colon U_\alpha \cap U_\beta \to \mathrm{GL}_n(\mathbb{R})$ be continuous maps such that $g_{\alpha\alpha}(x) = \mathrm{id}$ for all $x \in U_\alpha$ and they satisfy the cocycle condition $g_{\gamma\beta} \circ g_{\beta\alpha} = g_{\gamma\alpha}$ on $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$. Such a family of maps is called a cocycle.
Then there exists a vector bundle $p\colon E \to B$ and isomorphisms $g_\alpha\colon p_\alpha \to p|_{U_\alpha}$ making the following diagram commute



($g_{\beta\alpha}$ induces an isomorphism $E_\alpha|_{U_\alpha\cap U_\beta} \to E_\beta|_{U_\alpha\cap U_\beta}$. To simplify the notation we name it again $g_{\beta\alpha}$). Furthermore p is unique up to isomorphism.

Proof. Every $g_{\beta\alpha}$ induces an isomorphism $E_\alpha|_{U_\alpha\cap U_\beta} \to E_\beta|_{U_\alpha\cap U_\beta}$ again called $g_{\beta\alpha}$ by mapping a point $(b, v) \in (U_\alpha \cap U_\beta) \times \mathbb{R}^n$ to $(b, g_{\beta\alpha}(b)v)$.
These also satisfy the cocycle condition $g_{\gamma\beta} \circ g_{\beta\alpha} = g_{\gamma\alpha}$ on $U_\alpha \cap U_\beta \cap U_\gamma$.
Consider the disjoint union $\bigsqcup_\alpha E_\alpha$. For $e_\alpha \in E_\alpha|_{U_\alpha\cap U_\beta}$ and $e_\beta \in E_\beta|_{U_\alpha\cap U_\beta}$ we define the relation $e_\alpha \sim e_\beta$ if and only if $g_{\beta\alpha}(e_\alpha) = e_\beta$.
This is indeed an equivalence relation. Transitivity is just the cocycle condition. Reflexivity follows from $g_{\alpha\alpha}(x) = \mathrm{id}$ for all $x \in U_\alpha$. Using the cocycle condition we get $g_{\alpha\beta} \circ g_{\beta\alpha}(x) = g_{\alpha\alpha}(x) = x$ for all $x \in U_\alpha \cap U_\beta$ showing symmetry.
We define E to be the quotient space $(\bigsqcup_\alpha E_\alpha)/\sim$. The p_α induce a continuous map $p\colon E \to B$. We now show that this is a vector bundle having the desired properties.
For $b \in U_\alpha$ we have $E_b = p^{-1}(b) = p_\alpha^{-1}(b) = (E_\alpha)_x$ and therefore E_b carries a vector space structure. If b is also in U_β then $(E_\alpha)_b = (E_\beta)_b$ and the vector space structures coincide since $g_{\beta\alpha}$ is linear on each fibre.
Let $g_\alpha\colon E_\alpha \to p^{-1}(U_\alpha)$ be the map that sends a point to its equivalence class in E. This map is a homeomorphism inducing a vector space isomorphism on each fibre. Since being locally trivial is a local condition and E_α is locally trivial we get that E is locally trivial and thus a vector bundle. By Lemma 2.11 g_α is an isomorphism from p_α to $p|_{U_\alpha}$. ☐

We now come back to the special case of spheres.

Construction 2.33 (Clutching function)
Consider the upper and lower hemisphere $D^k_+, D^k_- \subset S^k$ of the k dimensional sphere and their intersection $D^k_+ \cap D^k_- = S^{k-1}$. Lets choose vector bundles on D^k_+, D^k_-. Because of Corollary 2.30 we can up to isomorphism only choose the trivial bundles $D^k_+ \times \mathbb{R}^n$, $D^k_- \times \mathbb{R}^n$ for some $n \in \mathbb{N}$.
Let $f\colon S^{k-1} \to \mathrm{GL}_n(\mathbb{R})$ be a continuous map. Since our cover only consists of two subsets the cocycle condition is trivially fulfilled. Such a f is called clutching function.
Now we are ready to apply the above construction. We take $D^k_+ \times \mathbb{R}^n \sqcup D^k_- \times \mathbb{R}^n$ and glue together $(x, v) \in \partial D^k_+ \times \mathbb{R}^n$ and $(x, f(x)(v)) \in \partial D^k_- \times \mathbb{R}^n$ obtaining a vector bundle $E_f \to S^k$.
There is a small technical detail to resolve.
The construction requires a cover by open sets which D^k_+ and D^k_- are not. This is not a problem since we can just enlarge both hemispheres by a small ε over the equator. The intersection is then homeomorphic to $S^{k-1} \times (-\varepsilon, \varepsilon)$ and in every slice $S^{k-1} \times \{t\}$ as a map we take f.

In this construction we never used that our vector bundles are real thus everything above works the same in the complex case.

Example 2.34 (Clutching function of the canonical line bundle H over S^2)
The space $\mathbb{C}\mathrm{P}^1 \cong S^2$ may be written as the quotient of $\mathbb{C}^2 \setminus \{0\}$ modulo the equivalence relation $(z_0, z_1) \sim \lambda(z_0, z_1)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. We write $[z_0, z_1]$ for the equivalence class of (z_0, z_1) and identify $[z_0, z_1]$ with $\frac{z_0}{z_1} \in \mathbb{C} \cup \{\infty\} = S^2$.
The equatorial circle in S^2 are then exactly the points $[z, 1]$ where $|z| = 1$. The points in the lower hemisphere D^2_- are exactly the points $[z, 1]$ where $|z| \leq 1$. A local trivialisation of the canonical line bundle H over D^2_- is given by $[z, 1] \mapsto z$.
Similarly the points on the upper hemisphere D^2_+ are the points $[1, z]$ for $|z| \leq 1$. A local trivialisation is given by $[1, z] \mapsto \frac{1}{z}$.
For these local trivialisations the transition function $S^2 \to \mathrm{GL}_1(\mathbb{C})$ is given by $z \mapsto z$.

Proposition 2.35
The isomorphism class of E_f only depends on the homotopy class of $f\colon S^{k-1} \to \mathrm{GL}_n(\mathbb{C})$.

Proof. Given a homotopy $F\colon S^{k-1} \times I \to \mathrm{GL}_n(\mathbb{C})$ from f to g we use the clutching construction to obtain the vector bundle $E_F \to S^k \times I$. Then E_F restricted to $S^k \times \{0\}$ is E_f and restricted to $S^k \times \{1\}$ it is E_g. Now we use Theorem 2.28 to conclude $E_f \cong E_g$. □

Example 2.36 (An equation for H)
We can now show that the canonical line bundle H over S^2 satisfies the equation $\mathrm{H}^2 \oplus \varepsilon^1 \cong \mathrm{H} \oplus \mathrm{H}$.
Using Example 2.34 we see that the clutching function for the left hand side is given by $z \mapsto \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix}$ and for the right hand side it is $z \mapsto \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$. Since $\mathrm{GL}_2(\mathbb{C})$ is path connected these two clutching functions lie in the same homotopy class and therefore define isomorphic bundles by the previous proposition.

Theorem 2.37
The map $\phi\colon \pi_{k-1}(\mathrm{GL}_n(\mathbb{C})) \to \mathrm{Vect}^n_{\mathbb{C}}(S^k), f \mapsto E_f$ *is a bijection for* $k \geq 1$.

Proof. By Proposition 2.35 the map in the theorem is well defined. We construct a inverse of ϕ.
Let $E \to S^k$ be a n-dimensional vector bundle. Then $E_{D^k_+}$ and $E_{D^k_-}$ are trivial since $D^k_\pm$ are contractible (see Corollary 2.30). Let $h_\pm\colon E_{D^k_\pm} \to D^k_\pm \times \mathbb{C}^n$ be local trivialisations. The transition function $h_+h_-^{-1}$ induces a map $\psi(E)\colon S^{k-1} \to \mathrm{GL}_n(\mathbb{C})$.
Since D^k is contractible and $\mathrm{GL}_n(\mathbb{C})$ is path-connected the maps $h_\pm$ are unique up to homotopy. Thus $h_+h_-^{-1}$ is unique up to homotopy.
Since $\mathrm{GL}_n(\mathbb{C})$ is path-connected every map is homotopic to a basepoint preserving map. Indeed let $f\colon S^{k-1} \to \mathrm{GL}_n(\mathbb{C})$ be a map with $f(e) = a \in \mathrm{GL}_n(\mathbb{C})$, where e is the basepoint in S^{k-1}. Choose a path $\gamma\colon [0,1] \to \mathrm{GL}_n(\mathbb{C})$ from the identity to a^{-1} and define

$$g\colon S^{k-1} \to \mathrm{GL}_n(\mathbb{C}), x \mapsto f(x)a^{-1}.$$

Then g is continuous and basepoint preserving since $g(e) = f(e)a^{-1} = aa^{-1} = \mathrm{id}$.
We further define $H\colon S^{k-1}\times[0,1] \to \mathrm{GL}_n(\mathbb{C}), (x,t) \mapsto f(x)\gamma(t)$. The map H is continuous and for all $x \in S^{k-1}$ we have

$$\begin{aligned} H(x,0) &= f(x)\gamma(0) = f(x), \\ H(x,1) &= f(x)\gamma(1) = f(x)a^{-1} = g(x). \end{aligned}$$

Thus H is a homotopy between f and g.
Putting all the above together we obtain a well defined map

$$\psi\colon \mathrm{Vect}^n_{\mathbb{C}}(S^k) \to \pi_{k-1}(\mathrm{GL}_n(\mathbb{C})), E \mapsto \psi(E).$$

which is the inverse of ϕ by construction. □

The unitary group $\mathrm{U}(n)$ is a deformation retract of $\mathrm{GL}_n(\mathbb{C})$. Thus we get the following

Corollary 2.38
Let $k \geq 1$. Then we have $\mathrm{Vect}^n_{\mathbb{C}}(S^k) \cong \pi_{k-1}(\mathrm{U}(n))$.

Corollary 2.39
Every complex vector bundle over S^1 is isomorphic to a trivial bundle.

Proof. This is a consequence of the path-connectedness of $\mathrm{GL}_n(\mathbb{C})$ respectively $\mathrm{U}(n)$ for all $n \geq 1$ and the previous theorem. □

The proof of Theorem 2.37 does not carries over to the real case, since $\mathrm{GL}_n(\mathbb{R})$ is not path-connected. But there is a similar result.
Let $a\colon S^0 \to \mathrm{GL}_n(\mathbb{R})$ be a representative of an element in $\pi_0(\mathrm{GL}_n(\mathbb{R}))$. Since a is basepoint preserving and S^0 is the disjoint union of two points a is determined by the image of the non basepoint. Therefore we view a just as an matrix in $\mathrm{GL}_n(\mathbb{R})$.
Let $f\colon S^{k-1} \to \mathrm{GL}_n(\mathbb{R})$ be a representative of an element in $\pi_{k-1}(\mathrm{GL}_n(\mathbb{R}))$. The map

$$f_a\colon S^{k-1} \to \mathrm{GL}_n(\mathbb{R}), x \mapsto af(x)a^{-1}$$

defines an element in $\pi_{k-1}(\mathrm{GL}_n(\mathbb{R}))$ and

$$\pi_0(\mathrm{GL}_n(\mathbb{R})) \times \pi_{k-1}(\mathrm{GL}_n(\mathbb{R})) \to \pi_{k-1}(\mathrm{GL}_n(\mathbb{R})), (a, f) \mapsto f_a$$

defines a group action of $\pi_0(\mathrm{GL}_n(\mathbb{R}))$ on $\pi_{k-1}(\mathrm{GL}_n(\mathbb{R}))$.
The next Proposition says that there is a bijection between the quotient by this action and the set of isomorphism classes of real rank n vector bundles over S^k. A proof can be found in [Kar09] Theorem I.7.6.

Theorem 2.40
The map $\pi_{k-1}(\mathrm{GL}_n(\mathbb{R}))/\pi_0(\mathrm{GL}_n(\mathbb{R})) \to \mathrm{Vect}^n_{\mathbb{R}}(S^k), f \mapsto E_f$ is a bijection for all $k \geq 1$.

Corollary 2.41
If n is odd then we have $\mathrm{Vect}^n_{\mathbb{R}}(S^k) \cong \pi_{k-1}(\mathrm{O}(n))$ for all $k \geq 1$.
If $k \geq 2$ we additionally have $\mathrm{Vect}^n_{\mathbb{R}}(S^k) \cong \pi_{k-1}(\mathrm{SO}(n))$.

Proof. $\mathrm{GL}_n(\mathbb{R})$ has two connected components. One connected component consists of all matrices with positive determinant and the other are all matrices of negative determinant. If n is odd then $-\mathrm{id}$ and id lie in different path components, since the determinant of $-\mathrm{id}$ is $(-1)^n = -1$. So we can always choose either id or $-\mathrm{id}$ as a representative for the action of $\pi_0(\mathrm{GL}_n(\mathbb{R}))$. But both id and $-\mathrm{id}$ act trivially and thus the hole action of $\pi_0(\mathrm{GL}_n(\mathbb{R}))$ is trivial.
Using the previous Theorem we have shown the first isomorphism.
Similarly to $\mathrm{GL}_n(\mathbb{R})$ also $\mathrm{O}(n)$ has two connected components one of them being $\mathrm{SO}(n)$. Let $k \geq 2$ then elements in $\pi_{k-1}(\mathrm{O}(n))$ are equivalence classes of basepoint-preserving maps thus attain at least one value (namely id) in $\mathrm{SO}(n)$. Since S^k is connected for $k \geq 1$ their image is completely contained in $\mathrm{SO}(n)$ and we obtain the second isomorphism. □

2.5 Inner products and Spin-structures on vector bundles

In this section we generalize inner products on vector spaces to vector bundles. We show that if the base space is compact Hausdorff then on every vector bundle we can always find a unique inner product. Further we will use this inner product to show that every vector bundle is a direct summand of a trivial vector bundle. This will later yield a convenient description of the reduced K-group.
We also introduce Spin structures of real vector bundles. We need them later for the real Thom isomorphism.

Definition 2.42 (Inner product on a vector bundle)
Let $p\colon E \to B$ be a vector bundle over B. A continuous map

$$\langle \cdot, \cdot \rangle\colon E \oplus E \to \mathbb{K}$$

is called inner product on E if for every $b \in B$ the restricted map $\langle \cdot, \cdot \rangle_b\colon E_b \oplus E_b \to \mathbb{K}$ is a inner product on the $\mathbb{K}$-vector space E_b.
We call two inner products $\langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2$ on E isomorphic if there is an automorphism f of E such that $\langle e, e' \rangle_1 = \langle f(e), f(e') \rangle_2$.

The following proposition is proven in [Kar09] Theorem I.8.8.

Proposition 2.43 (Uniqueness of inner products)
Any two inner products on a vector bundle $p\colon E \to B$ are isomorphic.

Proposition 2.44 (Existence of an inner product)
Let $p\colon E \to B$ be a vector bundle over a compact Hausdorff space. Then there exists an inner product on E.

Proof. Let $\{U_\alpha\}$ be a cover of B with local trivialisations $h_\alpha\colon p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$. On each $U_\alpha \times \mathbb{R}^n$ we have an inner product which on each fibre $\{b\} \times \mathbb{R}^n$ is just the standard inner product on $\mathbb{R}^n$. Let $\langle \cdot, \cdot \rangle_\alpha$ be the inner product on $p^{-1}(U_\alpha)$ induced via h_α by the inner product on $U_\alpha \times \mathbb{R}^n$. We choose a partition of unity $\{\varphi_\alpha\}$ for the cover $\{U_\alpha\}$ and obtain an inner product on E by setting $\langle x, y \rangle \coloneqq \sum_\alpha \varphi_\alpha(p(x))\langle x, y\rangle_\alpha$. The complex case is analogous. □

Proposition 2.45 (Every subbundle is a direct summand)
Let $p\colon E \to B$ be a vector bundle over a compact Hausdorff space and E_0 be a vector subbundle of E. Then there exists a vector subbundle $E_0^\perp$ of E such that $E_0 \oplus E_0^\perp \cong E$.

Proof. We choose an inner product on E (which exists by the previous Proposition) and let $E_0^\perp$ be the subspace of E which in each fibre consists of all the vectors orthogonal to the corresponding fibre of E_0.
We now show that the map $p|_{E_0^\perp}\colon E_0^\perp \to B$ is a vector bundle.
The map is continuous and the fibres are vector spaces, thus it remains to show that $p|_{E_0^\perp}$ is locally trivial.
Let $b_0 \in B$ and U be a neighbourhood over which E is trivial. Let m be the rank of E_0 and n the rank of $E_0^\perp$. For each $b \in U$ we choose a basis $s_i(b)$ of the fibre $(E_0)_b$. Since E_0 is locally trivial over U we can choose them in such a way that the map $s_i\colon B \to E_0, b \mapsto s_i(b)$ is continuous for $i = 1, \ldots, m$. Such a function is called a section of E_0 over U.
Using the Gram-Schmidt process we may assume that the $s_i(b)$ are a orthogonal basis of $(E_0)_b$ for each $b \in U$. We complement $s_1(b), \ldots, s_m(b)$ by vectors $s_{m+1}(b), \ldots, s_{m+n}(b)$ in $(E_0)_b^\perp$ to an basis of E_b for each $b \in U$. Again since E is locally trivial we may choose them in such a way that the map $s_i\colon U \to E_b, b \mapsto s_i(b)$ is continuous for $i = 1, \ldots, m+n$. Once again using Gram-Schmidt we obtain continuous maps s_i such that $s_1(b), \ldots, s_{m+n}(b)$ form an orthogonal basis of E_b for all $b \in U$.
The maps s_i induce a local trivialisation $h\colon p^{-1}(U) \to U \times \mathbb{R}^n$ of E by sending $(b, s_i(b))$ to (b, e_i) where e_i is the i-th standard basis vector of $\mathbb{R}^n$. By construction h identifies E_0 with $U \times \mathbb{R}^m$ and $E_0^\perp$ with $U \times \mathbb{R}^{n-m}$ and therefore $h|_{E_0^\perp}$ defines a local trivialisation of $E_0^\perp$.
Since we have $(E_0 \oplus E_0^\perp)_b = (E_0)_b \oplus (E_0^\perp)_b = E_b$ the obvious map $E_0 \oplus E_0^\perp \to E$ is an isomorphism on each fibre and thus an isomorphism of vector bundles by Lemma 2.11. □

Proposition 2.46 (Every bundle is a direct summand of a trivial bundle)
Let $p\colon E \to B$ be a vector bundle over a compact Hausdorff space. Then there exists a vector bundle $p'\colon E' \to B$ such that $E \oplus E' \cong \varepsilon^n$ for some $n \in \mathbb{N}$.

Proof. The idea is to show that the vector bundle E can be realized as a vector subbundle of a trivial bundle ε^n and then use the previous Proposition to obtain the vector bundle E' with $E \oplus E' \cong \varepsilon^n$.
For $b \in B$ let U_b be an open neighbourhood and $h_b\colon p^{-1}(U_b) \to U_b \times \mathbb{R}^n$ a local trivialization. By Urysohn's lemma there is a map $\varphi_b\colon B \to [0,1]$ that is non-zero at b and has support contained in U_b. The open sets $\{\varphi_b^{-1}((0,1])\}_{b\in B}$ form and open cover of B which by compactness of B has a finite subcover $\{\varphi_i^{-1}((0,1])\}_{i=1,\dots,m}$.
Let $\pi_i\colon U_i \times \mathbb{R}^n \to \mathbb{R}^n$ be the projection to the second component. We define maps

$$g_i\colon E \to \mathbb{R}^n, v \mapsto \varphi_i(p(v))\pi_i h_i(v).$$

If $v \notin p^{-1}(U_i)$ then $h_i(v)$ is not defined, but $\varphi_i(p(v)) = 0$, since $p(v) \notin U_i$ and φ_i has support in U_i. Therefore we interpret $\varphi_i(p(v))\pi_i h_i(v)$ as 0.
Every g_i is linear and injective if restricted to a fibre E_b for $b \in \varphi_i^{-1}(0,1]$. We now define

$$g\colon E \to \mathbb{R}^{n\cdot m}, v \mapsto (g_1(v), \dots, g_m(v)).$$

This is an injection on each fibre and we get a map

$$f\colon E \to B \times \mathbb{R}^{n\cdot m}, v \mapsto (p(v), g(v)).$$

The image of f is a subbundle since for the i-th copy of $\mathbb{R}^n$ in $\mathbb{R}^{n\cdot m}$ one can use the local trivializations h_i to obtain a local trivialization of the image of f. The image of f is isomorphic to E and thus E is isomorphic to a subbundle of the trivial bundle $B \times \mathbb{R}^{n\cdot m}$. □

At the beginning of Section 2.4 we have seen that every vector bundle can be constructed using trivial vector bundles and gluing them together using transition functions. We now impose more condition on the regularity of those transition functions.

Definition 2.47 (Orientation)
Let $p\colon E \to B$ be a real vector bundle. An orientation on E is a cocycle

$$\{g_{\beta\alpha}\colon U_\alpha \cap U_\beta \to \mathrm{SL}_n(\mathbb{R})\}_{\alpha,\beta}$$

such that if we apply the clutching construction (Theorem 2.32) to the trivial bundles over each U_α we obtain E.
A vector bundle that admits an orientation is called orientable.

Remark 2.48
If the base is compact Hausdorff then every vector bundle admits an inner product. One can ensure that the local trivialisations $p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ are compatible with the inner product on $U_\alpha \times \mathbb{R}^n$. Thus every real vector bundle of rank n over a compact space can be constructed from a $\mathrm{O}(n)$-cocycle (see also [Kar09] IV.4.21).
Nevertheless there are real vector bundles which cannot be constructed from a $\mathrm{SO}(n)$-cocycle.
If we are in the case of spheres then Corollary 2.41 shows that every real vector bundle over a sphere S^k for $k \geq 2$ can be constructed from a $\mathrm{SO}(n)$-cocycle and thus is orientable.

Definition 2.49 (Spin-group)
For $n \geq 3$ we define $\mathrm{Spin}(n)$ to be the universal cover of $\mathrm{SO}(n)$. This is a covering of degree two since the fundamental group of $\mathrm{SO}(n)$ is $\mathbb{Z}/2\mathbb{Z}$ (see for example [HH13] Proposition 13.10).
If $n = 2$ the special orthogonal group $\mathrm{SO}(2)$ is homeomorphic to S^1. Thus the fundamental group of $\mathrm{SO}(2)$ is $\mathbb{Z}$ and there is a unique covering of degree two of $\mathrm{SO}(2)$. We define $\mathrm{Spin}(2)$ to be this covering.

We generalize Definition 2.47 to the case of Spin groups.

Definition 2.50 (Spinorial structure)
Let $p\colon E \to B$ be a real vector bundle. A spinorial structure (or short $\mathrm{Spin}(n)$ structure) on E is a cocycle

$$\{g_{\beta\alpha}\colon U_\alpha \cap U_\beta \to \mathrm{Spin}(n)\}_{\alpha,\beta}$$

such that the corresponding cocycle
$\{g_{\beta\alpha}\colon U_\alpha \cap U_\beta \to \mathrm{SO}(n)\}_{\alpha,\beta}$ obtained by using the covering map $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ is an orientation on E.

We next show that every real vector bundle over a sphere of dimension larger than 2 may be equipped with a $\mathrm{Spin}(n)$-structure.

Proposition 2.51
Every real vector bundle of rank n over a sphere S^k for $k \geq 3$ can be equipped with a $\mathrm{Spin}(n)$-structure.

Proof. From Remark 2.48 we get that every real vector bundle of rank n over S^k for $k \geq 2$ can be constructed from a clutching function $f\colon S^{k-1} \to \mathrm{SO}(n)$. We now obtain a $\mathrm{Spin}(n)$-structure by choosing a preimage of the identity under the covering map $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ and lifting f to a map $S^{k-1} \to \mathrm{Spin}(n)$. This is possible since $\pi_1(S^{k-1})$ is trivial for $k \geq 3$. □

3 Topological K-Theory

In this chapter we introduce topological K-theory. Most of this chapter is based on [Hat03] and [Kar09]. We first define the K-group $K_{\mathbb{K}}(X)$ and the reduced K-group $\widetilde{K}_{\mathbb{K}}(X)$ for compact spaces X and a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We also define the higher K-groups $K^n_{\mathbb{K}}(X)$ for every $n \in \mathbb{Z}$. It turns out that these groups depend functorially on the space and constitute a generalized cohomology theory. We will not focus on the cohomological aspects but many results and definitions will be familiar if one knows cohomology.
The first section deals with the definition of $K_{\mathbb{K}}(X)$ and $\widetilde{K}_{\mathbb{K}}(X)$ and some rather simple calculations of K-groups. In the second section we will discuss Bott-periodicity which says that the reduced K-groups are periodic with period 2 if we are in the complex case and period 8 in the real case. After that we will also discuss several extra structures and tools related to the K-groups such as the Adams operations, the Chern character and the Thom isomorphism.

3.1 The functors $K(X)$ and $\widetilde{K}(X)$

Here we define the K-theory functor and the reduced K-theory functor. Furthermore we show that the groups $K(X)$ and $\widetilde{K}(X)$ can be equipped with a ring structure. We also define the stable orthogonal group and its analogues and express their homotopy groups conveniently in terms of the reduced K-theory of spheres. At the end we will introduce the cofibre sequence and see that it induces an exact sequence in K-theory which will be the starting point of the e-invariant in Chapter 4.

Definition 3.1 (Grothendieck group)
Let M be an abelian monoid. The Grothendieck group of M is the unique abelian group $G(M)$ together with a monoid homomorphism $\delta\colon M \to G(M)$ which satisfies the following universal property.
For every abelian group H and monoid homomorphism $f\colon M \to H$ there exists a unique group homomorphism $\widetilde{f}\colon G(M) \to H$ making the diagram

$$\begin{array}{ccc} M & \xrightarrow{\delta} & G(M) \\ & {\scriptstyle f}\searrow \qquad \swarrow{\scriptstyle \widetilde{f}} & \\ & H & \end{array}$$

commute.

As usually when dealing with universal properties it is not clear whether such a universal objects exists, but if there is one then it is unique up to unique isomorphism. We now explicitly construct $G(M)$ making the above definition valid.

Construction 3.2
$G(M)$ can be seen as adding inverses for every $m \in M$. We achieve this by considering the product $M \times M$. An element $(m, n) \in M \times M$ should be seen as $m - n$ and we embed M into $M \times M$ as the first component.
From this viewpoint the inverse of $(m, 0)$ should be $(0, m)$ and we should have the relation $(m, n) = (m', n')$ whenever $m + n' = m' + n$.
$G(M)$ will now be defined as the quotient space where we force this relation to be true. Therefore we consider the equivalence relation

$$(m, n) \sim (m', n') \iff \exists p \in M \text{ such that } m + n' + p = m' + n + p.$$

The element p is needed to ensure that this is always an equivalence relation on $M \times M$.
$G(M)$ is now defined as the quotient $(M \times M)/\sim$.
We define $\delta\colon M \to G(M)$ by setting $\delta(m) = (m, 0)$. This map will in general not be injective. It is clear that $G(M)$ is an abelian group where the inverse of (m, n) is (n, m). Note that from this description we get that every element of $G(M)$ can be written as $\delta(m) - \delta(n)$ for some $m, n \in M$.
We now show that $G(M)$ satisfies the universal property.
Let $f\colon M \to H$ be a monoid homomorphism. For $(m, n) \in G(M)$ we want to define $\widetilde{f}(m, n)$. Since $\widetilde{f}$ should be a group homomorphism it must satisfy the equations

$$\begin{aligned} \widetilde{f}(m, n) &= \widetilde{f}(m, 0) + \widetilde{f}(0, n) \\ &= \widetilde{f}(m, 0) - \widetilde{f}(n, 0) \\ &= (\widetilde{f} \circ \delta)(m) - (\widetilde{f} \circ \delta)(n). \end{aligned}$$

But for the last terms we do not have a choice since the commutative diagram says that $\widetilde{f} \circ \delta = f$. Therefore the only possible definition of $\widetilde{f}$ is $\widetilde{f}(m, n) := f(m) - f(n) \in H$ for $(m, n) \in G(M)$.
One can check that this gives a well defined group homomorphism $G(M) \to H$ which makes the diagram commute. Thus $G(M)$ is the Grothendieck group of M.

Example 3.3

1. $G(\mathbb{N}) = \mathbb{Z}$
2. $G((\mathbb{Z} \setminus \{0\}, \cdot)) = (\mathbb{Q} \setminus \{0\}, \cdot)$

Remark 3.4
Let $f\colon M \to N$ be a monoid homomorphism. Using the universal property of $G(M)$ for the monoid homomorphism $\delta_N \circ f\colon M \to G(N)$ we get a unique group homomorphism $\widetilde{f}\colon G(M) \to G(N)$. This induces a functor from the category of abelian monoids to the category of abelian groups which is left adjoint to the forgetful functor G from abelian groups to abelian monoids.

$\mathrm{Vect}_{\mathbb{K}}(X)$ together with the Whitney sum is an abelian monoid (Proposition 2.22). The K-group of X is the Grothendieck group of $\mathrm{Vect}_{\mathbb{K}}(X)$.

Definition 3.5 (K-theory)
Let X be a compact Hausdorff space. We define the K-theory of X to be the abelian group

$$K_{\mathbb{K}}(X) := G(\mathrm{Vect}_{\mathbb{K}}(X), \oplus).$$

Elements of $K_{\mathbb{K}}(X)$ are called virtual bundles.
We omit the $\mathbb{K}$ when the statements are true for both $\mathbb{R}$ and $\mathbb{C}$.

The following proposition gives a normal form of elements in $K(X)$ and describes when $\delta(E) = \delta(F)$ for $E, F \in \mathrm{Vect}(X)$. Recall that ε^n denotes the trivial bundle of rank n.

Proposition 3.6
Let X be compact Hausdorff. Then all elements of $K(X)$ are of the form $\delta(E) - \delta(\varepsilon^p)$ for some $p \in \mathbb{N}$ and $\delta(E) - \delta(\varepsilon^p) = \delta(F) - \delta(\varepsilon^q)$ if and only if there exists $n \in \mathbb{N}$ such that $E \oplus \varepsilon^p \oplus \varepsilon^n \cong F \oplus \varepsilon^q \oplus \varepsilon^n$.
In particular for $E, F \in \mathrm{Vect}(X)$ we have $\delta(E) = \delta(F)$ if and only if there exists $n \in \mathbb{N}$ such that $E \oplus \varepsilon^n \cong F \oplus \varepsilon^n$.

Proof. Every element in $K(X)$ can be written as $\delta(E_1) - \delta(E_2)$ for $E_1, E_2 \in \mathrm{Vect}(X)$. By Proposition 2.46 there exists a vector bundle F such that $E_2 \oplus F \cong \varepsilon^p$ for some $p \in \mathbb{N}$. Now

$$\begin{aligned}\delta(E_1) - \delta(E_2) &= \delta(E_1) - \delta(E_2) - \delta(F) + \delta(F) \\ &= \delta(E_1) + \delta(F) - \delta(\varepsilon^p) \\ &= \delta(E) - \delta(\varepsilon^p)\end{aligned}$$

where E is the vector bundle $E := E_1 \oplus F$.
Furthermore we have $\delta(E) - \delta(\varepsilon^p) = \delta(F) - \delta(\varepsilon^q)$ if and only if there exists $n \in \mathbb{N}$ such that $E \oplus \varepsilon^q \oplus \varepsilon^n \cong F \oplus \varepsilon^p \oplus \varepsilon^n$. ☐

Remark 3.7 (Functoriality of K)
Let Vect be the functor induced by $X \mapsto \mathrm{Vect}(X)$. Then K can be understood as the composition $G \circ \mathrm{Vect}$, hence is itself a contravariant functor.
Since Vect is homotopy invariant (see Corollary 2.30) we get that K is also homotopy invariant.

Lets calculate our first K-groups.

Example 3.8
Since vector bundles over a point P are just vector spaces we have $\mathrm{Vect}(P) \cong \mathbb{N}$ and therefore $K(P) \cong \mathbb{Z}$. The first isomorphism is given by mapping a vector space to its dimension and the second isomorphism is then given by mapping $V - W$ to $\dim(V) - \dim(W)$.
By homotopy invariance we get $K(X) \cong \mathbb{Z}$ whenever X is contractible.
In Corollary 2.39 we have seen that every complex vector bundle over S^1 is isomorphic to a trivial bundle. The same argument as above shows that $K_{\mathbb{C}}(S^1) \cong \mathbb{Z}$.

Proposition 3.9 (K-theory of a direct sums)
Let $X = X_1 \sqcup \cdots \sqcup X_n$ be a compact Hausdorff that is the disjoint union of compact Hausdorff spaces X_i. Then we have

$$K(X) \cong K(X_1) \oplus \cdots \oplus K(X_n)$$

where the isomorphism is induced by the inclusions $X_i \to X$.

Proof. A vector bundle over X is determined by its restrictions to each X_i. Conversely a collection of vector bundles $\{p_i\}_{i=1,\dots,n}$ where p_i is a vector bundle over X_i determines a vector bundle over X. In other words we have $\mathrm{Vect}(X) = \mathrm{Vect}(X_1) \times \cdots \times \mathrm{Vect}(X_n)$. Applying the Grothendieck group functor yields $K(X) \cong K(X_1) \oplus \cdots \oplus K(X_n)$. □

Remark 3.10 (Ring structure)

1. For $E_1 - F_1, E_2 - F_2 \in K(X)$ we define
$$(E_1 - F_1)(E_2 - F_2) \coloneqq E_1 \otimes E_2 - E_1 \otimes F_2 - E_2 \otimes F_1 + F_1 \otimes F_2.$$
Proposition 2.23 shows that this gives an associative and commutative product on $K(X)$ with unit element $\delta(\varepsilon^1)$. Proposition 2.24 also shows that $(K(X), \oplus, \otimes)$ is a commutative ring with unit $\delta(\varepsilon^1)$. We will now write $1 \coloneqq \delta(\varepsilon^1)$

2. Let $f\colon X \to Y$ be continuous. Then Proposition 2.23 (*iv*) shows that the induced map $f^*\colon K(Y) \to K(X)$ is a ring homomorphism.

We now define reduced K-theory.

Definition 3.11 (Reduced K-theory)
Let X be a pointed compact Hausdorff space with basepoint x_0. Then the inclusion $\iota\colon \{x_0\} \hookrightarrow X$ induces a homomorphism $\iota^*\colon K(X) \to K(\{x_0\}) \cong \mathbb{Z}$. We define the reduced K-theory of X to be

$$\widetilde{K}(X) \coloneqq \ker(\iota^*).$$

Proposition 3.12
Let X be a non-empty compact Hausdorff space with basepoint x_0. From the definition of $\widetilde{K}(X)$ we immediately get a short exact sequence

$$0 \longrightarrow \widetilde{K}(X) \longrightarrow K(X) \xrightarrow{\iota^*} \mathbb{Z} \longrightarrow 0 \ .$$

This sequence splits and therefore gives an isomorphism $K(X) \cong \widetilde{K}(X) \oplus \mathbb{Z}$ which is given by $\delta(E) - \delta(\varepsilon^n) \mapsto (\delta(E) - \delta(\varepsilon^{\dim(E_{x_0})}), \dim(E_{x_0}) - n)$.

Proof. The projection $X \to \{x_0\}$ induces a map $\mathbb{Z} \to K(X)$ which is a right split. The map ι is given by $\delta(E) - \delta(\varepsilon^n) \mapsto \dim(E_{x_0}) - n$. It is clear that the map in the Proposition is well defined and is an isomorphism. □

Proposition 3.13 (Reduced K-theory of a wedge)
Let A, B be compact Hausdorff pointed space. Then we have $\widetilde{K}(A \vee B) \cong \widetilde{K}(A) \oplus \widetilde{K}(B)$.

Proof. By [Kar09] Theorem II.2.42 we get an exact sequence

$$\widetilde{K}(A) \xrightarrow{j^*} \widetilde{K}(A \vee B) \xrightarrow{i^*} \widetilde{K}(B)$$

where i is the inclusion of B into $A \vee B$ and $j\colon A \vee B \to A$ is the map that collapses B to a point.
Let $q_1\colon A \vee B \to B$ be the map that collapses A to a point. Then we have $q_1 \circ i = \mathrm{id}_B$ and therefore $\mathrm{id}_{\widetilde{K}(B)} = (q_1 \circ i)^* = i^* \circ q_1^*$. This means i^* has a right inverse and thus is surjective.
Analogously for the inclusion $q_2\colon A \to A \vee B$ of A the induced map q_2^* is a left inverse of j^* and therefore j^* is injective.
Together this means we get a short exact sequence

$$0 \longrightarrow \widetilde{K}(A) \xrightarrow{j^*} \widetilde{K}(A \vee B) \xrightarrow{i^*} \widetilde{K}(B) \longrightarrow 0 \ .$$

Since q_1^* is a right split (and q_2^* is a left split) the splitting lemma shows the proposition. □

The next proposition gives an alternative description of the reduced K-group.

Proposition 3.14
$\widetilde{K}(X)$ is the quotient of $\mathrm{Vect}(X)$ by the equivalence relation

$$E \sim F \text{ if and only if there exists } n, q \in \mathbb{N} \text{ such that } E \oplus \varepsilon^n \cong F \oplus \varepsilon^q.$$

Such E and F are also called stably isomorphic.

Proof. Consider the homomorphism γ given by the composition

$$\mathrm{Vect}(X) \xrightarrow{\delta} K(X) \xrightarrow{p} \widetilde{K}(X)$$

where p is the projection. Every element of $K(X)$ can be written as $\delta(E) - \delta(\varepsilon^n)$. Since the class of ε^n is zero in $\widetilde{K}(X)$ we have $p(\delta(E) - \delta(\varepsilon^n)) = p(\delta(E)) - 0 = \gamma(E)$. Because p is surjective γ is also surjective.
Let $\gamma(E) = \gamma(F)$ this means $(p \circ \delta)(E - F) = 0$ which is equivalent to

$$\delta(E) - \delta(F) = \delta(\varepsilon^n) - \delta(\varepsilon^q)$$

where $n = \dim(E_{x_0})$ and $q = \dim(F_{x_0})$ (see Proposition 3.12). By Proposition 3.6 there exists a $m \in \mathbb{N}$ such that $E \oplus \varepsilon^q \oplus \varepsilon^m \cong F \oplus \varepsilon^n \oplus \varepsilon^m$. This finishes the proof. □

Remark 3.15
Let (X, x_0) and (Y, y_0) be a compact Hausdorff pointed spaces and let $f\colon X \to Y$ be a basepoint preserving map.

1. By mapping X to $\widetilde{K}(X)$ we get a contravariant functor from the category of compact Hausdorff pointed spaces to the category of abelian groups.
   Indeed we have the commutative diagram

$$\begin{array}{ccccc}
\widetilde{K}(X) & \longrightarrow & K(X) & \xrightarrow{\iota_{x_0}{}^*} & K(\{x_0\}) \\
\uparrow \vdots & & f^* \uparrow & & f^* \uparrow \\
\widetilde{K}(Y) & \longrightarrow & K(Y) & \xrightarrow{\iota_{y_0}^*} & K(\{y_0\})
\end{array}$$

and we want to define a homomorphism from $\widetilde{K}(Y)$ to $\widetilde{K}(X)$. We do a diagram chase to show that the restriction of $f^*\colon K(Y) \to K(X)$ to $\widetilde{K}(Y)$ has image in $\widetilde{K}(X)$.
Let $y \in \widetilde{K}(Y) \subset K(Y)$. Then by definition of $\widetilde{K}(Y)$ we get $\iota_{y_0}^*(y) = 0$ and thus $f^*(\iota_{y_0}^*(y)) = 0$. By commutativity we have $\iota_{x_0}^*(f^*(y)) = 0$ and therefore
$f^*(y) \in \ker(\iota_{x_0}^*) = \widetilde{K}(X)$.

2. Since K is homotopy invariant we immediately get that $\widetilde{K}$ is also homotopy invariant.

3. $\widetilde{K}(X)$ is the kernel of the ring homomorphism ι and thus an ideal in $K(X)$. In particular $\widetilde{K}(X)$ is closed under the multiplication induced by the tensor product. Therefore $\widetilde{K}(X)$ is also a commutative ring. But since $1 \notin \ker(\iota^*)$ the ring $\widetilde{K}(X)$ does not have a unit.

There is another interpretation of $\widetilde{K}(X)$ which will allow us to relate the reduced K-groups of spheres with certain homotopy groups.
Recall that $\mathrm{Vect}^n_{\mathbb{K}}(X)$ is the set which consists of all isomorphism classes of vector bundles of rank n.
The Whitney sum with ε^1 gives a map $\mathrm{Vect}^n_{\mathbb{K}}(X) \to \mathrm{Vect}^{n+1}_{\mathbb{K}}(X)$ and these assemble a directed system. We define

$$\Phi_{\mathbb{K}}(X) := \varinjlim_{n} \ \mathrm{Vect}^n_{\mathbb{K}}(X)$$

to be the direct limit of this system. The Whitney sum of vector bundles also induces maps $\mathrm{Vect}^n_{\mathbb{K}}(X) \times \mathrm{Vect}^m_{\mathbb{K}}(X) \to \mathrm{Vect}^{n+m}(X)$, thus providing $\Phi_{\mathbb{K}}(X)$ with an abelian monoid structure. It turns out that $\Phi_{\mathbb{K}}(X)$ is an abelian group that is isomorphic to $\widetilde{K}_{\mathbb{K}}(X)$.

Proposition 3.16
Let X be a compact Hausdorff pointed space. Then $\Phi_{\mathbb{K}}(X)$ is an abelian group which is isomorphic to $\widetilde{K}_{\mathbb{K}}(X)$.

Proof. We will prove this by defining a monoid morphism $\Phi_{\mathbb{K}}(X) \to \widetilde{K}_{\mathbb{K}}(X)$ and show that it is bijective.
Let $E \in \mathrm{Vect}^n_{\mathbb{K}}(X)$ be a representative of an element $\widetilde{E} \in \Phi_{\mathbb{K}}(X)$. Then $\delta(E) - \delta(\varepsilon^n)$ is an element of $\widetilde{K}_{\mathbb{K}}(X)$ which is independent of the choice of representative of $\widetilde{E}$. We now define

$$f\colon \Phi_{\mathbb{K}}(X) \to \widetilde{K}_{\mathbb{K}}(X), \widetilde{E} \mapsto \delta(E) - \delta(\varepsilon^n)$$

where $E \in \mathrm{Vect}^n(X)$ is a representative of $\widetilde{E}$. Clearly this is a monoid homomorphism. Every element of $\widetilde{K}_{\mathbb{K}}(X) \subset K_{\mathbb{K}}(X)$ can be written in the form $\delta(E) - \delta(\varepsilon^n)$ (see Proposition 3.6). Thus f is surjective.
Let $\widetilde{E}, \widetilde{F} \in \Phi_{\mathbb{K}}(X)$ such that $f(\widetilde{E}) = f(\widetilde{F})$. This means if $E \in \mathrm{Vect}^n(X)$ is a representative of $\widetilde{E}$ and $F \in \mathrm{Vect}^n(X)$ is a representative of $\widetilde{F}$ we have $\delta(E) - \varepsilon^n = \delta(F) - \varepsilon^m$ in $\widetilde{K}_{\mathbb{K}}(X) \subset K_{\mathbb{K}}(X)$.
By Proposition 3.6 there exists $p \in \mathbb{N}$ such that $(E \oplus \varepsilon^m) \oplus \varepsilon^p \cong (F \oplus \varepsilon^n) \oplus \varepsilon^p$. But this tells us that E, F define the same element in $\Phi_{\mathbb{K}}(X)$, therefore $\widetilde{E} = \widetilde{F}$ and thus f is injective. $\square$

To relate the above result to homotopy theory we need to define the infinite orthogonal groups and two analogues.

Definition 3.17 (The groups $\mathrm{O}, \mathrm{SO}, \mathrm{U}$)
We define the infinite orthogonal group O to be the direct limit

$$\mathrm{O} := \varinjlim_n \mathrm{O}(n)$$

where the maps $\mathrm{O}(n) \to \mathrm{O}(n+1), M \mapsto \left(\begin{array}{c|c} M & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \ \dots \ 0 & 1 \end{array}\right)$ are given by expanding a matrix with a one on the last diagonal entry and by zero elsewhere.
Analogously we have maps $\mathrm{SO}(n) \to \mathrm{SO}(n+1)$, $\mathrm{U}(n) \to \mathrm{U}(n+1)$ and define the infinite special orthogonal group to be

$$\mathrm{SO} := \varinjlim_n \mathrm{SO}(n)$$

and the infinite unitary group to be

$$\mathrm{U} := \varinjlim_n \mathrm{U}(n).$$

Lemma 3.18
In the cases of the groups above taking direct limit commutes with the homotopy functor. This means we have

$$\pi_k(\mathrm{O}) = \pi_k(\varinjlim_n \mathrm{O}(n)) \cong \varinjlim_n \pi_k(\mathrm{O}(n))$$

and the corresponding statements for SO and U.

Proof. We are going to define homomorphisms in both directions that are inverses of each other.
Let $\widetilde{g} \in \pi_k(\mathrm{O}(n))$ be a representative of an element in $\varinjlim_n \pi_k(\mathrm{O}(n))$ and let $g \colon S^k \to \mathrm{O}(n)$ be a representative map of g. Consider to the composition

$$S^k \xrightarrow{g} \mathrm{O}(n) \longrightarrow \varinjlim_n \mathrm{O}(n)$$

where the second map is the natural map into the direct limit. Its equivalence class in $\pi_k(\varinjlim_n \mathrm{O}(n))$ is independent of the choice of g and sending $\widetilde{g}$ to this element defines a homomorphism $\pi_k(\mathrm{O}(n)) \to \pi_k(\varinjlim_n \mathrm{O}(n))$.
This is compatible with the map $\pi_k(\mathrm{O}(n)) \to \pi_k(\mathrm{O}(n+1))$ and thus defines a homomorphism $\varinjlim_n \pi_k(\mathrm{O}(n)) \to \pi_k(\varinjlim_n \mathrm{O}(n))$.
Conversely let $f\colon S^r \to \varinjlim_n \mathrm{O}(n)$ be an representative of an element in $\pi_r(\varinjlim_n \mathrm{O}(n))$.
For every n the space $\mathrm{O}(n)$ is Hausdorff and closed in $\mathrm{O}(n+1)$.
Since S^r is compact and f is continuous the image of f is compact. Therefore Proposition 1.4.5 from [TD08] implies that the image of f is contained in $\mathrm{O}(n)$ for some n. Thus $f \in \pi_k(\mathrm{O}(n))$ and we get a homomorphism $\pi_k(\varinjlim_n \mathrm{O}(n)) \to \varinjlim_n \pi_k(\mathrm{O}(n))$ that is obviously the inverse if the map defined above. □

Remark 3.19
Let $k \geq 1$. Then we have $\pi_k(\mathrm{O}(n)) \cong \pi_k(\mathrm{SO}(n))$ for all $n \in \mathbb{N}$ (see proof of Corollary 2.41) and the Lemma above shows

$$\pi_k(\mathrm{O}) = \pi_k(\varinjlim_n \mathrm{O}(n)) \cong \pi_k(\varinjlim_n \mathrm{SO}(n)) = \pi_k(\mathrm{SO}).$$

Corollary 3.20
Let $k \geq 1$, then we have $\widetilde{K}_{\mathbb{C}}(S^k) \cong \pi_{k-1}(\mathrm{U})$ and $\widetilde{K}_{\mathbb{R}}(S^k) \cong \pi_{k-1}(\mathrm{O})$. If $k \geq 2$ then we additionally have $\widetilde{K}_{\mathbb{R}}(S^k) \cong \pi_{k-1}(\mathrm{SO})$.

Proof. We have

$$\begin{aligned}\widetilde{K}_{\mathbb{C}}(S^k) \cong \Phi_{\mathbb{C}}(S^k) &= \varinjlim_n \mathrm{Vect}_{\mathbb{C}}(S^k) \\ &\cong \varinjlim_n \pi_{k-1}(\mathrm{U}(n)) \\ &\cong \pi_{k-1}(\varinjlim_n \mathrm{U}(n)) \\ &= \pi_{k-1}(\mathrm{U}).\end{aligned}$$

The first isomorphism is due to Proposition 3.16. The second isomorphism uses Corollary 2.38 and for the last isomorphism we use Lemma 3.18.
To prove the real case we use the following fact about direct limits. If $\{A_i\}_{i\in I}$ is a directed system and $J \subset I$ is a subset with the property that for every $i \in I$ there is a $j \in J$ such that $i \leq j$ then $\varinjlim_{i\in I} A_i \cong \varinjlim_{j\in J} A_j$. Such a subset J is called cofinal.
The subset of odd numbers in $\mathbb{N}$ is cofinal and thus we get

$$\varinjlim_n \mathrm{Vect}^n_{\mathbb{R}}(S^k) \cong \varinjlim_j \mathrm{Vect}^{2j+1}_{\mathbb{R}}(S^k).$$

The same argument also yields

$$\varinjlim_n \mathrm{O}(n) \cong \varinjlim_j \mathrm{O}(2j+1).$$

Further Corollary 2.41 gives us an isomorphism $\mathrm{Vect}_{\mathbb{R}}^{2j+1}(S^k) \cong \pi_{k-1}(\mathrm{O}(2j+1))$. Combining all those isomorphisms we get

$$\begin{aligned}
\Phi_{\mathbb{R}}(S^k) = \varinjlim_n \mathrm{Vect}_{\mathbb{R}}^n(S^k) &\cong \varinjlim_j \mathrm{Vect}_{\mathbb{R}}^{2j+1}(S^k) \\
&\cong \varinjlim_j \pi_{k-1}(\mathrm{O}(2j+1)) \\
&\cong \varinjlim_n \pi_{k-1}(\mathrm{O}(n)) \\
&\cong \pi_{k-1}(\varinjlim_n \mathrm{O}(n)) \\
&= \pi_{k-1}(\mathrm{O}).
\end{aligned}$$

To obtain the last isomorphism we again used Lemma 3.18.
The last statement of the Corollary is now a direct consequence of Remark 3.19. □

Next we introduce the cofibre sequence of a map. This is a sequence of maps of spaces which induces an exact sequence in K-theory. Besides being a generally useful tool to calculate the K-theory of a space this is also the starting point of the definition of the e-invariant we will define later (see Section 4.2).
To construct the cofibre sequence we need to define the reduced cone and the reduced suspension. Those are analogues for pointed spaces of the cone of X and the suspension. We will denote the cone of X by CX and the suspension of X by SX.

Definition 3.21 (Reduced cone, reduced suspension)
Let (X, x_0) be a pointed topological space. We define the reduced cone of X to be the quotient space

$$C'X := CX/(\{x_0\} \times I) = X \times I/(X \times \{0\} \cup \{x_0\} \times I).$$

As a basepoint we choose the equivalence class of $(x_0, 1)$.
Similarly we define the reduced suspension of X to be the quotient space

$$\Sigma X := SX/(\{x_0\} \times I) = X \times I/(X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I).$$

As the basepoint we again choose the equivalence class of $(x_0, 1)$.

Since $\{x_0\} \times I \subset CX$ is closed and contractible $C'X$ is homotopy equivalent to CX. Similarly ΣX and SX are homotopy equivalent.
The last ingredient for the cofibre sequence is the mapping cone.

Definition 3.22 (Mapping cone)
Let X and Y be two spaces and $f\colon X \to Y$ be a map. We define the mapping cone of f to be the quotient space of $CX \sqcup Y$ by the equivalence relation $(x, 1) \sim f(x)$ for $x \in X$. We will denoted it by $CX \cup_f Y$ or C_f.
There is a analogue for pointed spaces using the reduced cone.
Let $f\colon (X, x_0) \to (Y, y_0)$ be a map of pointed spaces. We define the reduced cone denoted by $C'X \cup_f Y$ or C'_f to be the quotient space of $C'X \sqcup Y$ by the equivalence relation $(x, 1) \sim f(x)$ for $x \in X$.
The basepoint of C'_f is the equivalence class of $(x_0, 1)$.

Construction 3.23
We get a sequence

$$X \xrightarrow{f} Y \xrightarrow{i} CX \cup_f Y$$

where $i\colon Y \to CX \cup_f Y$ is the inclusion. We can iterate the mapping cone construction by now considering the mapping cone of i obtaining the sequence

$$X \xrightarrow{f} Y \xrightarrow{i} CX \cup_f Y \xrightarrow{j} CY \cup_i (CX \cup_f Y)\ .$$

Taking this even one step further we get

$$X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{j} CY \cup_i Z \xrightarrow{k} CZ \cup_j (CY \cup_i Z)$$

where we set $Z = CX \cup_f Y$.
The space $W := CY \cup_i (CX \cup_f Y)$ is homotopy equivalent to the suspension SX. To see this first note that we have the cone CX inside W. To obtain SX we need to collapse the slice $X \times \{1\}$ in CX which is glued together with Y. Since CY is contractible we can collapse it to a point. This also collapsing Y and hence $X \times \{1\}$ to a point.
Similarly $CZ \cup_j (CY \cup_i Z)$ is homotopy equivalent to SY. Identifying these spaces the map $k\colon CY \cup_i Z \to CZ \cup_j (CY \cup_i Z)$ corresponds to the map $-Sf\colon SX \to SY$ which is defined by $-Sf(x, s) = (f(x), 1 - s)$ and we obtain the sequence

$$X \xrightarrow{f} Y \xrightarrow{i} CX \cup_f Y \xrightarrow{j} SX \xrightarrow{-Sf} SY\ .$$

We call this sequence the cofibre sequence or Puppe sequence of f.
Essentially the same argument also gives a reduced version for which we use the reduced cone and the reduced suspension to obtain the sequence

$$X \xrightarrow{f} Y \xrightarrow{i'} C'X \cup_f Y \xrightarrow{j'} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y\ .$$

We call this the reduced cofibre sequence of reduced Puppe sequence.

The next proposition says the reduced cofibre sequence induces an exact sequence in reduced K-theory. It is proven in [Kar09] Theorem II.3.29.

Proposition 3.24
Let X, Y be pointed compact spaces and $f\colon X \to Y$ be a basepoint preserving map. Then the reduced Puppe sequence of f induces an exact sequence

$$\widetilde{K}(\Sigma Y) \xrightarrow{(-\Sigma f)^*} \widetilde{K}(\Sigma X) \xrightarrow{(j')^*} \widetilde{K}(C'X \cup_f Y) \xrightarrow{(i')^*} \widetilde{K}(Y) \xrightarrow{f^*} \widetilde{K}(X)$$

To later prove properties about the e-invariant we will record two diagrams related to the reduced cofibre sequence of maps. They are proven in [Pup58] Section 2.

Lemma 3.25
Let X, Y be spaces and $f, g\colon X \to Y$ be homotopic. Then there is a homotopy equivalence φ making the diagram

$$\begin{array}{ccccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{i} & C'X \cup_f Y & \xrightarrow{j} & \Sigma X & \xrightarrow{-\Sigma f} & \Sigma Y \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\
X & \xrightarrow{g} & Y & \xrightarrow{i'} & C'X \cup_g Y & \xrightarrow{j'} & \Sigma X & \xrightarrow{-\Sigma g} & \Sigma Y
\end{array}$$

commute up to homotopy.

Lemma 3.26
Let $r \in \mathbb{N}$. Then there exists a homotopy equivalence φ such that the diagram

$$\begin{array}{ccccc}
\Sigma^r Y & \xrightarrow{\Sigma^r i} & \Sigma^r(C'X \cup_f Y) & \xrightarrow{\Sigma^r j} & \Sigma^{r+1} X \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle (-\mathrm{id})^r} \\
\Sigma^r Y & \xrightarrow{i'} & C'(\Sigma^r X) \cup_{S^r f} \Sigma^r Y & \xrightarrow{j'} & \Sigma^{r+1} X
\end{array}$$

is commutative up to homotopy.

Lemma 3.27
Let

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} \\
X' & \xrightarrow{f'} & Y'
\end{array}$$

be a commutative diagram in spaces. Then there is a map φ making the diagram

$$\begin{array}{ccccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{i} & C'X \cup_f Y & \xrightarrow{j} & \Sigma X & \xrightarrow{-\Sigma f} & \Sigma Y \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \Sigma h} & & \downarrow{\scriptstyle \Sigma k} \\
X & \xrightarrow{f'} & Y & \xrightarrow{i'} & C'X \cup_{f'} Y & \xrightarrow{j'} & \Sigma X & \xrightarrow{-\Sigma f'} & \Sigma Y
\end{array}$$

commute.

3.2 Bott Periodicity

The goal of this section is to establish an isomorphism $\widetilde{K}_{\mathbb{K}}(X) \to \widetilde{K}_{\mathbb{K}}(\Sigma^r X)$ where $r = 2$ if $\mathbb{K} = \mathbb{C}$ and $r = 8$ if $\mathbb{K} = \mathbb{R}$.

We start with a computation of $\widetilde{K}_{\mathbb{C}}(S^2)$.
Recall the canonical line bundle H over S^2 from Example 2.7 (d). In Example 2.36 we have seen that H satisfies the equation $\mathrm{H}^2 \oplus \varepsilon^1 \cong \mathrm{H} \oplus \mathrm{H}$. This means in $K_{\mathbb{C}}(S^2)$ we have the relation $(\mathrm{H} - 1)^2 = 0$ and therefore we have a well defined homomorphism

$$\mathbb{Z}[H]/(H-1)^2 \to K_{\mathbb{C}}(S^2).$$

The external tensor product for vector bundles from Definition 2.26 induces a homomorphism

$$K_{\mathbb{C}}(X) \otimes K_{\mathbb{C}}(S^2) \to K_{\mathbb{C}}(X \times S^2).$$

We obtain a homomorphism

$$K_{\mathbb{C}}(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K_{\mathbb{C}}(X \times S^2).$$

The fundamental product theorem says that this is an isomorphism.

Theorem 3.28 (Fundamental product theorem)
Let X be compact Hausdorff. Then the map

$$K_{\mathbb{C}}(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K_{\mathbb{C}}(X \times S^2)$$

defined above is a ring-isomorphism.

This theorem is proven in [Hat03] Section 2.1. The proof of this theorem takes a lot of effort. In [Hat03] there is an entire section devoted to the proof. For the proof there considers generalized clutching function that allows to clue two vector bundles over $X \times D^2$ together to vector bundle over $X \times S^2$. One then gradually reduces to simpler clutching functions.

Corollary 3.29
If we take X to be a point the above Theorem shows that the map

$$\mathbb{Z}[H]/(H-1)^2 \to K_{\mathbb{C}}(S^2)$$

is a ring isomorphism. Choosing a basepoint in S^2 we get a group isomorphism

$$\widetilde{K}_{\mathbb{C}}(S^2) \cong \langle H - 1 \rangle \leq \mathbb{Z}[H]/(H-1)^2.$$

Thus as a group $\widetilde{K}_{\mathbb{C}}(S^2)$ is isomorphic to $\mathbb{Z}$ generated by $H - 1$. Since $(H-1)^2 = 0$ the multiplication in $\widetilde{K}_{\mathbb{C}}(S^2)$ is trivial.

It can be shown (see [Hat03] Section 2.2) that the external tensor product

$$K_{\mathbb{C}}(X) \otimes K_{\mathbb{C}}(Y) \to K_{\mathbb{C}}(X \times Y)$$

induces a map

$$\widetilde{K}_{\mathbb{C}}(X) \otimes \widetilde{K}_{\mathbb{C}}(Y) \to \widetilde{K}_{\mathbb{C}}(X \wedge Y)$$

where $X \wedge Y := (X \times Y)/(X \times \{y_0\} \cup \{x_0\} \times Y)$ is the smash product of X and Y. We call this the reduced external tensor product. Furthermore the external tensor product is natural, i.e. if $f\colon X \to X'$ and $f'\colon Y \to Y'$ are basepoint preserving maps then we have a commutative diagram

$$\begin{array}{ccc} \widetilde{K}_{\mathbb{C}}(X') \otimes \widetilde{K}_{\mathbb{C}}(Y') & \longrightarrow & \widetilde{K}_{\mathbb{C}}(X' \wedge Y') \\ \big\downarrow {\scriptstyle f^* \otimes (f')^*} & & \big\downarrow {\scriptstyle (f \wedge f')^*} \\ \widetilde{K}_{\mathbb{C}}(X) \otimes \widetilde{K}_{\mathbb{C}}(Y) & \longrightarrow & \widetilde{K}_{\mathbb{C}}(X \wedge Y) \end{array}$$

where the horizontal maps are the reduced external tensor product and
$f \wedge f'\colon X \wedge Y \to X' \wedge Y'$ is the map induced by $X \times Y \to X' \times Y', (x, x') \mapsto (f(x), f(x'))$.
The fundamental product theorem generalizes to the reduced case, in the sense that the map

$$\widetilde{K}_{\mathbb{C}}(X) \otimes \widetilde{K}_{\mathbb{C}}(S^2) \to \widetilde{K}_{\mathbb{C}}(X \wedge S^2)$$

is an isomorphism.
The smash product with a n-sphere $S^n \wedge X$ is the reduced n-th suspension $\Sigma^n X$ so we obtain an isomorphism

$$\widetilde{K}_{\mathbb{C}}(X) \otimes \widetilde{K}_{\mathbb{C}}(S^2) \cong \widetilde{K}_{\mathbb{C}}(\Sigma^2 X).$$

The group $\widetilde{K}_{\mathbb{C}}(S^2)$ is isomorphic to $\mathbb{Z}$ generated by $\mathrm{H} - 1$. Therefore the multiplication with $\mathrm{H} - 1$ induces an isomorphism

$$\widetilde{K}_{\mathbb{C}}(X) \xrightarrow{\cdot(\mathrm{H} - 1)} \widetilde{K}_{\mathbb{C}}(X) \otimes \widetilde{K}_{\mathbb{C}}(S^2)\ .$$

Here multiplication means taking the reduced external tensor product. Using the isomorphism above we obtain the following theorem.

Theorem 3.30 (Bott-periodicity - complex case)
Let X be a compact Hausdorff pointed space. Then multiplication with $H - 1$ induces a ring isomorphism

$$\beta\colon \widetilde{K}_{\mathbb{C}}(X) \to \widetilde{K}_{\mathbb{C}}(\Sigma^2 X).$$

Example 3.31
If $X = S^1$ the above theorem yields $\widetilde{K}_{\mathbb{C}}(S^{2n+1}) = 0$ for $n \in \mathbb{N}$.
For $X = S^2$ we get $\widetilde{K}_{\mathbb{C}}(S^{2n}) \cong \mathbb{Z}$ generated by the n-times multiplication of $H - 1$ with itself.

Corollary 3.32
The reduced complex K-theory of spheres is given by

$$\widetilde{K}_{\mathbb{C}}(S^n) \cong \begin{Bmatrix} \mathbb{Z}, & n \text{ even} \\ 0, & n \text{ odd} \end{Bmatrix}.$$

In the real case we also have a natural external tensor product

$$\widetilde{K}_{\mathbb{R}}(X) \otimes \widetilde{K}_{\mathbb{R}}(Y) \to \widetilde{K}_{\mathbb{R}}(X \wedge Y).$$

Bott periodicity in the real case takes a similar form. A proof can be found in [Kar09] Chapter III Section 3 and Section 4. As in the complex case the proof takes a lot of effort. The proof in [Kar09] uses real Clifford algebras.

Theorem 3.33 (Bott-periodicity - real case)
Let X be a compact Hausdorff pointed space. Then $\widetilde{K}(S^8) \cong \mathbb{Z}$ and multiplication with a generator of $\widetilde{K}(S^8)$ induces an isomorphism

$$\widetilde{K}_{\mathbb{R}}(X) \to \widetilde{K}_{\mathbb{R}}(\Sigma^8 X).$$

Remark 3.34
Choosing a basepoint in X and using $K_{\mathbb{K}}(X) \cong \widetilde{K}_{\mathbb{K}}(X) \oplus \mathbb{Z}$ (Proposition 3.12) one also obtains the periodicity $K_{\mathbb{R}}(X) \cong K_{\mathbb{R}}(\Sigma^8 X)$ in the unreduced case.

Theorem 3.35 (Real K-theory of spheres)
The reduced real K-theory of spheres is given by

$$\widetilde{K}_{\mathbb{R}}(S^n) \cong \begin{Bmatrix} \mathbb{Z}, & n \equiv 0, 4 \bmod 8 \\ \mathbb{Z}/2\mathbb{Z}, & n \equiv 1, 2 \bmod 8 \\ 0, & n \equiv 3, 5, 6, 7 \bmod 8 \end{Bmatrix}.$$

Proof. This is Theorem III.5.19. in [Kar09]. Since the group $K_{\mathbb{R}}^{-n}(\{*\})$ is isomorphic to $\widetilde{K}_{\mathbb{R}}^{-n}(S^0) = \widetilde{K}_{\mathbb{R}}(S^n)$. See below for the definition of the higher K-groups. □

Now that we have determined the K-theory of spheres we also know the homotopy groups of $\mathrm{U}, \mathrm{O}, \mathrm{SO}$ (see Corollary 3.20).

Corollary 3.36 (Homotopy groups of $\mathrm{U}, \mathrm{O}, \mathrm{SO}$)
We have

$$\pi_n(\mathrm{U}) \cong \begin{Bmatrix} \mathbb{Z}, & n \text{ odd} \\ 0, & n \text{ even} \end{Bmatrix}$$

and

$$\pi_n(\mathrm{O}) \cong \begin{cases} \mathbb{Z}, & n \equiv 3, 7 \bmod 8 \\ \mathbb{Z}/2\mathbb{Z}, & n \equiv 0, 1 \bmod 8 \\ 0, & n \equiv 2, 4, 5, 6 \bmod 8 \end{cases}.$$

If $n \geq 1$ then $\pi_n(\mathrm{SO}) \cong \pi_n(\mathrm{O})$.
Since $\mathrm{SO}(k)$ is path-connected for all k we have $\pi_0(\mathrm{SO}) = 0$.

Definition 3.37 (Higher K-groups)
Let X be a pointed compact space. For $n \in \mathbb{N}$ we define the n-th reduced K-group of X to be

$$\widetilde{K}_{\mathbb{K}}^{-n}(X) := \widetilde{K}_{\mathbb{K}}(\Sigma^n X).$$

With this definition Bott Periodicity says that the K-groups are 2 respectively 8-periodic, therefore we extend this definition to the positive degrees via the formula

$$\widetilde{K}_{\mathbb{K}}^{n+r}(X) := \widetilde{K}_{\mathbb{K}}^{n}(X)$$

where $r = 2$ if $\mathbb{K} = \mathbb{C}$ and $r = 8$ if $\mathbb{K} = \mathbb{R}$.
Furthermore we define the higher unreduced K-groups of X to be

$$K_{\mathbb{K}}^{n}(X) := \widetilde{K}_{\mathbb{K}}^{n}(X_+)$$

where X_+ is the space X with a disjoint basepoint "$+$" attached. If $n = 0$ this is consistent with the earlier definition since $K_{\mathbb{K}}^0(X) = \widetilde{K}_{\mathbb{K}}^0(X_+) = \ker(K_{\mathbb{K}}(X_+) \to K_{\mathbb{K}}(+)) = K(X)$.
Lastly we define

$$K_{\mathbb{K}}^{*}(X) := \bigoplus_{n\in\mathbb{Z}} K_{\mathbb{K}}^{n}(X)$$

and

$$\widetilde{K}_{\mathbb{K}}^{*}(X) := \bigoplus_{n\in\mathbb{Z}} \widetilde{K}_{\mathbb{K}}^{n}(X).$$

Remark 3.38

1. We now have $\widetilde{K}_{\mathbb{K}}^{n+r}(X) \cong \widetilde{K}_{\mathbb{K}}^{n}(X)$ for all $n \in \mathbb{N}$, where r is as above.
2. With this definition it can be shown that the functors $\{K^n\}_{n\in\mathbb{Z}}$ (respectively $\{\widetilde{K}^n\}_{n\in\mathbb{Z}}$) constitute a generalized cohomology theory (respectively generalized reduced cohomology theory).
3. One can use the external tensor product to equip $K_{\mathbb{K}}^{*}(X)$ with the structure of a graded ring. This restricts to a graded ring structure on $\widetilde{K}_{\mathbb{K}}^{*}(X)$ (see [Kar09] Section II.5).

3.3 Adams Operations

For every $k \in \mathbb{Z}$ the ring $K_{\mathbb{K}}(X)$ admits ring homomorphisms ψ^k called Adams operations. We will show how to construct them and record further properties. These operations provide a finer invariant of spaces. For example we will be able to distinguish the K-groups of spheres of arbitrary dimension which is not possible with $K(X)$ alone.

Theorem 3.39
Let X be a compact Hausdorff space. For every $k \in \mathbb{N}$ there exists a homomorphism

$$\psi^k\colon K_{\mathbb{C}}(X) \to K_{\mathbb{C}}(X)$$

called Adams operation satisfying the following properties

(i) $\psi^0 = 0$, $\psi^1 = \mathrm{id}$ *and* ψ^{-1} *is induced by the conjugation of vector bundles (see Example 2.20 (e))*

(ii) $\psi^k(x+y) = \psi^k(x) + \psi^k(y)$ *for all* $x, y \in K_{\mathbb{C}}(X)$

(iii) $\psi^k(xy) = \psi^k(x)\psi^k(y)$ *for all* $x, y \in K_{\mathbb{C}}(X)$

(iv) $\psi^k(\varepsilon^1) = \varepsilon^1$

(v) $\psi^k(L) = L^k$ *for every line bundle* L

(vi) For every compact space Y *and a map* $f\colon X \to Y$ *the following diagram commutes*

$$\begin{array}{ccc} K_{\mathbb{C}}(Y) & \xrightarrow{f^*} & K_{\mathbb{C}}(X) \\ \downarrow{\scriptstyle \psi^k} & & \downarrow{\scriptstyle \psi^k} \\ K_{\mathbb{C}}(Y) & \xrightarrow{f^*} & K_{\mathbb{C}}(X) \end{array} .$$

(vii) $\psi^{kl}(x) = (\psi^k \circ \psi^l)(x)$ *for all* $x \in K_{\mathbb{C}}(X)$

Remark 3.40

1. For negative $k \in \mathbb{Z}$ the term L^k should be interpreted as $\bar{L}^{-k}$, where $\bar{L}$ is the conjugate bundle from Example 2.20 (e). This is justified by *(vii)* and the fact that ψ^{-1} is induced by the conjugation.

2. Properties *(ii)*-*(iv)* may be rephrased as ψ^k being a ring homomorphism of $K_{\mathbb{C}}(X)$.

3. Property *(vi)* is also called naturality of ψ^k.

4. From Property *(vii)* we also get that all Adams operations commute with each other.

5. Let (X, x_0) be a compact pointed space. By applying Property *(vi)* to the map $K_{\mathbb{C}}(X) \to K_{\mathbb{C}}(x_0)$, induced by the inclusion of x_0, we see that ψ^k restricts to a ring homomorphism $\psi^k\colon \widetilde{K}_{\mathbb{C}}(X) \to \widetilde{K}_{\mathbb{C}}(X)$.

Note that since ψ^{-1} is predetermined by property *(i)* it suffices to define the operations for $k > 1$ and we can use property *(vii)* to then determine ψ^k for all negative k. For $k > 1$ the Adams operations will be given by so called Newton polynomial which can be written down explicitly. To show that this definition satisfies the properties we will use a technical but very useful result that allows us to reduce arguments to the case of direct sums of line bundles. It is proven in [Kar09] Theorem IV.2.15.

Theorem 3.41 (Splitting Principle)
Let X be compact and let $p\colon E \to X$ be a complex vector bundle. Then there exists a compact space $F(E)$ and a map $p\colon F(E) \to X$ such that $p^*\colon K^*_{\mathbb{C}}(X) \to K^*_{\mathbb{C}}(F(E))$ is injective and $p^*(E)$ splits as the direct sum of line bundles.

Proof of Theorem 3.39. Let $n \in \mathbb{N}$ be a natural number. For $j \in \{1, \dots, n\}$ let

$$\sigma_j(X_1, \dots, X_n) \coloneqq \sum_{1 \le k_1 < \dots < k_j \le n} X_{k_1} \dots X_{k_j}$$

be the j-th elementary symmetric polynomial. These polynomials satisfy

$$\prod_{i=1,\dots,n} (1 + X_i) = 1 + \sigma_1(X_1, \dots, X_n) + \dots + \sigma_n(X_1, \dots, X_n)$$

and

$$\sigma_j(X_1Y, \dots, X_nY) = \sigma_j(X_1, \dots, X_n)Y^j.$$

Consider the power series

$$\lambda_t(E) \coloneqq \sum_{i=0}^{\infty} \lambda^i(E)t^i.$$

By Proposition 2.25 (iii) this is a finite series and by Proposition 2.25 (iv) we get

$$\lambda_t(E_1 \oplus E_2) = \lambda_t(E_1)\lambda_t(E_2)$$

for vector bundles E_1, E_2.
If L is a line bundle Proposition 2.25 (ii), (iv) yield

$$\lambda_t(L) = 1 + \lambda^1(E)t = 1 + Et.$$

Now let $E \coloneqq L_1 \oplus \dots \oplus L_n \in \mathrm{Vect}_{\mathbb{C}}(X)$ be a sum of line bundles L_i. Putting all of the above together we get

$$\begin{aligned} \sum_{i=0}^{n} \lambda^i(E)t^i = \lambda_t(E) = \prod_{i=0}^{n} \lambda_t(L_i) = \prod_{i=0}^{n}(1 + L_i t) \\ = 1 + \sum_{i=1}^{n} \sigma_i(\lambda^1(L_1)t, \dots, \lambda^1(L_n)t) \\ = 1 + \sum_{i=1}^{n} \sigma_i(\lambda^1(L_1), \dots, \lambda^1(L_n))t^i. \end{aligned}$$

Comparing the coefficients of the first and the last polynomial in this equation yields

$$\lambda^i(E) = \sigma_i(L_1, \ldots, L_n).$$

We need a last definition before we can define the Adams operations. For this we use the fundamental theorem of symmetric polynomials that says that every symmetric polynomial of degree k can uniquely be written as a polynomial in $\sigma_1, \ldots, \sigma_k$ (see [Lan12] Theorem 6.1).

Let s_k be this polynomial for the symmetric polynomial $t_1^k + \cdots + t_n^k$, i.e.

$$s_k(\sigma_1(t_1, \ldots, t_n), \ldots, \sigma_k(t_1, \ldots, t_n)) = t_1^k + \cdots + t_n^k.$$

The s_k are called Newton polynomials (not to be confused with the Newton polynomials from interpolation theory).

For $k \in \mathbb{N}$ and $E \in \mathrm{Vect}_{\mathbb{C}}(X)$ we define

$$\psi^k(E) \coloneqq s_k(\lambda^1(E), \ldots, \lambda^k(E)) \in \mathrm{Vect}_{\mathbb{C}}(X).$$

If E is the sum of line bundles $E = L_1 \oplus \cdots \oplus L_n$ we have

$$\begin{aligned}
\psi^k(E) &= s_k(\lambda^1(E), \ldots, \lambda^k(E)) \\
&= s_k(\sigma_1(L_1, \ldots, L_n), \ldots, \sigma_k(L_1, \ldots, L_n)) \\
&= L_1^k \oplus \cdots \oplus L_n^k.
\end{aligned}$$

We now show that these ψ^k satisfy the properties from the theorem.

(i) ψ^0 and ψ^{-1} are the correct maps by definition. Since $s_1 = \sigma_1$ we get $\psi^1 = \mathrm{id}$.

(ii) For $E_1, E_2 \in \mathrm{Vect}(X)$ we can use the splitting principle to first pull back $E_1 \oplus E_2$ such that E_1 splits as $L_1 \oplus \cdots \oplus L_n$ and then do another pullback such that E_2 splits as $\widetilde{L}_1 \oplus \cdots \oplus \widetilde{L}_m$. Then $\psi^k(L_1 \oplus \cdots \oplus L_n \oplus \widetilde{L}_1 \oplus \cdots \oplus \widetilde{L}_m) = L_1^k \oplus \cdots \oplus L_n^k \oplus \widetilde{L}_1^k \oplus \cdots \oplus \widetilde{L}_m^k$ and thus we get $\psi^k(E_1 \oplus E_2) = \psi^k(E_1) \oplus \psi^k(E_2)$. Therefore using the universal property of the Grothendieck group we obtain an additive operation on $K_{\mathbb{C}}(X)$ by setting $\psi^k(E - F) = \psi^k(E) - \psi^k(F)$.

(iii) For the multiplicativity of ψ^k we again use the splitting principle to reduce to the case where $E_1 = L_1 \oplus \cdots \oplus L_n$ and $E_2 = \widetilde{L}_1 \oplus \cdots \oplus \widetilde{L}_m$. Then by Proposition 2.24 we have $E_1 \otimes E_2 = \sum_{i,j} L_i \otimes \widetilde{L}_j$. Note that $L_i \otimes \widetilde{L}_j$ is also a line bundle. Therefore by the equation for the sum of line bundles above we have

$$\begin{aligned}
\psi^k(E_1 \otimes E_2) = \psi^k\Big(\sum_{i,j} L_i \otimes \widetilde{L}_j\Big) &= \sum_{i,j} (L_i \otimes \widetilde{L}_j)^k \\
&= \sum_{i,j} L_i^k \otimes \widetilde{L}_j^k = \Big(\sum_i L_i^k\Big)\Big(\sum_j \widetilde{L}_j^k\Big) \\
&= \psi^k(E_1)\psi^k(E_2).
\end{aligned}$$

(iv) This is a special case of *(v)*.

(v) This is a special case of the equation $\psi^k(L_1 \oplus \cdots \oplus L_n) = L_1^k \oplus \cdots \oplus L_n^k$ we have already seen above.

(vi) This follows directly from the naturality of the exterior power (see Proposition 2.25 (v)).

(vii) The splitting principle reduces this to the case of a direct sum of line bundles. From the equation $\psi^k(L_1 \oplus \cdots \oplus L_n) = L_1^k \oplus \cdots \oplus L_n^k$, additivity and multiplicativity we get the claim.

□

Let us compute the Adams operations in the case of spheres. If the dimension is odd then the Adams operations are zero since the reduced K-group is zero. If the dimension is even then the reduced K-group is $\mathbb{Z}$. As a ring homomorphism of $\mathbb{Z}$ the map ψ^k is the multiplication with an integer.

Proposition 3.42 (Adams operation on even dimensional spheres)
If $X = S^{2n}$ then we have $\psi^k(x) = k^n x$ for all $x \in \widetilde{K}_{\mathbb{C}}(X)$.

Proof. Let H be the canonical line bundle over S^2. Then $H - 1$ is a generator of $\widetilde{K}_{\mathbb{C}}(S^2)$ and $(H-1)^2 = 0$. We get

$$\psi^k(H-1) = \psi^k(H) - \psi^k(1) = H^k - 1 = (1 + (H-1))^k - 1 = k(H-1).$$

This means ψ^k is multiplication by k.
By Bott periodicity $\widetilde{K}_{\mathbb{C}}(S^n)$ is generated by $(H-1)^n = (H-1)\cdots(H-1)$ and thus ψ^k is multiplication by k^n. □

In the real case one might also define operations ψ^k for $k > 1$ on $K_{\mathbb{R}}(X)$ by setting $\psi^k(E) := s_k(\lambda^1(E), \ldots, \lambda^k(E))$ for $E \in \mathrm{Vect}_{\mathbb{R}}(X)$. Here we define $\psi^{-1} := \mathrm{id}$ and use property *(vii)* from Theorem 3.39 to obtain Adams operations for all integers $k \in \mathbb{Z}$. As in the complex case ψ^k is additive and induces a homomorphism on $K_{\mathbb{R}}(X)$. These operations also satisfy *(i)*-*(vii)* from Theorem 3.39, but since one does not have the Splitting principle their proofs take a different form which can be found in [Kar09] IV.7.25 and IV.7.40.

Proposition 3.43 (Compatibility of real and complex Adams Operations)
Let X, Y be compact Hausdorff spaces. The real and complex Adams operations are compatible which means we have commutative diagrams

$$\begin{array}{ccc} K_{\mathbb{C}}(Y) & \xrightarrow{r} & K_{\mathbb{R}}(X) \\ \downarrow{\scriptstyle\psi^k} & & \downarrow{\scriptstyle\psi^k} \\ K_{\mathbb{C}}(Y) & \xrightarrow{r} & K_{\mathbb{R}}(X) \end{array} \qquad \begin{array}{ccc} K_{\mathbb{R}}(Y) & \xrightarrow{c} & K_{\mathbb{C}}(X) \\ \downarrow{\scriptstyle\psi^k} & & \downarrow{\scriptstyle\psi^k} \\ K_{\mathbb{R}}(Y) & \xrightarrow{c} & K_{\mathbb{C}}(X) \end{array}$$

where r is the realification and c the complexification. This also extends to the reduced case.

Proposition 3.44
The composition

$$K_{\mathbb{R}}(X) \xrightarrow{\ c\ } K_{\mathbb{C}}(X) \xrightarrow{\ r\ } K_{\mathbb{R}}(X)$$

is multiplication by 2. This is also true in the reduced case.

Proof. This is a consequence of the fact that we have an isomorphism of $\mathbb{R}$-vector spaces $V \otimes_{\mathbb{R}} \mathbb{C} \cong V \oplus V$. □

Remark 3.45
Let $n \equiv 0, 4 \bmod 8$. Then $\widetilde{K}_{\mathbb{R}}(S^n) \cong \mathbb{Z}$ and the previous proposition implies that the complexification $c\colon \widetilde{K}_{\mathbb{R}}(S^n) \to \widetilde{K}_{\mathbb{C}}(S^n)$ is injective.
One can further show that in the case $q \equiv 0 \bmod 8$ the image of c is $\mathbb{Z}$ and in the case $q \equiv 4 \bmod 8$ it is $2\mathbb{Z}$ (see proof of Corollary 5.2 in [Ada62]).

For spheres where the real K-theory is $\mathbb{Z}$ we have the same formula for the real Adams operations as in the complex case.

Proposition 3.46
Let $n \in \mathbb{N}$ be even and $X = S^{2n}$. Then we have $\psi^k(x) = k^n x$ for all $x \in \widetilde{K}_{\mathbb{R}}(X)$.

Proof. We have $\widetilde{K}_{\mathbb{R}}(S^{2n}) \cong \mathbb{Z} \cong \widetilde{K}_{\mathbb{R}}(S^{2n})$ and therefore the complexification $\widetilde{K}_{\mathbb{R}}(S^{2n}) \to \widetilde{K}_{\mathbb{C}}(S^{2n})$ is injective by Proposition 3.44. The Proposition now follows from the compatibility of the real and the complex Adams operations (see Proposition 3.43) using the formula for the complex Adams operations (see Proposition 3.42). □

By Bott periodicity we have $\widetilde{K}_{\mathbb{K}}(\Sigma^r X) \cong \widetilde{K}_{\mathbb{K}}(X)$ ($r = 2$ if $\mathbb{K} = \mathbb{C}$ and $r = 8$ if $\mathbb{K} = \mathbb{R}$), but the Adams operations are not necessarily preserved under this isomorphism. Fortunately they are still related. The next Proposition makes the relation precise by calculating the Adams operations on $\widetilde{K}_{\mathbb{K}}(\Sigma^r X)$ in terms of the Adams operations on $\widetilde{K}_{\mathbb{K}}(X)$. A proof can be found in [Ada62] Corollary 5.3.

Proposition 3.47 (Adams operations and suspension)
Let X be a compact Hausdorff space, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $r = 2$ if $\mathbb{K} = \mathbb{C}$ or $r = 8$ if $\mathbb{K} = \mathbb{R}$. Further let $\beta_{\mathbb{K}}\colon \widetilde{K}_{\mathbb{K}}(X) \to \widetilde{K}_{\mathbb{K}}(\Sigma^r X)$ be the isomorphism given by Bott periodicity.
Let ψ^k denote the Adams operations on $\widetilde{K}_{\mathbb{K}}(X)$ and let ψ^k_r denote the Adams operations on $\widetilde{K}_{\mathbb{K}}(\Sigma^r X)$. For $x \in \widetilde{K}_{\mathbb{K}}(X)$ we have

$$\psi^k_r(\beta_{\mathbb{K}}(x)) = k^{\frac{1}{2}r} \beta_{\mathbb{K}}(\psi^k(x)).$$

Using this Proposition we can complete the calculation of the Adams operations on spheres of even dimension.

Proposition 3.48
Let $n \equiv 2 \bmod 8$. The Adams operation ψ^k on $\widetilde{K}_{\mathbb{R}}(S^n) \cong \mathbb{Z}/2\mathbb{Z}$ is given by

$$\psi^k x = \begin{Bmatrix} x, & k \text{ odd} \\ 0, & k \text{ even} \end{Bmatrix}.$$

Proof. We first calculate the Adams operations on S^2.
We have $\widetilde{K}_{\mathbb{R}}(S^2) \cong \pi_1(\mathrm{SO}) \cong \mathbb{Z}/2\mathbb{Z}$ and a generator is given by the element

$$S^1 \to \mathrm{SO}, \theta \mapsto A_\theta := \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 & \dots \\ \sin(\theta) & \cos(\theta) & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The matrix A_θ is the image of $\cos(\theta) + i\sin(\theta)$ under the map $\mathrm{U} \to \mathrm{SO}$ which is induced by $\mathrm{U}(n) \to \mathrm{SO}(2n)$. Therefore the complexification

$$c\colon \widetilde{K}_{\mathbb{R}}(S^2) \cong \pi_1(\mathrm{SO}) \to \pi_1(\mathrm{U}) \cong \widetilde{K}_{\mathbb{C}}(S^2)$$

is surjective.
Now let $x \in \widetilde{K}_{\mathbb{R}}(S^2)$ and let $x' \in \widetilde{K}_{\mathbb{C}}(S^2)$ be a preimage of x under c. Using the compatibility of the real and complex Adams operations (see Proposition 3.43) and our previous calculation of the Adams operations on complex K-theory of spheres (see Proposition 3.42) we obtain $\psi^k_{\mathbb{R}}(x) = \psi^k_{\mathbb{R}}(c(x')) = c(\psi^k_{\mathbb{C}}(x')) = c(kx') = kx$. Since this is an equation in $\mathbb{Z}/2\mathbb{Z}$ we have

$$\psi^k x = \begin{Bmatrix} x, & k \text{ odd} \\ 0, & k \text{ even} \end{Bmatrix}$$

for all $x \in \widetilde{K}_{\mathbb{R}}(S^2)$.
The general case now follows from Proposition 3.47 since $k^4 \cong k \bmod 2$. □

In preparation for the next section we record the following technical result. A proof can be found in [Ada62] Theorem 5.1.

Proposition 3.49
Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. For $x \in K_{\mathbb{K}}(X)$ and $m \in \mathbb{Z}$ the value of $\psi^k(x)$ in $K_{\mathbb{K}}(X)/mK_{\mathbb{K}}(X)$ only depends on the residue class of k modulo m^e for some $e \in \mathbb{N}$.

3.4 The category $\mathscr{A}$

By definition the functor $\widetilde{K}_{\mathbb{K}}$ takes values in the category of abelian groups. It turns out that this invariant is to 'coarse' for our purposes.
For example for every sphere of even dimension the $\widetilde{K}_{\mathbb{C}}$-group is $\mathbb{Z}$. It is therefore not possible to distinguish them by just knowing the reduced K-group.
The idea is to additionally consider the Adams operations. We then can distinguish even dimensional spheres since the corresponding Adams operations on $\mathbb{Z}$ depend on the dimension of the sphere by Proposition 3.42.
The purpose of this section is to follow Adams approach in [Ada66] Section 6 and introduce a suitable category $\mathscr{A}$ where the pairs $(\widetilde{K}_{\mathbb{K}}(X), \{\psi^k\}_{k\in\mathbb{Z}})$ live in as objects. As it turns out the category $\mathscr{A}$ is abelian which is crucial, since we later want to study short exact sequences in this category.

We start by defining the category $\mathscr{A}$. The category actually depends on $\mathbb{K} = \mathbb{R}, \mathbb{C}$ but we omit the $\mathbb{K}$ in the notation.

Definition 3.50 (The category $\mathscr{A}$)
We define the category $\mathscr{A}$ to be the category with objects $(M, \{\psi^k\}_{k\in\mathbb{Z}})$, where M is a finitely generated abelian group and $\psi^k \colon M \to M$ is an endomorphism of $\mathscr{A}$ for every $k \in \mathbb{Z}$. We also require that the endomorphisms ψ^k satisfy the following axioms

(i) $\psi^0 = 0, \psi^1 = \mathrm{id}$ (and $\psi^{-1} = \mathrm{id}$ if $\mathbb{K} = \mathbb{R}$)

(ii) $\psi^k \circ \psi^l = \psi^l \circ \psi^k = \psi^{kl}$ for all $k, l \in \mathbb{Z}$

(iii) For every $x \in M$ and $q \in \mathbb{Z}$ the value of $\psi^k(x)$ in M/qM only depends on the residue class of k modulo q^e for some $e = e(x, q)$.

The morphisms between in $(M, \{\psi^k_M\}_{k\in\mathbb{Z}}), (N, \{\psi^k_N\}_{k\in\mathbb{Z}}) \in \mathscr{A}$ are group homomorphisms $\varphi \colon M \to N$ commuting with the operations ψ^k. This means we have the following commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ {\scriptstyle\psi^k_M}\downarrow & & \downarrow{\scriptstyle\psi^k_N} \\ M & \xrightarrow{\varphi} & N \end{array}$$

for every $k \in \mathbb{Z}$.

Remark 3.51
The functor $\widetilde{K}_{\mathbb{K}}$ takes values in $\mathscr{A}$. As operations we take the Adams operations ψ^k. The axioms are satisfied due to Theorem 3.39 and Proposition 3.49.
Bott periodicity induces an isomorphism of groups $\widetilde{K}_{\mathbb{K}}(X) \cong \widetilde{K}_{\mathbb{K}}(\Sigma^r X)$, where $r = 2$ if $\mathbb{K} = \mathbb{C}$ and $r = 8$ if $\mathbb{K} = \mathbb{R}$. The Adams operations on $\widetilde{K}_{\mathbb{K}}(\Sigma^r X)$ are $k^{\frac{1}{2}r}$ times the Adams operations on $\widetilde{K}_{\mathbb{K}}(X)$ (see Proposition 3.47).
Motivated by this we define an endofunctor T on $\mathscr{A}$, which maps an object $(M, \psi^k) \in \mathscr{A}$

to $(M, k^{\frac{1}{2}r}\psi^k)$. The operations $k^{\frac{1}{2}r}\psi^k$ satisfy the axiom so this is indeed an object in $\mathscr{A}$. For a map $f\colon M \to N \in \mathrm{Hom}(M,N)$ we define $Tf := f$. This map obviously commutes with $k^{\frac{1}{2}r}\psi^k$.
If we view $\widetilde{K}_{\mathbb{K}}(X)$ as an object in $\mathscr{A}$ the functor T gives an isomorphism

$$\widetilde{K}_{\mathbb{K}}(\Sigma^r X) \cong T\widetilde{K}_{\mathbb{K}}(X)$$

in $\mathscr{A}$ realizing Bott Periodicity.
If we use the original definition of the groups $\widetilde{K}_{\mathbb{K}}(\Sigma^r X)$ and $\widetilde{K}_{\mathbb{K}}(X)$, then for every map of spaces $f\colon X \to Y$ we have $Tf^* = (\Sigma^r f)^*$.
Furthermore we have $\widetilde{K}_{\mathbb{K}}\Sigma^r = T\widetilde{K}_{\mathbb{K}}$ as functors from the category of compact Hausdorff pointed spaces to $\mathscr{A}$.

Proposition 3.52
The category $\mathscr{A}$ is abelian.

We will not prove the Proposition (it is proven in [Ada66] Proposition 6.5.) but instead illustrate what quotients, images and kernels look like in $\mathscr{A}$.
The idea is to use the forgetful functor from $\mathscr{A}$ to abelian groups that forgets the operations. We can then first take the corresponding object in the category of abelian groups and then figure out what the operations should be.
We start with quotients.
Let (M, ψ^k) be an object in $\mathscr{A}$. A subobject is a subgroup N that is closed under the operations ψ^k. The operations on N are the restrictions of ψ^k to N.
Now we want to define operations on the quotients M/N. Let $x + N \in M/N$ be an equivalence class. Then we have $\psi^k(x+N) = \psi^k(x) + \psi^k(N) \subset \psi^k(x) + N$. Therefore the value of $\psi^k(x+N)$ modulo N does not depend on the choice of representative and we get well defined operations on the quotient M/N.
Let (M, ψ^k_M) and (N, ψ^k_N) be objects in $\mathscr{A}$ and let $\varphi\colon (M, \psi^k_M) \to (N, \psi^k_N)$ be a morphism. The image $\mathrm{im}(\varphi)$ of the group homomorphism φ is closed under ψ^k_N, since if $x = \varphi(y) \in N$ is in the image of φ then $\psi^k_N(x) = \psi^k_N(\varphi(y)) = \varphi(\psi^k_N(y)) \in \mathrm{im}(\varphi)$. Therefore the operations restrict to operations on $\mathrm{im}(\varphi)$.
This is indeed the image of φ in the category $\mathscr{A}$.
Similarly the kernel $\ker(\varphi)$ is closed under ψ^k_M, since $\varphi(x) = 0$ implies
$\varphi(\psi^k_M(x)) = \psi^k_M(\varphi(x)) = \psi^k_M(0) = 0$.

Remark 3.53
From the above description of kernel and images and the fact that T is the identity on the underlying groups we get that the functor T is exact. That means if

$$0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$$

is a short exact sequence in $\mathscr{A}$ then the sequence

$$0 \longrightarrow TN \longrightarrow TE \longrightarrow TM \longrightarrow 0$$

is also exact.

To finish this section we show how to add short exact sequences. We introduce the abelian group $\mathrm{Ext}^n(A, B)$ for objects $A, B \in \mathscr{A}$. The notation is justified by the fact that if $\mathscr{A}$ has enough projectives or enough injectives then this definition coincides with the definition of the Ext-groups as the right derivative of the covariant Hom-functor. Proofs can be found for example in [Wei94].

Definition 3.54
Let $\mathscr{A}$ be an abelian category, $n > 0$ and A, B be objects in $\mathscr{A}$. Exact sequences of the form

$$0 \longrightarrow B \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow A \longrightarrow 0$$

are called n-extensions.
We call two such extensions isomorphic if there are homomorphisms ι_i, $i = 0, \dots, n+1$ such that we have the commutative diagram

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & B & \longrightarrow & X_n & \longrightarrow & \cdots & \longrightarrow & X_1 & \longrightarrow & A & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \iota_{n+1}=\mathrm{id}} & & \Big\downarrow{\scriptstyle \iota_n} & & & & \Big\downarrow{\scriptstyle \iota_1} & & \Big\downarrow{\scriptstyle \iota_0=\mathrm{id}} & & \\ 0 & \longrightarrow & B & \longrightarrow & X'_n & \longrightarrow & \cdots & \longrightarrow & X'_1 & \longrightarrow & A & \longrightarrow & 0 \end{array} .$$

We define $\mathrm{Ext}^n(A, B)$ to be the set of isomorphism classes of n-extensions of $\mathscr{A}$ by B and $\mathrm{Ext}^0(A, B) := \mathrm{Hom}(A, B)$.

Let $f\colon A' \to A$ be a morphism in $\mathscr{A}$. For $n > 0$ consider the diagram

$$\begin{array}{ccccccccccccc} & & & & & & & & & & A' & & \\ & & & & & & & & & & \Big\downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & B & \longrightarrow & X_n & \longrightarrow & \cdots & \longrightarrow & X_1 & \longrightarrow & A & \longrightarrow & 0 \end{array} .$$

Let E be the pullback of the diagram $X_1 \longrightarrow A \xleftarrow{f} A'$. Then the universal property of E gives a map from X_2 to E (the map from X_2 to A' is just the zero map). One checks that the sequence

$$0 \longrightarrow B \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_2 \longrightarrow E \longrightarrow A \longrightarrow 0$$

is exact and thus an element of $\mathrm{Ext}^n(A, B)$.
There is the dual case. Let $g\colon B \to B'$ be a morphism in $\mathscr{A}$ and let E be the pushout of the diagram $B' \xleftarrow[g]{} B \longrightarrow X_n$.
Then

$$0 \longrightarrow B \longrightarrow E \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow A \longrightarrow 0$$

is an element of $\mathrm{Ext}^n(A, B)$.
The above means that $\mathrm{Ext}^n(-,-)$ is contravariant in the first component and covariantly in the second component.
Since $\mathrm{Ext}^0(-,-)$ is just $\mathrm{Hom}(-,-)$ we can also interpret the first case as a pairing

$$\mathrm{Ext}^n(A, B) \times \mathrm{Ext}^0(A', A) \to \mathrm{Ext}^n(A', B)$$

and the second case as a pairing

$$\mathrm{Ext}^0(B, B') \times \mathrm{Ext}^n(A, B) \to \mathrm{Ext}^n(A, B').$$

Next we introduce the group structure on $\mathrm{Ext}^n(A, B)$.

Construction 3.55 (Baer sum)
Let

$$\xi := 0 \longrightarrow B \longrightarrow X_n \longrightarrow \dots \longrightarrow X_1 \longrightarrow A \longrightarrow 0$$

.

$$\xi' := 0 \longrightarrow B \longrightarrow X_n' \longrightarrow \dots \longrightarrow X_1' \longrightarrow A \longrightarrow 0$$

be two n-extensions. We first take their direct sum $\xi_1 \oplus \xi_2$ to be

$$0 \longrightarrow B \oplus B \longrightarrow X_n \oplus X_n' \longrightarrow \dots \longrightarrow X_1 \oplus X_1' \longrightarrow A \oplus A \longrightarrow 0 \ .$$

This is a n-extension of $A \oplus A$ by $B \oplus B$.
To again obtain a n-extension of A by B we use the diagonal map

$$\Delta \colon A \to A \times A, a \mapsto (a, a)$$

and the addition map

$$\nabla \colon B \times B \to B, (b_1, b_2) \mapsto b_1 + b_2$$

to define

$$\xi_1 + \xi_2 := \nabla(\xi_1 \oplus \xi_2)\Delta$$

using the pairing from above. The n-extension $\xi_1 + \xi_2$ is called Baer sum of ξ_1 and ξ_2.
The Baer sum turns $\mathrm{Ext}^n(A, B)$ into an abelian group. If the category $\mathscr{A}$ has enough projectives or enough injectives then we obtain the same group structure when using the definition as a left derivative of the Hom-functor (see [Wei94] Section 3.4).

The next proposition is a result which turns a short exact sequence in $\mathscr{A}$ into an exact sequence of the corresponding Ext-groups. If one uses the definition of Ext as a left derivative then this is a part of the usual long exact sequence. One can formulate the proof in terms of extensions and see that it still works in the general case.

Proposition 3.56
Let $A \in A$ and let $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ be a short exact sequence in $\mathscr{A}$. Then the following sequence of Ext-groups

$$\mathrm{Ext}^1(A, B') \longrightarrow \mathrm{Ext}^1(A, B) \longrightarrow \mathrm{Ext}^1(A, B'')$$

is exact.

3.5 Chern character

The next tool from K-theory, we will need, is the Chern character. It relates the K-theory of a space X to the singular cohomology of X. The definition of the Chern character makes use of the Chern classes.
As for the Adams operations we define the Chern classes by imposing axioms which uniquely determine the desired object. A proof of the theorem can be found in [Hat03] Theorem 3.2.

Theorem 3.57 (Chern classes)
Let X be a compact space and E a complex vector bundle over X. For every $i \in \mathbb{N}$ there exist unique classes $c_i(E) \in H^{2i}(X;\mathbb{Z})$ called Chern classes satisfying the following properties

(i) *For every map $f\colon Y \to X$ we have $c_i(f^*(E)) = f^*(c_i(E))$.*

(ii) $c_n(E_1 \oplus E_2) = \sum_{i+j=n} c_i(E_1) \smile c_j(E_2)$, *where $\smile$ denotes the cup-product in $H^{2i}(X;\mathbb{Z})$*

(iii) $c_i(E) = 0$ *for* $i > \dim(E)$

(iv) *Let $E \to \mathbb{C}\mathrm{P}^\infty$ be the canonical line bundle. Then $c_1(E)$ is a fixed generator of $H^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$.*

Remark 3.58

1. *(i)* is called naturality and *(iv)* normalization
2. *(ii)* might also be formulated in terms of the so called total Chern class $c \coloneqq 1 + c_1 + \cdots \in H^*(X;\mathbb{Z})$ as $c(E_1 \oplus E_2) = c(E_1) \smile c(E_2)$.

Recall the Newton polynomial s_k from the proof of the existence of the Adams operations (see Theorem 3.39). We will use the polynomials s_k to define the Chern character.

Definition 3.59 (Chern character)
Let X be a compact space and $E \to X$ a complex vector bundle over X. We define

$$s_k(E) \coloneqq s_k(c_1, \ldots, c_n)$$

for $k > 0$ and s_0 is defined to be the rank of E.
We define the Chern character of E to be

$$\mathrm{ch}(E) \coloneqq \sum_{k=0}^{\infty} \frac{1}{k!} s_k(E) \in H^{\mathrm{even}}(X;\mathbb{Q}) \coloneqq \bigoplus_{i=0}^{\infty} H^{2i}(X;\mathbb{Q}).$$

This is actually a finite sum because of axiom *(iii)* in Theorem 3.57. Note that since the Chern classes are elements of cohomology groups of even degree the Chern character only takes value in even degrees.

The next Proposition shows that ch is a homomorphism. A proof can be found in [Hat03] Proposition 4.2.

Proposition 3.60 (Properties of the Chern character)
Let X be a compact Hausdorff space and E_1, E_2 be complex vector bundles over X. Then we have

$$\begin{aligned} \mathrm{ch}(E_1 \oplus E_2) &= \mathrm{ch}(E_1) + \mathrm{ch}(E_2), \\ \mathrm{ch}(E_1 \otimes E_2) &= \mathrm{ch}(E_1) \smile \mathrm{ch}(E_2). \end{aligned}$$

For line bundles L we have $\mathrm{ch}(L) = e^{c_1(L)}$.

Remark 3.61 (Chern character – K-theory version)
By the previous proposition we obtain a ring homomorphism $\mathrm{ch}\colon K_{\mathbb{C}}(X) \to H^{\mathrm{even}}(X; \mathbb{Q})$. We also call this map the Chern character.
The reduced groups $\widetilde{K}_{\mathbb{C}}(X)$ and $\widetilde{H}^{\mathrm{even}}(X; \mathbb{Q})$ are the kernel of the respective restriction to a point. Thus by the naturality of the Chern classes we get an induces ring homomorphism $\mathrm{ch}\colon \widetilde{K}_{\mathbb{C}}(X) \to \widetilde{H}^{\mathrm{even}}(X; \mathbb{Q})$.

In the special case of spheres we get that the Chern character is injective. A proof can be found in [Hat03] Proposition 4.3.

Proposition 3.62
$\mathrm{ch}\colon \widetilde{K}_{\mathbb{C}}(S^{2n}) \to \widetilde{H}^{\mathrm{even}}(S^{2n}; \mathbb{Q})$ is injective and its image is $\widetilde{H}^{2n}(S^{2n}; \mathbb{Z}) \subset \widetilde{H}^{2n}(S^{2n}; \mathbb{Q})$.

Remark 3.63
Let ξ be a generator of $\widetilde{K}_{\mathbb{C}}(S^{2n})$. Then the preceding Proposition implies that $\mathrm{ch}(\xi)$ is a generator of $\widetilde{H}^{2n}(S^{2n}; \mathbb{Z})$.

We define an algebra homomorphism $\psi_H^k\colon \widetilde{H}^{\mathrm{even}}(X; \mathbb{Q}) \to \widetilde{H}^{\mathrm{even}}(X; \mathbb{Q})$ by setting $\psi_H^k(x) = k^r x$ for $x \in \widetilde{H}^{2r}(X; \mathbb{Q})$.

Proposition 3.64 (Chern character and Adams operations)
We have

$$\mathrm{ch}(\psi^k(x)) = \psi_H^k(x)$$

for all $x \in \widetilde{K}_{\mathbb{C}}(X)$.

Definition 3.65 (Real Chern character)
We define the real Chern character $\mathrm{ch}_{\mathbb{R}}\colon \widetilde{K}_{\mathbb{R}}(X) \to H^*(X; \mathbb{Q})$ to be the composition

$$\widetilde{K}_{\mathbb{R}}(X) \xrightarrow{\ c\ } \widetilde{K}_{\mathbb{C}}(X) \xrightarrow{\ \mathrm{ch}\ } H^*(X; \mathbb{Q})$$

where $c\colon \widetilde{K}_{\mathbb{R}}(X) \to \widetilde{K}_{\mathbb{C}}(X)$ is the complexification.

Remark 3.66
For even n the complexification $c\colon \widetilde{K}_{\mathbb{R}}(S^{2n}) \to \widetilde{K}_{\mathbb{C}}(S^{2n})$ is injective (see Remark 3.45). Combining this with Proposition 3.62 we get that the real Chern character $\mathrm{ch}_{\mathbb{R}}\colon \widetilde{K}_{\mathbb{R}}(S^{2n}) \to \widetilde{H}^{2n}(S^{2n}; \mathbb{Z})$ is injective for n even. Furthermore Proposition 3.64 generalizes to the real case.

3.6 Thom isomorphism

In this section we will introduce the Thom space $T(E)$ of a vector bundle $p\colon E \to X$. Multiplication with a certain element $U_E \in \widetilde{K}_{\mathbb{K}}(T(E))$, called the Thom class, gives an isomorphism from $K_{\mathbb{K}}(X)$ to $\widetilde{K}_{\mathbb{K}}(T(E))$. We omit the proofs and redirect the interested reader to [Kar09] Section IV.1 and IV.5.

Definition 3.67 (Thom space)
Let X be compact Hausdorff and $p\colon E \to X$ be a vector bundle over X. By Proposition 2.44 and Proposition 2.43 we have a unique inner product on E and therefore a norm on each fibre. Let $D(E)$ be all the vectors of length less or equal 1 of each fibre and similarly let $S(E)$ be all the vectors of length 1 of each fibre.
We define the Thom space of E to be

$$T(E) \coloneqq D(E)/S(E).$$

$D(E)$ is called the unit disk bundle of E and $S(E)$ is called the sphere bundle of E.

Remark 3.68
One might view $T(E)$ as the one-point compactification of E.

Let $f\colon E \to X$ be a map of compact Hausdorff spaces, then $K_{\mathbb{K}}(E)$ is a $K_{\mathbb{K}}(X)$-module. The module structure is defined as follows. Let $G \in K_{\mathbb{K}}(X)$ and $F \in K_{\mathbb{K}}(E)$. We first pullback G along f and then use the ring structure in $K_{\mathbb{K}}(E)$ to multiply F with $f^*(G)$ obtaining an element in $K_{\mathbb{K}}(E)$.
We want to apply this to the case where $p\colon E \to X$ is a vector bundle. But we cannot apply this directly since if the rank of E is not zero then E contains several copies of $\mathbb{R}$ and thus is not compact.
To resolve this issue we use a generalization of the theory we did for compact space. Namely we want to also allow locally compact spaces (see [Kar09] for a detailed treatment). A space is called locally compact if every point has a compact neighbourhood.
Let Z be locally compact, $\dot{Z}$ be its one-point compactification and let ∞ be the new point. We define $K_{\mathbb{K}}(Z) \coloneqq \widetilde{K}_{\mathbb{K}}(\dot{Z})$ where we view $\dot{Z}$ as a pointed space with basepoint ∞.
If Z is compact then $\dot{Z}$ is just Z with the disjoint point ∞ therefore this definition agrees with the definition for compact spaces.
The module structure explained above generalizes to the case of locally compact spaces. If we now apply this to a vector bundle $p\colon E \to X$ then we get that $K(E)$ is a $K(X)$-module. Since the Thom space is the one-point compactification of E we get that $\widetilde{K}_{\mathbb{K}}(T(E))$ is a $K(X)$-module.

Theorem 3.69 (Thom isomorphism - complex case)
Let X be compact and $p\colon E \to X$ be a complex vector bundle over X. Then $\widetilde{K}_{\mathbb{C}}(T(E))$ is a one-dimensional $K_{\mathbb{C}}(X)$-module generated by the so called Thom class $U_E \in \widetilde{K}_{\mathbb{C}}(T(E))$. Reformulated this means that multiplication with the Thom class induces an isomorphism $\varphi_K\colon K_{\mathbb{C}}(X) \to \widetilde{K}_{\mathbb{C}}(T(E)), x \mapsto x \cdot U_E$.

Remark 3.70 (Property of the Thom class)
The Thom class U_E has the property that its restriction to a fibre of E is a generator. Concretely this means for every point $x \in X$ the image of the Thom class under the map $\widetilde{K}_{\mathbb{C}}(T(E)) \to \widetilde{K}_{\mathbb{C}}(S^n)$ induced by the inclusion $E_x = S^n \to E$ of the fibre is a generator of $\widetilde{K}_{\mathbb{C}}(S^n)$.

The corresponding statements of Theorem 3.69 are also true (but harder to prove) in the real case if we require the vector bundle to carry a $\mathrm{Spin}(n)$-structure where n is a multiple of 8.

Theorem 3.71 (Thom isomorphism - real case)
Let X be compact and $p\colon E \to X$ be a real vector bundle over X with a $\mathrm{Spin}(n)$ structure where n is divisible by 8. As in the complex case there exists a Thom class $U_E \in \widetilde{K}_{\mathbb{R}}(T(E))$ such that $\widetilde{K}_{\mathbb{R}}(T(E))$ is a one-dimensional $K_{\mathbb{R}}(X)$-module generated by U_E.
The Thom class also has the corresponding property from the remark.

4 The image of the J-homomorphism

In the first section we introduce the stable homotopy groups of spheres and the J-homomorphism. The rest of this chapter deals with the proof of following main theorem.

Theorem
Let $r \equiv 0, 1 \bmod 8$ and $r > 1$. Then the J-homomorphism $J\colon \pi_r(\mathrm{SO}) \to \pi_r^s$ is injective and its image is a direct summand in π_r^s which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

We will follow the approach by Adams from the paper [Ada66]. The paper also deals with the other cases mentioned in the introduction but we will only treat the results needed for the case $r \equiv 0, 1 \bmod 8$. We also give arguments in greater detail and fill in pieces left out by Adams.
We start the preparation of the proof of the above theorem in Section 4.2 by defining for maps $f\colon X \to Y$ invariants $d(f)$ and $e(f)$. We will show some properties of these invariants and see how to obtain their 'stable' version. In Section 4.3 we will see that in the special case of spheres of suitable dimension the e-invariant possess a description using the Chen character. This will enable us to do calculations and proof the main theorem in the last section.

4.1 The stable homotopy groups of spheres and the J-homomorphism

This section introduces the stable homotopy groups of spheres and the J-homomorphism.

Definition 4.1 (Stable homotopy groups)
Let X be a pointed topological space and $r > 0$. The reduced suspension Σ induces a sequence

$$\pi_r(X) \xrightarrow{\Sigma} \pi_{r+1}(\Sigma X) \xrightarrow{\Sigma} \pi_{r+2}(\Sigma^2 X) \xrightarrow{\Sigma} \dots .$$

We define the r-th stable homotopy groups to be the limit of this sequence

$$\pi_r^s(X) := \varinjlim_n \pi_{r+n}(\Sigma^n X).$$

For $X = S^0$ we get the stable homotopy group of spheres

$$\pi_r^s := \varinjlim_n \pi_{r+n}(S^n).$$

π_r^s is also called r-stem.

The next theorem will lead to an alternative definition of the stable homotopy group. A space X is called n-connected if $\pi_i(X) = 0$ for all $i \leq n$.

Theorem 4.2 (Freudenthal suspension theorem)
Let $n \in \mathbb{N}$ and let X be a n-connected pointed CW complex. Then the map

$$\pi_r(X) \to \pi_{r+1}(\Sigma X)$$

induced by the reduced suspension is an isomorphism for all $r \leq 2n$.

Remark 4.3
Let X be a n-connected CW-complex. For $r \leq n$ the Freudenthal suspension theorem implies that ΣX is $(n+1)$-connected. Since the suspension ΣY of any CW-complex Y is path-connected, i.e. 0-connected we get by induction that $\Sigma^{n+1} Y$ is n-connected.
If n is large enough then $r + n \leq 2(n-1)$ and we can apply the Freudenthal suspension theorem to the $(n-1)$-connected space $\Sigma^n X$ and get $\pi_{r+n}(\Sigma^n X) \cong \pi_{r+n+1}(\Sigma^{n+1} X)$. Therefore we get that the sequence from the definition of the stable homotopy group stabilises for large n which means that all later maps are isomorphisms. So we have

$$\pi_r^s(X) \cong \pi_{r+n}(\Sigma^n X)$$

for large enough n.

Construction 4.4 (J-homomorphism)
The J-homomorphism is a map $\pi_r(\mathrm{SO}) \to \pi_r^s$. Recall from Corollary 3.36 that for $r \equiv 2, 4, 5, 6 \bmod 8$ the homotopy group $\pi_r(\mathrm{SO})$ is zero so the J-homomorphism is not very interesting in these cases.
The strategy to defining the J-homomorphism is to first define a homomorphism $\pi_r(\mathrm{SO}(n)) \to \pi_{r+n}(S^n)$ for every natural number n and then show that they induce a homomorphism between the direct limits on both sides.
Let $H(n)$ denote the homotopy equivalences from S^n to S^n that preserve the basepoint which we call ∞. We further define the n-th loopspace $\Omega^n X$ of a pointed space X to be the set of all continuous base-preserving maps $S^n \to X$. $H(n)$ is naturally a subspace of $\Omega^n S^n$.
We view S^n as the one-point-compactification of $\mathbb{R}^n$ where the new point is ∞. Every element of $\mathrm{SO}(n)$ may now be seen as a homeomorphism of S^n fixing ∞. Thus there is a natural map $\mathrm{SO}(n) \to H(n)$ and consequently a map $\mathrm{SO}(n) \to \Omega^n S^n$.
Applying the r-th homotopy functor we get a map $\pi_r(\mathrm{SO}(n)) \to \pi_r(\Omega^n S^n)$. The reduced suspension functor is left adjoint to the loop space functor Ω (see [Hat02] Section 4.3), so we get an isomorphism $\pi_r(\Omega^n S^n) \cong \pi_{r+n}(S^n)$ and therefore the maps from above induce maps

$$\pi_r(\mathrm{SO}(n)) \to \pi_{r+n}(S^n).$$

These maps fit into the commutative diagram

$$\begin{array}{ccc} \pi_r(\mathrm{SO}(n)) & \xrightarrow{J} & \pi_{r+n}(S^n) \\ \downarrow & & \downarrow \Sigma \\ \pi_r(\mathrm{SO}(n+1)) & \xrightarrow{J} & \pi_{r+n+1}(S^{n+1}) \end{array}$$

where the map on the left is induced by the inclusion

$$\mathrm{SO}(n) \to \mathrm{SO}(n+1), M \mapsto \left(\begin{array}{c|c} M & \begin{smallmatrix} 0 \\ \vdots \\ 0 \end{smallmatrix} \\ \hline \begin{smallmatrix} 0 & \dots & 0 \end{smallmatrix} & 1 \end{array}\right).$$

Using the Lemma below and $\varinjlim_n \pi_r(\mathrm{SO}(n)) \cong \pi_r(\mathrm{SO})$ (see Lemma 3.18) we obtain the stable J-homomorphism $J\colon \pi_r(\mathrm{SO}) \to \pi_r^s$.

Lemma 4.5
Let $\{X_i\}_{i\in\mathbb{N}}$, $\{Y_i\}_{i\in\mathbb{N}}$ be groups and $\varphi_i\colon X_i \to X_{i+1}$ and $\psi_i\colon Y_i \to Y_{i+1}$ be group homomorphisms for all $i \in \mathbb{N}$. Let further $f_i\colon X_i \to Y_i$ be a group homomorphism for all $i \in I$ such that the diagram

$$\begin{array}{ccc} X_i & \xrightarrow{\varphi_i} & X_{i+1} \\ \downarrow f_i & & \downarrow f_{i+1} \\ Y_i & \xrightarrow{\psi_i} & Y_{i+1} \end{array}$$

commutes.
Then the maps $\{f_i\}_{i\in I}$ induce a 'stable' homomorphism $f\colon \varinjlim_i X_i \to \varinjlim_i Y_i$.

Proof. Let $\widetilde{f}_i\colon X_i \to \varinjlim_i Y_i$ be the composition of f_i with the natural map $Y_i \to \varinjlim_i Y_i$. Using the diagram from the assumption we get that the diagram

$$\begin{array}{ccc} X_i & \xrightarrow{\varphi_i} & X_{i+1} \\ & \searrow^{\widetilde{f}_i} \quad \swarrow_{\widetilde{f}_{i+1}} & \\ & \varinjlim_i Y_i & \end{array}$$

commutes. Using the universal property of the direct limit we obtain the desired homomorphism $f\colon \varinjlim_i X_i \to \varinjlim_i Y_i$. □

There is also a more explicit description of the J-homomorphism. For this we introduce the join and the Hopf construction.

Definition 4.6 (Join)
Let X, Y be topological spaces. The join $X \star Y$ is the quotient space of $X \times Y \times I$ by the equivalence relation $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$ for all $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$.

Remark 4.7

(a) In short the join is obtained from $X \times Y \times I$ by identifying the slice $X \times Y \times \{0\}$ to X and the slice $X \times Y \times \{1\}$ to Y.

(b) The join of X with a single point is isomorphic to the cone CX of X.

(c) The join of X with the 0-sphere S^0 is homeomorphic to the reduced suspension ΣX of X.

(d) The join of two spheres is again a sphere. Concretely $S^n \star S^m \cong S^{n+m+1}$.

Definition 4.8 (Hopf construction)
Let $f\colon X \times Y \to Z$ be a map of spaces. The Hopf construction of f is the map

$$H_f\colon X \star Y \to \Sigma Z, (x, y, t) \mapsto (f(x, y), t).$$

Remark 4.9 (J-homomorphism via the Hopf construction)
Let $\varphi \in \pi_r(SO(n))$. Since we view elements in $\mathrm{SO}(n)$ as maps $S^n \to S^n$ the map φ corresponds to a map $f\colon S^r \times S^n \to S^n$ by currying.
Explicitly we have $f(x, y) = \phi(x)y \in S^n$ where on the right $\phi(x)y$ means matrix vector multiplication. Applying the Hopf construction to f we precisely get $J\phi$.

4.2 The invariants d and e

In this section we introduce the invariants d and e and prove some useful properties about them. The theory developed in this section works for general half exact functors $k\colon CW_{fin}^{op} \to A$, where CW_{fin}^{op} is the category of finite CW-complexes and $\mathscr{A}$ is any abelian category. Here half exact means that for every finite CW-complex W and subcomplex $V \subset W$ the sequence $k(W/V) \longrightarrow k(W) \longrightarrow k(V)$ is an exact sequence in $\mathscr{A}$.
But we only formulate it for the case of the reduced K-theory functor. The abelian category we will use is the category $\mathscr{A}$ defined in Section 3.4. To simplify the notation $\widetilde{K}$ means either $\widetilde{K}_{\mathbb{R}}$ or $\widetilde{K}_{\mathbb{C}}$. Unless stated otherwise space means finite pointed CW-complex.
We begin with the definition of the d-invariant.

Definition 4.10 (d-invariant - unstable version)
Let $f\colon X \to Y$ be a map of spaces. We define the d-invariant to be

$$d(f) \coloneqq f^* \in \mathrm{Hom}(\widetilde{K}(Y), \widetilde{K}(X)).$$

From the definition we immediately get that the d-invariant is homotopy invariant and $d(fg) = d(g)d(f)$.

We apply the reduced K-theory functor to the reduced cofibre sequence of f (see Construction 3.23)

$$X \xrightarrow{f} Y \xrightarrow{i} C'X \cup_f Y \xrightarrow{j} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \ .$$

and obtain the sequence

$$\widetilde{K}(\Sigma Y) \xrightarrow{d(\Sigma f)} \widetilde{K}(\Sigma X) \xrightarrow{j^*} \widetilde{K}(C'X \cup_f Y) \xrightarrow{i^*} \widetilde{K}(Y) \xrightarrow{d(f)} \widetilde{K}(X)$$

which is exact by Proposition 3.24.
If we now assume that $d(f) = 0$ and $d(\Sigma f) = 0$, then we obtain the short exact sequence

$$0 \longrightarrow \widetilde{K}(\Sigma X) \xrightarrow{j^*} \widetilde{K}(C'X \cup_f Y) \xrightarrow{i^*} \widetilde{K}(Y) \longrightarrow 0 \ .$$

This short exact sequence is the e-invariant.

Definition 4.11 (e-invariant - unstable version)
Let $f\colon X \to Y$ be a map of spaces such that $d(f) = 0$ and $d(\Sigma f) = 0$. We define $e(f) \in \operatorname{Ext}^1(\widetilde{K}(Y), \widetilde{K}(\Sigma X))$ to be the short exact sequence

$$0 \longrightarrow \widetilde{K}(\Sigma X) \xrightarrow{j^*} \widetilde{K}(C'X \cup_f Y) \xrightarrow{i^*} \widetilde{K}(Y) \longrightarrow 0 \ .$$

We say the e-invariant is defined if $d(f) = 0$ and $d(Sf) = 0$

Lets show some elementary properties of these invariants. We start by showing that both invariants only depend on the homotopy class of f.

Proposition 4.12
Let $f, g\colon X \to Y$ be a homotopic maps of spaces. Then we have

(i) $d(f) = d(g)$

(ii) Let further $e(f)$ be defined. Then $e(g)$ is also defined and $e(f) = e(g)$.

Proof. (i) is a direct consequence of the homotopy invariance of $\widetilde{K}$.
For *(ii)* we apply the homotopy invariant functor $\widetilde{K}$ to the diagram from Lemma 3.25 and obtain the commutative diagram

$$\begin{array}{ccccccccc}
\widetilde{K}(\Sigma Y) & \xrightarrow{(-\Sigma g)^*} & \widetilde{K}(\Sigma X) & \xrightarrow{(j')^*} & \widetilde{K}(C'X \cup_g Y) & \xrightarrow{(i')^*} & \widetilde{K}(Y) & \xrightarrow{g^*} & \widetilde{K}(X) \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \varphi^*} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\
\widetilde{K}(\Sigma Y) & \xrightarrow{(-\Sigma f)^*} & \widetilde{K}(\Sigma X) & \xrightarrow{j^*} & \widetilde{K}(C'X \cup_f Y) & \xrightarrow{i^*} & \widetilde{K}(Y) & \xrightarrow{f^*} & \widetilde{K}(X)
\end{array} .$$

Since $e(f)$ is defined f^* and $(\Sigma f)^*$ are zero. By commutativity we get that g^* and $(\Sigma g)^*$ are also zero and therefore $e(g)$ is defined. The five lemma implies that φ^* is an isomorphism, so $e(f) = e(g)$. □

Justified by this Proposition we view the d-invariant as a map

$$[X, Y]_* \to \operatorname{Hom}(\widetilde{K}(Y), \widetilde{K}(X))$$

where $[X, Y]_*$ denotes homotopy classes of pointed maps from X to Y. Similarly we view the e invariant as a map

$$[X, Y]_* \to \operatorname{Ext}^1(\widetilde{K}(Y), \widetilde{K}(\Sigma X)).$$

For the next proposition we need a lemma that allows us to detect pullbacks and pushout.

Lemma 4.13 (Detecting pullbacks/pushouts)
Suppose we have a diagram in the category of groups

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B' & \xrightarrow{\iota'} & M' & \xrightarrow{\pi'} & A' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & B & \xrightarrow{\iota} & M & \xrightarrow{\pi} & A & \longrightarrow & 0
\end{array}$$

where the rows are exact. Then M' is the pullback of the diagram $M \xrightarrow{\pi} A \xleftarrow{g} A'$ if and only if f is an isomorphism.

Dually M is the pushout of the diagram $B \xleftarrow{f} B' \xrightarrow{\iota'} M'$ if and only if g is an isomorphism.

Proof. We will only show the first statement, since the second follows by duality from the first.
The pullback of $M \xrightarrow{\pi} A \xleftarrow{g} A'$ is $M \times_A A' = \{(m, a') \in M \times A' \mid \pi_2(m) = g(a')\}$ and the map to M respectively A' is the projection onto the first respectively second coordinate. The universal property of the pullback yields a unique homomorphism $\phi \colon B \to M \times_A A'$.

$$\begin{array}{ccccc}
B & & \xrightarrow{\quad 0 \quad} & & A' \\
 & \overset{\exists!\phi}{\searrow} & & & \\
 & & M \times_A A' & \xrightarrow{\pi_2} & A' \\
 & \iota \searrow & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle g} \\
 & & M & \xrightarrow{\pi} & A
\end{array}$$

Concretely we have $\phi \colon B \to M \times_A A', b \mapsto (\iota(b), 0)$.
We now show that

$$0 \longrightarrow B \xrightarrow{\phi} M \times_A A' \xrightarrow{\pi_2} A' \longrightarrow 0$$

is a short exact sequence.

π_2 is surjective:

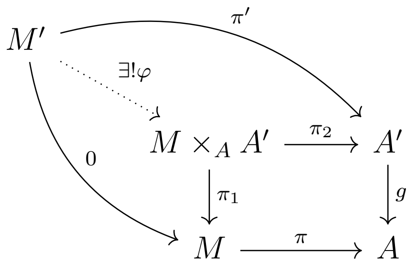
Let $a' \in A'$. Since π is surjective there exists an element $m \in M$ such that $\pi(m) = g(a')$. Now $(m, a') \in M \times_A A'$ and $\pi_2((m, a')) = a'$.

ϕ is injective:

Let $(\iota(b), 0) = \phi(b) = (0, 0)$. Then $\iota(b) = 0$ which means b is zero since ι is injective.

$\ker(\pi_2) = \operatorname{im}(\phi)$:

Elements in $\ker(\pi_2)$ are precisely the elements $(m, 0) \in M \times_A A'$. Since this is an element in $M \times_A A'$ we have $\pi(m) = g(0) = 0$. This means $m \in \ker(\pi) = \operatorname{im}(\iota)$. Consequently $\ker(\pi_2)$ consists of all elements $(m, 0) \in M \times_A A'$ such that $m \in \operatorname{im}(\iota)$. But these are exactly the elements in $\operatorname{im}(\phi)$.
Similar to before the universal property of the pullback yields a unique homomorphism $\varphi \colon M' \to M \times_A A'$



We now have the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B' & \xrightarrow{\iota'} & M' & \xrightarrow{\pi'} & A' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \varphi} & & \| & & \\ 0 & \longrightarrow & B & \xrightarrow{\phi} & M \times_A A' & \xrightarrow{\pi_2} & A' & \longrightarrow & 0 \end{array}$$

where the rows are exact.
To prove the Lemma we now show that f is an isomorphism if and only if φ is an isomorphism.
If f is an isomorphism then the 5-Lemma directly shows that φ is also an isomorphism.
Now assume φ is an isomorphism.

f is injective:

Let $b' \in B'$ with $f(b') = 0$. We have $0 = \Phi(f(b')) = \varphi(\iota'(b'))$. But since both ι and φ are injective b' must be zero.

f is surjective:

Let $b \in B$. The surjectivity of φ yields an element $m' \in M'$ such that $\varphi(m') = \phi(b)$. Then we have $\pi'(m') = \pi_2(\phi(b)) = 0$. This means there exists an element $b' \in B'$ such that $\iota'(b') = m'$. Now we have $\phi(b) = \varphi(m') = \varphi(\iota'(b')) = \phi(f(b'))$. Since ϕ is injective we get $b = f(b')$. ☐

The next proposition describes the behaviour of the invariants under composition. The multiplications on the right hand side are the pairing defined in Section 3.4.

Proposition 4.14 (d, e and composition)
Let $f\colon X \to Y$ and $g\colon Y \to Z$ be maps of spaces. Then we have

(i) $d(gf) = d(f)d(g)$

(ii) Let further $e(f)$ be defined. Then $e(gf)$ is defined and $e(gf) = e(f)d(g)$.

(iii) Let further $e(g)$ be defined. Then $e(gf)$ is defined and $e(gf) = d(Sf)e(g)$.

Proof. (i) is the functoriality of $\widetilde{K}$
For (ii) we apply $\widetilde{K}$ to the diagram obtained from applying Lemma 3.27 to

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle g} \\ X & \xrightarrow{gf} & Z \end{array} .$$

We end up with the commutative diagram

$$\begin{array}{ccccccccc} \widetilde{K}(\Sigma Z) & \xrightarrow{(-\Sigma(gf))^*} & \widetilde{K}(\Sigma X) & \xrightarrow{(j')^*} & \widetilde{K}(C'X \cup_{gf} Z) & \xrightarrow{(i')^*} & \widetilde{K}(Z) & \xrightarrow{(gf)^*} & \widetilde{K}(X) \\ \big\downarrow{\scriptstyle (\Sigma g)^*} & & \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle \varphi^*} & & \big\downarrow{\scriptstyle g^*} & & \big\downarrow{\scriptstyle \mathrm{id}} \\ \widetilde{K}(\Sigma Y) & \xrightarrow{(-\Sigma f)^*} & \widetilde{K}(\Sigma X) & \xrightarrow{j^*} & \widetilde{K}(C'X \cup_{f} Y) & \xrightarrow{i^*} & \widetilde{K}(Y) & \xrightarrow{f^*} & \widetilde{K}(X) \end{array} .$$

The top row belongs to $e(gf)$ and the bottom row to $e(f)$. By commutativity we see that if $e(f)$ is defined then $e(gf)$ is also defined.
Since $\mathrm{id}\colon \widetilde{K}(\Sigma X) \to \widetilde{K}(\Sigma X)$ is an isomorphism

$$\begin{array}{ccc} \widetilde{K}(C'X \cup_{gf} Z) & \xrightarrow{(i')^*} & \widetilde{K}(Z) \\ \big\downarrow{\scriptstyle \varphi^*} & & \big\downarrow{\scriptstyle g^*} \\ \widetilde{K}(C'X \cup_{f} Y) & \xrightarrow{i^*} & \widetilde{K}(Y) \end{array}$$

is a pullback square by Lemma 4.13. This means $e(gf) = e(f)g^*$.
We obtain (iii) by the same procedure applied to the diagram

$$\begin{array}{ccc} X & \xrightarrow{gf} & Z \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle \mathrm{id}} \\ Y & \xrightarrow{g} & Z \end{array}$$

and using the dual statement of Lemma 4.13. ☐

We now want to say that our invariants are homomorphisms. For this to make sense we need to endow $[X, Y]_*$ with a monoid structure. Therefore we impose further conditions on the spaces X and Y. We will assume X and Y are suspensions, which is not very restrictive since we are interested in the case of spheres. One can generalize this to the case where X is a so called co-H-space (see [Ada66] Proposition 3.3).
Now let $X = \Sigma Z$ with basepoint $Z \times \{1\}$ and $Y = \Sigma W$ with basepoint $W \times \{1\}$. Consider the space $\widetilde{X}$ that we obtain from X by collapsing $Z \times \{\frac{1}{2}\}$ to a point. Furthermore consider $X_d \coloneqq Z \times [0, \frac{1}{2}]/(Z \times \{0\} \cup Z \times \{\frac{1}{2}\})$ and $X_u \coloneqq Z \times [\frac{1}{2}, 1]/(Z \times \{\frac{1}{2}\} \cup Z \times \{1\})$. For both X_d and X_u we take the point $Z \times \{\frac{1}{2}\}$ as the basepoint. Then $\widetilde{X}$ is the wedge sum $X_d \vee X_u$.
We can identify X_d with X via the map $i_d \colon Z_d \to X, (z, t) \mapsto (z, 2t)$. We further identify X_u with X via the map $i_u \colon Z_u \to X, (z, t) \mapsto (z, 2 - 2t)$. We needed to reverse the direction in the second component since in X the north pole is the basepoint but in Z_u the south pole is the basepoint. The composition

$$X \longrightarrow \widetilde{X} = Z_d \vee Z_u \xrightarrow{i_d \vee i_u} X \vee X$$

gives a map $\Delta \colon X \to X \vee X$.
Let $j_u \colon Y = \Sigma W \to Y, (w, t) \mapsto (w, 1 - t)$ be the map that reverses the direction in the second component of Y. We also define the map $\nabla = \mathrm{id}_Y \vee j_u \colon Y \vee Y \to Y$ to be the identity in the first part and j_u in the second part of the wedge sum.
Now if $f, g \colon X \to Y$ are basepoint preserving maps then we define $f + g \in [X, Y]_*$ to be the composition

$$X \xrightarrow{\Delta} X \vee X \xrightarrow{f \vee g} Y \vee Y \xrightarrow{\nabla} Y.$$

With this definition we get a well defined monoid structure on $[X, Y]_*$. In the case where $X = S^r$ is a sphere of dimension $r \geq 1$ this is the same structure as the usual group structure on $\pi_r(Y)$.

Proposition 4.15 (d and e are homomorphisms)
Let $f, g \colon X \to Y$ be maps of spaces. Then we have

(i) $d(f + g) = d(f) + d(g)$

(ii) If $e(f)$ and $e(g)$ are defined then $e(f+g)$ is defined and we have $e(f+g) = e(f)+e(g)$.

In (ii) the sum on the right hand side is the Bear sum in $\mathrm{Ext}^1(\widetilde{K}(Y), \widetilde{K}(\Sigma X))$ (see Construction 3.55).

Proof. Let $h_1, h_2 \colon H \to G$ be group homomorphisms. Then the map $f + g$ is the composition

$$H \longrightarrow H \oplus H \xrightarrow{f \oplus g} G \oplus G \longrightarrow G$$

where the first map is the diagonal map and the last map is the addition in G.
In Proposition 3.13 we have seen that $\widetilde{K}(X \vee X) \cong \widetilde{K}(X) \oplus \widetilde{K}(X)$ for every space X. Under this identification for the maps defined above we get

- Δ^* is the addition map $\widetilde{K}(X) \oplus \widetilde{K}(X) \to \widetilde{K}(X)$.
- ∇^* is the diagonal map $\widetilde{K}(Y) \to \widetilde{K}(Y) \oplus \widetilde{K}(Y)$.
- $(f \vee g)^*$ is the map $f^* \oplus g^* \colon \widetilde{K}(Y) \oplus \widetilde{K}(Y) \to \widetilde{K}(X) \oplus \widetilde{K}(X)$.

Therefore we obtain

$$\begin{aligned}(f+g)^* &= (\nabla \circ (f \vee g) \circ \Delta)^* \\ &= \Delta^* \circ (f^* \oplus g^*) \circ \nabla^*.\end{aligned}$$

This proves (i).
We know need the following facts about the reduced cone and the reduced suspension. For spaces x, Y we have $C'(X \vee Y) = C'X \vee C'Y$ and $\Sigma(X \vee Y) = \Sigma X \vee \Sigma Y$. For maps $f, g \colon X \to Y$ we have $\Sigma(f \vee g) = \Sigma f \vee \Sigma g$ as a map $\Sigma X \vee \Sigma X \to \Sigma Y \vee \Sigma Y$.
The reduced cofibre sequence of $f \vee g$ is

$$X \vee X \xrightarrow{f \vee g} Y \vee Y \longrightarrow C'(X \vee X) \cup_{f \vee g} Y \vee Y \longrightarrow \Sigma(X \vee X) \xrightarrow{\Sigma(f \vee g)} \Sigma(Y \vee Y) \ .$$

By the above we write it in the following form

$$X \vee X \xrightarrow{f \vee g} Y \vee Y \longrightarrow (C'(X) \cup_f Y) \vee (C'(X) \cup_g Y) \longrightarrow \Sigma X \vee \Sigma X \xrightarrow{\Sigma f \vee \Sigma g} \Sigma Y \vee \Sigma Y \ .$$

Applying reduced K-theory and using that reduced K-theory turns wedges into direct sums we obtain a sequence which is too long to be displayed in a single line. But fortunately the map $(f \vee g)^*$ becomes $f^* \oplus g^*$ and $(\Sigma f \vee \Sigma g)^*$ becomes $(\Sigma f)^* \vee (\Sigma g)^*$ and thus both maps are zero, since $e(f)$ and $e(g)$ are defined. Therefore $e(f \vee g)$ is defined by the short exact sequence

$$0 \to \widetilde{K}(\Sigma X) \oplus \widetilde{K}(\Sigma X) \to \widetilde{K}(C'(X) \cup_f Y) \oplus \widetilde{K}(C'(X) \cup_g Y) \to \widetilde{K}(Y) \oplus \widetilde{K}(Y) \to 0 \ .$$

This is precisely the direct sum of the short exact sequence $e(f)$ with the short exact sequence $e(g)$. Now we use Proposition 4.14 and get

$$\begin{aligned}e(f+g) &= e(\nabla \circ (f \vee g) \circ \Delta) \\ &= (\Sigma\Delta)^* e(f \vee g) \nabla^* \\ &= (\Sigma\Delta)^* e(f \oplus g) \nabla^*.\end{aligned}$$

As before $(\Sigma\Delta)^* \colon \widetilde{K}(\Sigma X) \oplus \widetilde{K}(\Sigma X) \to \widetilde{K}(\Sigma X)$ is the addition in $\widetilde{K}(\Sigma X)$ and
$\nabla^* \colon \widetilde{K}(Y) \to \widetilde{K}(Y) \oplus \widetilde{K}(Y)$ is the diagonal map.
Therefore the last term in the equation above is the formula for the Bear sum of $e(f)$ and $e(g)$ from Construction 3.55. □

We want to compose the stable J-homomorphism with our invariants. Since the image of J lies in π_r^s we need stable versions of our invariants. Therefore we describe how they behave under suspension.
Let $t = 2$ in the complex case and $t = 8$ in the real case. Recall the functor T from $\mathscr{A}$ to itself which realizes Bott periodicity (see Remark 3.51). In Remark 3.53 we have seen that T is exact and thus defines a homomorphism

$$\operatorname{Ext}^1(\widetilde{K}(Y), \widetilde{K}(\Sigma X)) \to \operatorname{Ext}^1(T\widetilde{K}(Y), T\widetilde{K}(\Sigma X)).$$

The next proposition describes the behaviour of the invariants under the t-fold reduced suspension.

Proposition 4.16
Let $f\colon X \to Y$ be a map of spaces. Then we have

(i) $d(\Sigma^t f) = Td(f)$

(ii) Let further $e(f)$ be defined. Then $e(\Sigma^t f)$ is defined and $e(\Sigma^t f) = Te(f)$.

Proof. (i) is the definition of the functor T (see Remark 3.51).
For (ii) observe that by (i) we have $d(\Sigma^t f) = Td(f) = 0$ since $d(f) = 0$ and also $d(\Sigma^t \Sigma f) = Td(\Sigma f) = 0$ since $d(\Sigma f) = 0$. This means that $e(\Sigma^t f)$ is defined whenever $e(f)$ is. To calculate $e(\Sigma^t f)$ we apply $\widetilde{K}$ to the diagram from Lemma 3.26 and get the diagram

$$\begin{array}{ccccc}
\widetilde{K}(\Sigma^{t+1}X) & \xrightarrow{(j')^*} & \widetilde{K}(C'(\Sigma^t X) \cup_{\Sigma^t f} \Sigma^t Y) & \xrightarrow{(i')^*} & \widetilde{K}(\Sigma^t Y) \\
\downarrow{\scriptstyle (-\operatorname{id})^t} & & \downarrow{\scriptstyle \varphi^*} & & \downarrow{\scriptstyle \operatorname{id}} \\
\widetilde{K}(\Sigma^{t+1}X) & \xrightarrow{(\Sigma^t j)^*} & \widetilde{K}(\Sigma^t(C'X \cup_f Y)) & \xrightarrow{(\Sigma^t i)^*} & \widetilde{K}(\Sigma^t Y)
\end{array}$$

The top row is the non-zero part of the short exact sequence defining $e(\Sigma^t f)$ and since $\widetilde{K}\Sigma^t = T\widetilde{K}$ (see Remark 3.51) the bottom row is the non-zero part of the short exact sequence defining $Te(f)$. Therefore the two short exact sequences $e(\Sigma^t f)$ and $Te(f)$ are isomorphic. □

Definition 4.17
For spaces X, Y and objects $M, N \in A$ we define the stable track groups

$$\operatorname{Map}_s(X, Y) \coloneqq \varinjlim_n \, [\Sigma^{nt}X, \Sigma^{nt}Y]_*$$

where the maps $[\Sigma^{nt}X, \Sigma^{nt}Y]_* \to [\Sigma^{(n+1)t}X, \Sigma^{(n+1)t}Y]_*$ are induced by applying the suspension to both sides.
In the category $\mathscr{A}$ we further define the stabilised Hom groups

$$\operatorname{Hom}_s(M, N) \coloneqq \varinjlim_n \, \operatorname{Hom}(T^n M, T^n N)$$

and the stabilised Ext^1 groups

$$\mathrm{Ext}^1_s(M,N) \coloneqq \varinjlim_n \mathrm{Ext}^1(T^nM, T^nN).$$

In both cases the maps are induced by the functor T.

We now show how to define the stable invariants.

Construction 4.18 (d, e-invariant - stable version)
We start with the d invariant. For every $n \in \mathbb{N}$ we have a diagram of the form

$$\begin{array}{ccc} \mathrm{Map}(\Sigma^{nt}X, \Sigma^{nt}Y) & \xrightarrow{\Sigma^t} & \mathrm{Map}(\Sigma^{(n+1)t}X, \Sigma^{(n+1)t}Y) \\ \downarrow{\scriptstyle d(\cdot)} & & \downarrow{\scriptstyle d(\cdot)} \\ \mathrm{Hom}(T^n\widetilde{K}(Y), T^n\widetilde{K}(X)) & \xrightarrow{T} & \mathrm{Hom}(T^{n+1}\widetilde{K}(Y), T^{n+1}\widetilde{K}(X)) \end{array}$$

which is commutative because of Proposition 4.16 (i). By Lemma 4.5 we get an induced stable map

$$\mathrm{Map}_s(X,Y) \to \mathrm{Hom}_s(\widetilde{K}(Y), \widetilde{K}(X)).$$

We define the stable d invariant to be this homomorphism.
The same argument works for the e-invariant since Proposition Proposition 4.16 (ii) ensures that $e(\Sigma^{nt}f)$ is defined for all $n \in \mathbb{N}$ whenever $e(f)$ is. The e-invariant gives a map from $\ker(d) \cap \ker(d\Sigma) \subset \mathrm{Map}_s(X,Y)$ to $\mathrm{Ext}^1_s(\widetilde{K}(Y), \widetilde{K}(\Sigma X))$.
We will denote the stable invariants again by $d(f)$ and $e(f)$. It will be clear from the context which variant we use. For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ we will also write $d_{\mathbb{K}}$, $e_{\mathbb{K}}$ for the invariants when $\widetilde{K} = \widetilde{K}_{\mathbb{K}}$.

To obtain an invariant on the stable homotopy group π_r^s we can set $X = S^r$ and $Y = S^0$. Then $\mathrm{Map}_s(S^r, S^0) = \varinjlim_n [S^{nt+r}, S^{nt}]_* \cong \varinjlim_l [S^{l+r}, S^l]_* = \pi_r^s$ since $t\mathbb{N} \subset \mathbb{N}$ is cofinal. Using this isomorphism we view the stable d-invariant as a homomorphism

$$d\colon \pi_r^s \to \mathrm{Hom}_s(\widetilde{K}(S^0), \widetilde{K}(S^r)).$$

Actually for every choice of $l \in \mathbb{N}$ setting $X = S^{l+r}$ and $Y = S^l$ gives an isomorphism $\mathrm{Map}_s(S^{l+r}, S^l) \cong \pi_r^s$ and therefore an invariant d_l. It is clear by Bott periodicity that if $l \equiv l' \bmod t$ then the stable invariants d_l and $d_{l'}$ on π_r^s coincide.
But a priori it is unclear that the resulting invariants are independent of the congruence class of l modulo t. Before proving that it suffices to consider the case $l \equiv 0 \bmod t$ we illustrate why the invariants might differ.
Lets say $l = 0$. Then in the original definition of the d-invariant we considered elements in $\mathrm{Map}_s(S^r, S^0)$. The maps occurring in the direct limit $\varinjlim_n [S^{nt+r}, S^{nt}]_* = \mathrm{Map}_s(S^r, S^0)$ are of the form $S^{q+r} \to S^q$ with $q \equiv 0 \bmod t$ and we have $d(f) = f^*$.
Therefore calculating the d-invariant for an element $\widetilde{f} \in \pi_r^s$ takes the following form. We first choose an representative map $f\colon S^{q+r} \to S^q$ of $\widetilde{f}$.
If $q \equiv 0 \bmod t$ then we are in the original case of the definition of the d-invariant and

can use the formula $d(f) = f^*$.
But if $q \not\equiv 0 \bmod t$ then we need to choose a number $s \in \mathbb{N}$ such that $q + s \equiv 0 \bmod t$. We then consider the map $S^s f\colon S^{q+s+r} \to S^{s+l}$. Note that $S^l f$ is also a representative of $\widetilde{f}$. Since $q + s \equiv 0 \bmod t$ we have $d(\widetilde{f}) = d(S^l f) = (S^l f)^*$. Here we see that the formula $d(f) = f^*$ might not hold if $q \not\equiv 0 \bmod t$.
This is a consequence of our choices of l since the residue class of l modulo t determines for which maps $f\colon S^{q+r} \to q^l$ we are allowed to use the formula $d(f) = f^*$.
The same discussion also carries over to the e-invariant. Since the e-invariant is essentially determined by induced maps on reduced K-theory, proving that for d_l it suffices to know the case $l \equiv 0 \bmod t$ shows the same for the e-invariant.

We will now prove that if we know the invariant d_l for $l \equiv 0 \bmod t$ then we know the invariant d_l for every $l \in \mathbb{N}$.

Proposition 4.19
Let $f\colon S^a \to S^b$ be a basepoint-preserving map and let $c \in \mathbb{N}_0$ such that $c + b \equiv 0 \bmod t$. Then $f^*\colon \widetilde{K}(S^b) \to \widetilde{K}(S^a)$ is determined by $(S^c f)^*\colon \widetilde{K}(S^{c+b}) \to \widetilde{K}(S^{c+a})$.
In particular if there exists a $k \in \mathbb{N}_0$ such that $k + a \equiv 0 \bmod t$ and $(S^k f)^* = 0$ then $(S^l f)^* = 0$ for all $l \in \mathbb{N}_0$.

Proof. We choose a number e such that $e + c \equiv 0 \bmod t$. This means knowing f is the same as knowing $\Sigma^{c+e} f$ by Bott periodicity. We also choose a number e such that $c + b \equiv 0 \bmod t$.
From the naturality of the reduced external tensor product (see Section 3.2) we get a commutative diagram

$$\begin{array}{ccc} \widetilde{K}(S^e) \otimes \widetilde{K}(S^c \wedge S^a) & \longrightarrow & \widetilde{K}(S^e \wedge S^c \wedge S^a) \\ {\scriptstyle \mathrm{id}_{\widetilde{K}(S^e)} \otimes (\mathrm{id}_{S^c} \wedge f)^*} \uparrow & & \uparrow {\scriptstyle (\mathrm{id}_{S^e} \wedge \mathrm{id}_{S^c} \wedge f)^*} \\ \widetilde{K}(S^e) \otimes \widetilde{K}(S^c \wedge S^b) & \longrightarrow & \widetilde{K}(S^e \wedge S^c \wedge S^b) \end{array}$$

where the horizontal maps are the reduced external product.
Recall that the smash product with a S^n is the n-th reduced suspension Σ^n. Therefore the map on the right is $\Sigma^{e+c} f$. By the choice of c the bottom map is precisely Bott periodicity which is induced by the reduced external tensor product with a generator u of $\widetilde{K}(S^c \wedge S^b) \cong \mathbb{Z}$. This means knowing $\Sigma^{c+e} f$ is the same as knowing the composition

$$\widetilde{K}(S^e) \otimes \widetilde{K}(S^c \wedge S^b) \longrightarrow \widetilde{K}(S^e \wedge S^c \wedge S^b) \longrightarrow \widetilde{K}(S^e \wedge S^c \wedge S^a)$$

where the second map is $(\mathrm{id}_{S^e} \wedge \mathrm{id}_{S^c} \wedge f)^*$.
By commutativity of the diagram above this is the same as knowing the composition

$$\widetilde{K}(S^e) \otimes \widetilde{K}(S^c \wedge S^b) \longrightarrow \widetilde{K}(S^e) \otimes \widetilde{K}(S^c \wedge S^a) \longrightarrow \widetilde{K}(S^e \wedge S^c \wedge S^a)$$

where the first map is $\mathrm{id}_{\widetilde{K}(S^e)} \otimes (\mathrm{id}_{S^c} \wedge f)^*$. But the map $\mathrm{id}_{\widetilde{K}(S^e)} \otimes (\mathrm{id}_{S^c} \wedge f)^*$ is completely determined by the image of the generator u under f^*.

In particular if the image of u is zero then $\mathrm{id}_{\widetilde{K}(S^e)} \otimes (\mathrm{id}_{S^c} \wedge f)^*$ is zero and thus $\Sigma^{c+e} f$ is zero. □

Justified by this Proposition from now on we will set $X = S^r$ and $Y = S^0$ and use the isomorphism $\mathrm{Map}_s(S^r, S^0) \cong \pi_r^s$ to consider the d and e-invariant as invariants on π_r^s. This means to calculate the stable invariants of $\widetilde{f} \in \pi_r^s$ we will choose a representative $f \colon S^{q+r} \to S^q$ with $q \equiv 0 \bmod t$ and use the original definition of the unstable invariants to calculate it.

Remark 4.20
Also note that for a map $f \colon S^{q+r} \to S^q$ with $q \equiv 0 \bmod t$ the previous proposition says that if $d(f) = f^* = 0$ then $(\Sigma f)^* = 0$. This means that the stable e-invariant of an element $\widetilde{f} \in \pi_r^s$ is defined whenever the stable d-invariant $d(\widetilde{f})$ is zero.

The next proposition characterizes the d-invariant in certain cases.

Proposition 4.21
Let $r > 0$ and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. If we are not in the case ($\mathbb{K} = \mathbb{R}$ and $r \equiv 1, 2 \bmod 8$) then $d_{\mathbb{K}}$ is zero on π_r^s.
In particular $e_{\mathbb{K}}$ is defined in these cases.

Proof. For an element in π_r^s choose a representative $f \colon S^{q+r} \to S^q$ with $q \equiv 0$(mod 2 if $\mathbb{K} = \mathbb{C}$ and mod 8 if $\mathbb{K} = \mathbb{R}$). This means that $\widetilde{K}_{\mathbb{K}}(S^q) \cong \mathbb{Z}$.
If $\mathbb{K} = \mathbb{C}$ and $r \equiv 1 \bmod 2$ or $\mathbb{K} = \mathbb{R}$ and $r \equiv 3, 5, 6, 7 \bmod 8$ then $\widetilde{K}_{\mathbb{K}}(S^{q+r}) = 0$ and thus $d(f) = 0$.
The only cases left to consider are $\mathbb{K} = \mathbb{C}$, $r \equiv 0 \bmod 2$ and $\mathbb{K} = \mathbb{R}$, $r \equiv 0, 4 \bmod 8$. In all of these cases we have $\widetilde{K}_{\mathbb{K}}(S^{q+r}) \cong \mathbb{Z}$ and the Adams operations are given by $\psi^k(x) = k^{\frac{1}{2}(q+r)} x$ (see Proposition 3.42 and Proposition 3.46). The Adams operations on $\widetilde{K}_{\mathbb{K}}(S^q)$ are given by $\psi^k(x) = k^{\frac{1}{2}q} x$. By naturality of the Adams operations (see Theorem 3.39 *(vi)*) we know that $d(f)$ commutes with the operations. This means $d(f)(\psi^k(x)) = \psi^k(d(f)(x))$ and consequently $k^{\frac{1}{2}q} d(f)(x) = k^{\frac{1}{2}(q+r)} d(f)(x)$ since $d(f)$ is a homomorphism. If $r > 0$ the only possible homomorphism satisfying this equation is the zero homomorphism. □

4.3 Elementary description of the e-invariant

Now we will treat the special case of spheres. We will construct another invariant $\lambda(f)$ taking values in $\mathbb{Q}/\mathbb{Z}$ and prove that λ is equivalent to e (see Proposition 4.29). The invariant λ will enable us to actually compute the e-invariant.

Construction 4.22 (λ-invariant)
We start by constructing λ. As for the invariants d, e we have a real and a complex version of λ. For $n > q > 0$ let $f \colon S^{2n-1} \to S^{2q}$ be a map. If $\mathbb{K} = \mathbb{R}$ we further assume that n, q are even.
Under these assumptions $\widetilde{K}_{\mathbb{K}}(S^{2n-1})$, $\widetilde{K}_{\mathbb{K}}(S^{2q+1})$ are zero and $\widetilde{K}_{\mathbb{K}}(S^{2n})$, $\widetilde{K}_{\mathbb{K}}(S^{2q})$ are $\mathbb{Z}$.
Now consider the cofibre sequence of f

$$S^{2n-1} \xrightarrow{f} S^{2q} \xrightarrow{i} \mathbb{R}^{2n} \cup_f S^{2q} \xrightarrow{j} S^{2n} \xrightarrow{-Sf} S^{2q+1} \ .$$

Applying $\widetilde{K}_{\mathbb{K}}$ we get the short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{K}_{\mathbb{K}}(S^{2n}) & \xrightarrow{j^*} & \widetilde{K}_{\mathbb{K}}(\mathbb{R}^{2n} \cup_f S^{2q}) & \xrightarrow{i^*} & \widetilde{K}_{\mathbb{K}}(S^{2q}) & \longrightarrow & 0 \\ & & \wr\| & & & & \wr\| & & \\ & & \mathbb{Z} & & & & \mathbb{Z} & & \end{array} .$$

Up to this point this is just the construction of the e invariant. The idea now is to characterize the Adams operations on $\widetilde{K}_{\mathbb{K}}(\mathbb{R}^{2n} \cup_f S^{2q})$.
Since $\mathbb{Z}$ is free we have $\mathrm{Ext}^1(\mathbb{Z}, \mathbb{Z}) = 0$ in the category of abelian groups, thus the underlying group of $\widetilde{K}_{\mathbb{K}}(\mathbb{R}^{2n} \cup_f S^{2q})$ is $\mathbb{Z} \oplus \mathbb{Z}$.
We fix generators $\langle \widetilde{\eta} \rangle \cong \widetilde{K}_{\mathbb{K}}(S^{2n})$ and $\langle \widetilde{\xi} \rangle \cong \widetilde{K}_{\mathbb{K}}(S^{2q})$ and choose ξ, η corresponding generators in $\widetilde{K}_{\mathbb{K}}(\mathbb{R}^{2n} \cup_f S^{2q})$, that means $i^*(\xi) = \widetilde{\xi}$ and $j^*(\widetilde{\eta}) = \eta$.
Then we have $\langle \eta \rangle \oplus \langle \xi \rangle \cong \widetilde{K}_{\mathbb{K}}(\mathbb{R}^{2n} \cup_f S^{2q})$ and under this identification j^* can be viewed as the inclusion into the first component and i^* can be viewed as the projection onto the second component.
The situation may be best memorized with the following picture in mind

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{K}_{\mathbb{K}}(S^{2n}) & \xrightarrow{j^*} & \widetilde{K}_{\mathbb{K}}(\mathbb{R}^{2n} \cup_f S^{2q}) & \xrightarrow{i^*} & \widetilde{K}_{\mathbb{K}}(S^{2q}) & \longrightarrow & 0 \\ & & \widetilde{\eta} & \longmapsto & \eta & & & & \\ & & & & \xi & \longmapsto & \widetilde{\xi} & & \end{array} .$$

Note that η is determined by $\widetilde{\eta}$ since j^* is injective but we have several choices of ξ for the same $\widetilde{\xi}$. Concretely all choices are given by $\xi + N\eta$, $N \in \mathbb{Z}$, since $i^*(\eta) = 0$.
Recall the complex Chern character

$$ch_{\mathbb{C}} \colon \widetilde{K}_{\mathbb{C}}(X) \to \widetilde{H}^{\mathrm{even}}(X; \mathbb{Q})$$

and its real version $ch_{\mathbb{R}}$

$$\widetilde{K}_{\mathbb{R}}(X) \xrightarrow{c} \widetilde{K}_{\mathbb{C}}(X) \xrightarrow{ch_{\mathbb{C}}} \widetilde{H}^{\mathrm{even}}(X; \mathbb{Q})$$

from Definition 3.65.
We have a diagram

$$\begin{array}{ccccc} \widetilde{K}_{\mathbb{K}}(S^{2n}) & \xrightarrow{j^*} & \widetilde{K}_{\mathbb{K}}(\mathbb{R}^{2n} \cup_f S^{2q}) & \xrightarrow{i^*} & \widetilde{K}_{\mathbb{K}}(S^{2q}) \\ \downarrow{\scriptstyle ch_{\mathbb{K}}} & & \downarrow{\scriptstyle ch_{\mathbb{K}}} & & \downarrow{\scriptstyle ch_{\mathbb{K}}} \\ \widetilde{H}^{2n}(S^{2n}; \mathbb{Z}) & \xrightarrow{j^*} & \widetilde{H}^{\mathrm{even}}(\mathbb{R}^{2n} \cup_f S^{2q}; \mathbb{Q}) & \xrightarrow{i^*} & \widetilde{H}^{2q}(S^{2q}; \mathbb{Z}) \end{array}$$

Similar to before we choose generators $h^{2q} \in \widetilde{H}^{2q}(\mathbb{R}^{2n} \cup_f S^{2q}; \mathbb{Z})$, $h^{2n} \in \widetilde{H}^{2n}(\mathbb{R}^{2n} \cup_f S^{2q}; \mathbb{Z})$ corresponding to generators in $\widetilde{H}^{2n}(S^{2n}; \mathbb{Z})$, $\widetilde{H}^{2q}(S^{2q}; \mathbb{Z})$ under the maps i^*, j^*.
In $\widetilde{H}^{\text{even}}(\mathbb{R}^{2n} \cup_f S^{2q}; \mathbb{Z})$ we obtain the following formulas

$$\begin{aligned} ch_{\mathbb{K}}(\xi) &= a_{2q}h^{2q} + \lambda(f)a_{2n}h^{2n} \\ ch_{\mathbb{K}}(\eta) &= a_{2n}h^{2n} \end{aligned} \tag{$*$}$$

where $\lambda(f) \in \mathbb{Q}$ and $a_{2q}, a_{2n} \in \mathbb{Z}$. By a suitable choice of the generators h^{2q}, h^{2n} we may assume that if $\mathbb{K} = \mathbb{R}$ and $2r \equiv 4 \bmod 8$ then $a_{2r} = 2$ and in all other cases $a_{2r} = 1$. This is a consequence of Remark 3.45 in combination with Remark 3.63.
If we take a different ξ, lets say $\xi + N\eta$ then λ is replaced by $\lambda + N$, since $ch_{\mathbb{K}}(\xi + N\eta) = ch_{\mathbb{K}}(\xi) + ch_{\mathbb{K}}(\eta) = a_{2q}h^{2q} + (\lambda(f) + N)a_{2n}h^{2n}$.
Therefore $\lambda(f)$ is well defined as an element in $\mathbb{Q}/\mathbb{Z}$.

The next Proposition calculates the Adams operations on $\widetilde{K}_{\mathbb{K}}(\mathbb{R}^{2n} \cup_f S^{2q})$ in terms of $\lambda = \lambda(f)$. This also means that if we know λ we know $e(f)$.

Proposition 4.23
The Adams operations on $\widetilde{K}_{\mathbb{K}}(\mathbb{R}^{2n} \cup_f S^{2q})$ are given by

$$\begin{aligned} \psi^k(\xi) &= k^q\xi + \lambda(k^n - k^q)\eta, \\ \psi^k(\eta) &= k^n\eta. \end{aligned}$$

Proof. Because of $\widetilde{K}_{\mathbb{K}}(\mathbb{R}^{2n} \cup_f S^{2q}) \cong \widetilde{K}_{\mathbb{K}}(S^{2n}) \oplus \widetilde{K}_{\mathbb{K}}(S^{2q})$ and Proposition 3.62 (respectively Remark 3.66) the image of the Chern character

$$ch_{\mathbb{K}} \colon \widetilde{K}_{\mathbb{K}}(\mathbb{R}^{2n} \cup_f S^{2q}) \to \widetilde{H}^{\text{even}}(\mathbb{R}^{2n} \cup_f S^{2q}; \mathbb{Q})$$

is actually contained in $\widetilde{H}^{2n}(S^{2n}; \mathbb{Z}) \oplus \widetilde{H}^{2q}(S^{2q}; \mathbb{Z})$ and the Chern character is given as

$$\mathrm{ch}_{\mathbb{K}} \colon \widetilde{K}_{\mathbb{K}}(S^{2n}) \oplus \widetilde{K}_{\mathbb{K}}(S^{2q}) \to \widetilde{H}^{2n}(S^{2n}; \mathbb{Z}) \oplus \widetilde{H}^{2q}(S^{2q}; \mathbb{Z}), (x, y) \mapsto (\mathrm{ch}_{\mathbb{K}}(x), \mathrm{ch}_{\mathbb{K}}(y)).$$

Note that the Chern characters on the right hand side are the Chern character on $\widetilde{K}_{\mathbb{K}}(S^{2n})$ respectively $\widetilde{K}_{\mathbb{K}}(S^{2q})$ which are injective by Proposition 3.62 and Remark 3.66. Thus the Chern character on $\widetilde{K}_{\mathbb{K}}(S^{2n}) \oplus \widetilde{K}_{\mathbb{K}}(S^{2q})$ is injective.
Therefore to prove the Proposition it suffices to show that the equations hold if we apply $\mathrm{ch}_{\mathbb{K}}$ to both sides.
Using $(ch_{\mathbb{K}} \circ \psi^k)(\eta) = k^n ch_{\mathbb{K}}(\eta)$ (see Proposition 3.64) we directly obtain the second equation.
For the first equation observe that ξ might have a non zero part in $\widetilde{K}_{\mathbb{K}}(S^{2n})$. Let $\xi = \xi_{2n} + \xi_{2q}$ where ξ_{2n} is a generator in $\widetilde{K}_{\mathbb{K}}(S^{2n})$ and $\xi_{2q} \in \widetilde{K}_{\mathbb{K}}(S^{2q})$. As above we get $ch_{\mathbb{K}}(\xi_{2q}) = a_{2q}h^{2q}$ and $ch_{\mathbb{K}}(\xi_{2n}) = \lambda a_{2n}h^{2h}$. Again using Proposition 3.64 and Equation ($*$) we calculate

$$\begin{aligned} (ch_{\mathbb{K}} \circ \psi^k)(\xi) &= ch_{\mathbb{K}}(\psi^k(\xi_{2q})) + ch_{\mathbb{K}}(\psi^k(\xi_{2n})) \\ &= k^q ch_{\mathbb{K}}(\xi_{2q}) + k^n ch_{\mathbb{K}}(\xi_{2n}) \\ &= k^q a_{2q}h^{2q} + \lambda k^n a_{2n}h^{2n} \end{aligned}$$

and

$$\begin{aligned}ch_{\mathbb{K}}(k^q\xi+\lambda(f)(k^n-k^q)\eta) &= k^q ch_{\mathbb{K}}(\xi)+\lambda(k^n-k^q)ch_{\mathbb{K}}(\eta)\\ &= k^q a_{2q}h^{2q}+\lambda k^q a_{2n}h^{2n}+\lambda(k^n-k^q)a_{2n}h^{2n}\\ &= k^q a_{2q}h^{2q}+\lambda k^n a_{2n}h^{2n}.\end{aligned}$$

Thus $(ch_{\mathbb{K}}(\psi^k))(\xi)=\mathrm{ch}(k^q\xi+\lambda(f)(k^n-k^q)\eta)$ proving the first equation. □

Since $\lambda(k^n-k^q)$ must be an integer for all $k\in\mathbb{Z}$ we get the following Corollary.

Corollary 4.24
λ is of the form $\frac{z}{h}$, where $z\in\mathbb{Z}$ and h is the highest common factor of the numbers (k^n-k^q) with $k\in\mathbb{Z}$.

We now show that the e-invariant and λ-invariant are equivalent. To shorten notation we set $M:=\widetilde{K}_{\mathbb{K}}(S^{2q})\in\mathscr{A}$ and $N:=\widetilde{K}_{\mathbb{K}}(S^{2n})\in\mathscr{A}$. Recall that $n>q>0$ and n,q are even if $\mathbb{K}=\mathbb{R}$.

Proposition 4.25
There exists a injective homomorphism

$$\theta\colon\ \mathrm{Ext}^1(M,N)\to\mathbb{Q}/\mathbb{Z}$$

with the property that for every $f\colon S^{2n-1}\to S^{2q}$ we have

$$\theta(e(f))=\lambda(f).$$

The image of θ is precisely the residue classes of the elements $\frac{z}{h}$ where $z\in\mathbb{Z}$ and h as in the previous corollary.

Proof. We start by defining the map θ. For this we adapt the construction of the λ invariant to the general case of extensions. Let

$$0\longrightarrow N\longrightarrow E\longrightarrow M\longrightarrow 0$$

be an element of $\mathrm{Ext}^1(M,N)$. We choose elements $\xi,\eta\in E$ such that ξ projects to the generator in M and η is the image of the generator in N. Again the operations in E are given by

$$\begin{aligned}\psi^k(\xi) &= k^q\xi+c(k)\eta\\ \psi^k(\eta) &= k^n\eta.\end{aligned}$$

<u>Claim 1:</u> The $c(k)$ are of the form $\lambda(k^n-k^q)$ for some $\lambda\in\mathbb{Q}$.

By definition the operations ψ^k satisfy $\psi^k\psi^l=\psi^{kl}$ for all $k,l\in\mathbb{Z}$. Calculating

$$\begin{aligned}\psi^k(\psi^l(\xi)) &= \psi^k(l^q\xi+c(l)\eta)\\ &= l^q\psi^k(\xi)+c(l)\psi^k(\eta)\\ &= l^qk^q\xi+(c(k)l^q+c(l)k^n)\eta\end{aligned}$$

we obtain $c(kl) = c(k)l^q + c(l)k^n$. Doing the same calculation for $\psi^l\psi^k = \psi^{kl}$ we get $c(kl) = c(l)k^q + c(k)l^n$.
Choosing $l \in \mathbb{Z}$ such that $l^n - l^q \neq 0$ these two equation yield

$$c(k) = \tfrac{c(l)(k^n - k^q)}{l^n - l^q}.$$

for all $k \in \mathbb{Z}$. Note that $\frac{c(l)}{l^n - l^q}$ is actually independent of l (whenever $l^n - l^q \neq 0$). Setting $\lambda = \frac{c(l)}{l^n - l^q} \in \mathbb{Q}$ shows Claim 1.

Again $c(k)$ depends on the choice of ξ. If we choose $\xi + N\eta$ as a generator the calculation

$$\begin{aligned}\psi^k(\xi + N\eta) &= k^q\xi + c(k)\eta + Nk^n\eta \\ &= k^q(\xi + N\eta) + (c(k) + N(k^n - k^q))\eta\end{aligned}$$

shows that $c(k)$ gets replaced by $c(k) + N(k^n - k^q)$ and therefore λ gets replaced by $\lambda + N$. Thus λ is well defined as an element in $\mathbb{Q}/\mathbb{Z}$. We define the image of our extension under θ to be λ.
The same argument that we used to show Corollary 4.24 also applies here, so λ is of the desired form.
When $c(k)$ is zero for all k we get the trivial extension. This can only happen if λ is zero which means that θ is injective.

Claim 2: θ is a homomorphism.

Let

$$0 \longrightarrow N \xrightarrow{\iota_1} E_1 \xrightarrow{\pi_1} M \longrightarrow 0$$

and

$$0 \longrightarrow N \xrightarrow{\iota_2} E_2 \xrightarrow{\pi_2} M \longrightarrow 0$$

be two elements of $\mathrm{Ext}^1(M, N)$. Writing out the definition of the Baer sum from Construction 3.55 we see that the sum of these extensions is the short exact sequence

$$0 \longrightarrow N \xrightarrow{\iota} D \xrightarrow{\pi} M \longrightarrow 0$$

where the underlying group of D is the quotient of

$$\{(e_1, e_2, m, n) \mid e_i \in E_i, \pi_1(e_1) = \pi_2(e_2) = m, n \in N\}$$

by the subgroup

$$\{(\iota_1(n), \iota_2(n'), 0, -(n + n')) \mid n, n' \in N\}.$$

The map ι is given by $n \mapsto (0, 0, 0, n)$ and π is given by

$$(e_1, e_2, m, n) \mapsto m = \pi_1(e_1) = \pi_2(e_2).$$

We choose the elements $(0,0,0,\iota_1^{-1}(\eta_1)) = (0,0,0,\iota_2^{-1}(\eta_2))$ and $(\xi_1, \xi_2, \pi_1(\xi_1), 0) = (\xi_1, \xi_2, \pi_2(\xi_2), 0)$ in D where $\eta_i, \xi_i \in E_i$ are the above chosen elements. The operations on D are given component wise since D is a quotient of a subgroup of the product $E_1 \oplus E_2 \oplus M \oplus N$. We can now calculate the operations in D. Let $c_i(k)$ determine the operations on E_i. Then we get

$$\begin{aligned}\psi^k(0,0,0,\iota_1^{-1}(\eta_1)) &= (0,0,0,\psi^k(\iota_1^{-1}(\eta_1))) \\ &= k^n(0,0,0,\iota_1^{-1}(\eta_1))\end{aligned}$$

and

$$\begin{aligned}\psi^k(\xi_1, \xi_2, \pi_1(\xi_1), 0) &= (\psi^k(\xi_1), \psi^k(\xi_2), \psi^k(\pi_1(\xi_1)), \psi^k(0)) \\ &= (k^q\xi_1 + c_1(k)\eta, k^q\xi_2 + c_2(k)\eta, k^q\pi_1(\xi_1), 0) \\ &= k^q(\xi_1, \xi_2, \pi_1(\xi_1), 0) + (c_1(k)\eta, c_2(k)\eta, 0, 0) \\ &= k^q(\xi_1, \xi_2, \pi_1(\xi_1), 0) + (c_1(k) + c_2(k))(0,0,0,\iota_1^{-1}(\eta_1).\end{aligned}$$

For the last equation we used that $(-c_1(k)\eta, -c_2(k)\eta, 0, c_1(k)\iota_1^{-1}(\eta_1) + c_2(k)\iota_1^{-1}(\eta_1)) = 0$ in D.
This shows that the operations in D are determined by $c_1(k) + c_2(k)$ and thus θ is a homomorphism.

To determine the image of θ we prove the converse. Let $\frac{z}{h} \in \mathbb{Q}/\mathbb{Z}$ have the form stated in the Theorem. As a group let E be $M \oplus N$. We define operations ψ^k on E via the obvious formulas namely $\psi^k(\xi) = k^q\xi + \frac{z}{h}(k^n - k^q)\eta$ and $\psi^k(\eta) = k^n\eta$. These operations satisfy the axioms turning E into an object in $\mathscr{A}$. We get the short exact sequence $0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$ in $\mathscr{A}$ which is by construction the desired preimage of $\frac{z}{h}$ under θ. $\square$

Before we treat the stable case we define the Bernoulli numbers.

Definition 4.26 (Bernoulli numbers and the function m)
Consider the function $f\colon \mathbb{R} \to \mathbb{R}, x \mapsto \frac{x}{e^x - 1}$ (for $x = 0$ we define $f(0) = \frac{0}{0}$ to be 1). The numbers β_k are defined via the power series representation of f

$$f(x) = \frac{x}{e^x-1} = \sum_{k=0}^{\infty} \frac{\beta_k}{k!} x^k.$$

For $s \in \mathbb{N}_0$ we define the s-th Bernoulli number B_s to be

$$B_s := (-1)^{1-s}\beta_{2s}.$$

We further define the function $m(2s)$ to be the denominator of $(-1)^{s-1}\frac{B_s}{4s}$ when expressed in its reduced form.

Remark 4.27
We chose to use the same convention for the indexing of the Bernoulli numbers as Adams did in his paper [Ada66]. Nowadays people usually define the s-th Bernoulli number to be β_s.

Lemma 4.28
For even numbers n, q we define $h(n,q)$ to be the highest common factor of the numbers $(k^n - k^q)$ for $k \in \mathbb{Z}$. If both n and q tend to infinity such that they are always even and their difference $t \coloneqq n - q$ is constant then $h(n,q)$ attains the constant value $m(t)$.

Proof. We have $(k^n - k^q) = k^q(k^t - 1)$. With $f(k) \coloneqq q$ for $k \in \mathbb{Z}$ and $t \coloneqq n - q$ [Ada65] Theorem 2.7 yields that for q large enough we have $h(n,q) = m(t)$. Actually the theorem only states the existence of a function f but the last part of the proof gives a condition on the function f which is satisfied for our f if q is large. $\square$

We now generalise Proposition 4.25 to the stable case.

Proposition 4.29
There exists a injective homomorphism

$$\theta_s \colon \operatorname{Ext}^1_s(M,N) \to \mathbb{Q}/\mathbb{Z}$$

with the property that for every $f \colon S^{2n-1} \to S^{2q}$ we have

$$\theta_s(e(f)) = \lambda(f).$$

The image of θ_s is precisely the residue classes of the elements $\frac{z}{m(t)}$ where $z \in \mathbb{Z}$, $t \coloneqq n - q$ and m is the function defined above (recall that both n and q are even so t is even too). This means that $\operatorname{Ext}^1_s(M,N)$ is the cyclic group of order $m(t)$.

Proof of Proposition 4.29. Since for every object P in $\mathscr{A}$ the Adams operations on TP are $k^{\frac{1}{2}r} \in \mathbb{Z}$ times the operations on P (see Remark 3.51) we have the commutative diagram

$$\begin{array}{ccc} \operatorname{Ext}^1(M,N) & \xrightarrow{\quad T \quad} & \operatorname{Ext}^1(TM,TN) \\ & {\scriptstyle\theta}\searrow \quad \swarrow{\scriptstyle\theta} & \\ & \mathbb{Q}/\mathbb{Z} & \end{array} .$$

Therefore we get a injective map $\theta_s \colon \operatorname{Ext}^1_s(M,N) \to \mathbb{Q}/\mathbb{Z}$ with $\theta_s(e(f)) = \lambda(f)$. The number h from Proposition 4.25 depends on n and q. For the stable case both n and q go to infinity (in steps of 4 to be precise) but their difference $t = n - q$ is constant. Therefore Lemma 4.28 yields that the corresponding $h = h(n,q)$ from Proposition 4.25 attains the constant value $m(t)$. $\square$

4.4 Proof of the main theorem in the case $r \equiv 0, 1 \bmod 8$

In this section we use the previous results to finally prove the main theorem.

Theorem 4.30
Let $r \equiv 0, 1 \bmod 8$ and $r > 1$. Then the J-homomorphism $J \colon \pi_r(\mathrm{SO}) \to \pi_r^s$ is injective and its image is a direct summand in π_r^s which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

The main theorem is a consequence of the following theorem which will take the remaining section to prove.

Theorem 4.31
Let $r \equiv 0, 1 \bmod 8$*,* $r > 1$*. Then*

$$e_{\mathbb{R}}J\colon \pi_r(SO) \to \mathbb{Z}/2\mathbb{Z}$$

is an isomorphism.

Before proving this theorem let us first show how to deduce the main theorem from this result.

proof of Theorem 4.30. Since $e_{\mathbb{R}}J$ is an isomorphism J is injective and $\operatorname{im} J$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ via $e_{\mathbb{R}}$. What is left to show is that $\operatorname{im} J$ is a direct summand in π_r^s. We have short exact sequence

$$0 \longrightarrow \ker(e_{\mathbb{R}}) \longrightarrow \ker(d_{\mathbb{R}}) \xrightarrow{e_{\mathbb{R}}} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

where the first map is the inclusion. Let $\phi := e_{\mathbb{R}}J$ be the isomorphism from the previous theorem, then $J \circ \phi^{-1}$ is a right split, since $e_{\mathbb{R}} \circ J \circ \phi^{-1} = \phi \circ \phi^{-1} = \mathrm{id}$. Thus we get $\ker(d_{\mathbb{R}}) = \ker(e_{\mathbb{R}}) \oplus \mathbb{Z}/2\mathbb{Z}$ by the splitting lemma. Here $\mathbb{Z}/2\mathbb{Z} \subset \ker(d_{\mathbb{R}})$ is identified with $(J \circ \phi^{-1})(\mathbb{Z}/2\mathbb{Z}) = \operatorname{im} J$ since ϕ is an isomorphism. Therefore $\operatorname{im} J$ is a direct summand in $\ker(d_{\mathbb{R}})$.
If $r \equiv 0 \bmod 8$ then $\ker(d_{\mathbb{R}}) = \pi_r^s$ by Proposition 4.21 and we are done. In the case $r \equiv 1 \bmod 8$ there is a decomposition of the stable homotopy group $\pi_r^s \cong \langle \mu_r \rangle \oplus \ker d_{\mathbb{R}}$ where μ_r is a certain element of order 2 in π_r^s. Constructing the element μ_r and proving that it has the desired property uses theory and techniques like Massey products and Toda brackets not introduced in this thesis. We point the interested reader towards the paper by Adams [Ada66]. The result there is stated in Theorem 7.2. □

Proving Theorem 4.31 will take the remaining section. So lets break down the major steps of this proof.
We start with Proposition 4.32 which computes the Ext group in which $e(f)$ takes values in. After that we will show that the e-invariant is defined on the image of the J-homomorphism by showing that the d-invariant is zero on the image of the J-homomorphism (see Theorem 4.35). For this we prove the technical Lemma 4.34 which allows us to make use of the Thom isomorphism. Finally we then prove Theorem 4.31.

As mentioned above we start by calculating the Ext in which the e-invariant takes values. We therefore consider maps $f\colon S^{2q+r} \to S^{2q}$ where $r \equiv 0, 1 \bmod 8$ and $2q \equiv 0 \bmod 8$. Then we have $\widetilde{K}_{\mathbb{R}}(S^{2q}) \cong \mathbb{Z}$ and $\widetilde{K}_{\mathbb{R}}(S^{2q+r+1}) \cong \mathbb{Z}/2\mathbb{Z}$. The operations on $M := \widetilde{K}_{\mathbb{R}}(S^{2q}) \in \mathscr{A}$ are given by $\psi^k(x) = k^q x$ and in $N' := \widetilde{K}_{\mathbb{R}}(S^{2q+r+1}) \in \mathscr{A}$ they are $\psi^k x = \begin{Bmatrix} x, & k \text{ odd} \\ 0, & k \text{ even} \end{Bmatrix}$ (see Proposition 3.48).
We now have $e_{\mathbb{R}}(f) \in \operatorname{Ext}^1_s(M, N')$.

In Proposition 4.29 we have already determined a Ext^1_s group. We cannot apply this result directly since N' is not infinite cyclic, but we use our previous calculation to express $\mathrm{Ext}^1_s(M,N')$ in terms of $\mathrm{Ext}^1_s(M,N)$.
We use that N' may also be described as the quotient object $N/2N \in \mathscr{A}$ where $N \cong \mathbb{Z}$ as a group and the operations are $\psi^k(x) = k^n x$ for a $n \in \mathbb{N}$.

Proposition 4.32
Let $\mathbb{K} = \mathbb{R}$ and M, N, N' be as above. The quotient map $N \to N'$ induces an isomorphism

$$\mathrm{Ext}^1_s(M,N)/2\,\mathrm{Ext}^1_s(M,N) \to \mathrm{Ext}^1_s(M,N').$$

Using Proposition 4.29 this means that $\mathrm{Ext}^1_s(M,N')$ is the group of elements $\frac{z}{m(t)}$ mod (1 and $\frac{2}{m(t)}$), where $z \in \mathbb{Z}$ and $t = n - q$ i.e. $\mathrm{Ext}^1_s(M,N')$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Proof. The short exact sequence of groups

$$0 \longrightarrow N \xrightarrow{\cdot 2} N \longrightarrow N' \longrightarrow 0$$

induced an exact sequence (see Proposition 3.56)

$$\mathrm{Ext}^1(M,N) \xrightarrow{\cdot 2} \mathrm{Ext}^1(M,N) \longrightarrow \mathrm{Ext}^1(M,N') \ .$$

The functor T is exact so we get an exact sequence

$$\mathrm{Ext}^1_s(M,N) \xrightarrow{\cdot 2} \mathrm{Ext}^1_s(M,N) \longrightarrow \mathrm{Ext}^1_s(M,N') \ .$$

Since the first map is injective and its image is $2\,\mathrm{Ext}^1_s(M,N)$ we only need to show that the second map is surjective.
Let

$$0 \longrightarrow N' \longrightarrow E \longrightarrow M \longrightarrow 0$$

be an element in $\mathrm{Ext}^1_s(M,N')$. As a group E is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$, since $\mathbb{Z}$ is free and therefore $\mathrm{Ext}^1(M,N') = 0$ in the category of abelian groups.
We use the description of the e-invariant from Section 4.3. Let η, ξ be elements in E corresponding to the generators in N' and M and let $c(k) \in \mathbb{Z}/2\mathbb{Z}$ determine the operation ψ^k on E via the equations

$$\psi^k(\xi) = k^q \xi + c(k)\eta$$
$$\psi^k(\eta) = k^n \eta.$$

We now want to find numbers $\widetilde{c}(k) \in \mathbb{Z}$ such that their residue classes modulo 2 are $c(k)$ and they define operations on $N \oplus M$.
By property (iii) from Definition 3.50 we know that the value of $c(k)$ is periodic with period 2^e for some $e \in \mathbb{Z}$. Due to Bott periodicity we can take $e = 3$. Then $c(k)$ is 8-periodic.
Now let G be the multiplicative group of integers prime to 8, i.e. the invertible elements of $\mathbb{Z}/8\mathbb{Z}$. G is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ where one summand is generated by -1 and

the other by 3.
We define $l := 3$ and repeat the calculations we did to prove Claim 1 in the proof of Proposition 4.29 to get

$$c(kl) = l^q c(k) + k^n c(l) \bmod 2$$

for all $k \in G$ and therefore

$$c(l^r) = \frac{l^{rn} - l^{rq}}{l^n - l^q} c(l) \bmod 2$$

for all $r \in \mathbb{Z}$.
Since we are in the real case we have $\psi^k = \psi^{-k}$ which means

$$c(-l^r) = \frac{l^{rn} - l^{rq}}{l^n - l^q} c(l) \bmod 2$$

for all $r \in \mathbb{Z}$.
All together we have shown

$$c(k) = (k^n - k^q) \frac{c(l)}{l^n - l^q} \bmod 2$$

for all $k \in G$ and due to the periodicity for all integers k prime to 8, i.e. all odd numbers.
We may also write

$$c(k) = (k^n - k^q)\lambda \bmod 2$$

where the denominator of λ is a power of 2, since every odd number is invertible in $\mathbb{Z}/2\mathbb{Z}$.
What we would like to do now is taking the right hand side as $\widetilde{c}(k)$. For this to work we need to ensure two things, first the right hand side is an integer and second the equation is true for all $k \in \mathbb{Z}$.
For this observe that the functor T does not change the class of the extension we started with. This means if we apply t times the functor T the equation

$$k^{4t} c(k) = (k^{n+4t} - k^{q+4t})\lambda \bmod 2$$

still describes $c(k)$ for all odd k.
Choosing a large enough t we can ensure that for all k even

- $k^{4t} c(k) = 0 \bmod 2$
- $(k^{n+4t} - k^{q+4t})\lambda$ is an even integer and thus zero modulo 2

The second point is possible because the denominator of λ is a power of 2.
Now we can define

$$\widetilde{c}(k) := (k^{n+4t} - k^{q+4t})\lambda \text{ for all } k \in \mathbb{Z}.$$

In the previous section we have seen that $c(k)$'s of this form indeed define operations on $N \oplus M$. By construction the residue class of $\widetilde{c}(k)$ modulo 2 is $c(k)$ which shows that the map $\mathrm{Ext}^1_s(M, N) \to \mathrm{Ext}^1_s(M, N')$ is surjective. □

Remark 4.33

(i) It is also possible to extend the above proof to the case where 2 is replaced by a number $\nu \in \mathbb{N}$ and N' is the quotient $N/\nu N \in A$ (see [Ada66] Proposition 9.1).

(ii) From the equation $c(kl) = l^q c(k) + k^n c(l) \bmod 2$ we see that $c(\cdot)$ defines a group homomorphism from G to $(\mathbb{Z}/2\mathbb{Z}, +)$. Therefore the $c(k)$ are determined by the image of the generator $l = 3$. For its image we have two choices. Sending l to 0 we get $c(k) = 0$ for all $k \in \mathbb{Z}$. Sending l to 1 we get $c(k) = \begin{Bmatrix} 0, & k \equiv \pm 1 \bmod 8 \\ 1, & k \equiv \pm 3 \bmod 8 \end{Bmatrix}$.
The later describes the non-zero element in $\operatorname{Ext}^1_s(M, N') \cong \mathbb{Z}/2\mathbb{Z}$.

Next we discuss in which cases $e_{\mathbb{R}} J$ is defined.
If $r \equiv 0 \bmod 8$ then Proposition 4.21 tells us that $d_{\mathbb{R}}$ is zero, therefore $d_{\mathbb{R}} J$ is zero and $e_{\mathbb{R}}$ is defined on the image of the J-homomorphism.
For $r \equiv 1 \bmod 8$ the $e_{\mathbb{R}}$ is also defined on the image of the J-homomorphism but the proof requires more work.
Let $\phi\colon S^r \to \mathrm{SO}(q)$ be a representative of an element in $\pi_r(\mathrm{SO})$. We consider two constructions using the map ϕ.
For the first we apply the J-homomorphism to ϕ. From $J\phi\colon S^{q+r} \to S^r$ we get the mapping cone $X \coloneqq \mathbb{R}^{q+r+1} \cup_{J\phi} S^q$. This space appears in the definition of $e_{\mathbb{R}}(J\phi)$.
For the second construction we view the map ϕ as a clutching function which corresponds to a real vector bundle E of rank q over S^{r+1} (see Theorem 2.40). This vector bundle has a Thom space $T(E)$.
The next Lemma says that X and $T(E)$ are homotopy equivalent and therefore

$$\widetilde{K}_{\mathbb{K}}(X) \cong \widetilde{K}_{\mathbb{K}}(T(E)).$$

The advantage of this identification is that for the later group we can make use of the Thom isomorphism (if $q \equiv 0 \bmod 8$).

Lemma 4.34
Let $\phi \in \pi_r(\mathrm{SO}(q))$ and let E be the vector bundle obtained by using ϕ as a clutching function.
Then the mapping cone $\mathbb{R}^{q+r+1} \cup_{J\phi} S^q$ is homotopy equivalent to the Thom space $T(E)$.
Therefore we have $\widetilde{K}_{\mathbb{R}}(\mathbb{R}^{q+r+1} \cup_{J\phi} S^q) \cong \widetilde{K}_{\mathbb{R}}(T(E))$.

Proof. The idea is to formulate both constructions in such a way that one sees that the results are homotopy equivalent.
We start with the construction using ϕ as a clutching function obtaining the vector bundle E. We adapt the clutching construction a little bit. Instead of taking trivial bundles of rank q on the two hemispheres of S^{r+1} we shrink one hemisphere to a point. The clutching construction now is taking a trivial bundle on D^{r+1} and glueing the fibres over points in $\partial D^{r+1} = S^r$ to an extra copy of $\mathbb{R}^q$ using ϕ. We take $(\mathbb{R}^{r+1} \times \mathbb{R}^q) \sqcup \mathbb{R}^q$ and glue a point $(x, y) \in S^r \times \mathbb{R}^q$ to the point $\phi(x)y$ in the extra copy E^q.
This has indeed the homotopy type of E since $\mathbb{R}^{r+1}$ deformation retracts onto D^{r+1}.

We obtain the Thom space $T(E)$ by identifying $(\mathbb{R}^{r+1} \times S^{q-1}) \sqcup S^{q-1}$ (or rather this subset after the glueing) to a point.
We now show how to construct the mapping cone $\mathbb{R}^{q+r+1} \cup_{J\phi} S^q$. The map $J\phi\colon S^{q+r} \to S^q$ may be obtained in the following way (compare to the description of J via the Hopf construction). We view S^{q+r} as the boundary of $\mathbb{R}^{r+1} \times \mathbb{R}^q$ and map $S^r \times S^{q-1}$ to S^{q-1} via $(J\phi)(x, y) = \phi(x)y$. We can extend $J\phi$ to a map $S^r \times \mathbb{R}^q$ to the upper hemisphere $\mathbb{R}^q_+$ of S^q again via $(J\phi)(x, y) = \phi(x)y$. Finally we extend $J\phi$ to a map from $\mathbb{R}^{r+1} \times S^{q-1}$ to the lower hemisphere $\mathbb{R}^q_-$ of S^q.
Because of $\partial(\mathbb{R}^{r+1} \times \mathbb{R}^q) = (\partial\mathbb{R}^{r+1} \times \mathbb{R}^q) \cup (\mathbb{R}^{r+1} \times \partial\mathbb{R}^q) = S^r \times \mathbb{R}^q \cup \mathbb{R}^{r+1} \times S^{q-1}$ we have defined a map from $S^{r+1} \times S^q$ to S^q and obtain the mapping cone
$C(S^{r+1} \times S^q) \cup_{J\phi} S^q = (\mathbb{R}^{r+1} \times \mathbb{R}^q) \cup_{J\phi} S^q$. We do not change the homotopy type if we identify the lower hemisphere $\mathbb{R}^q_-$ to a point (that is the reason why we did not specify the second extension of $J\phi$).
Comparing the two constructions we see that we obtained homotopy equivalent spaces. □

We can now show that $d_{\mathbb{R}}$ is zero on the image of the J-homomorphism.

Theorem 4.35
Let $r \equiv 1 \bmod 8$, $r > 1$. Then $d_{\mathbb{R}}J$ is zero on $\pi_r(\mathrm{SO})$.

Proof. Let $\beta\colon S^r \to \mathrm{SO}(q)$ be a representative of a generator in $\pi_r(\mathrm{SO})$. By possibly taking another representative we can assure that $q = 8n$ for some $n \in \mathbb{N}$. Interpreting β as a clutching function we also view β as a real vector bundle over S^{r+1} of rank $8n$. We choose β in such a way that its class in $\widetilde{K}_{\mathbb{R}}(S^{r+1})$ is a generator. We equip β with a Spin$(8n)$-structure. This is possible by Proposition 2.51, since $r + 1 \geq 3$.
We apply the J-homomorphism to β and from the cofibre sequence of $J\beta$ we obtain the exact sequence

$$\widetilde{K}_{\mathbb{R}}(\mathbb{R}^{q+r+1} \cup_{J\beta} S^q) \xrightarrow{i^*} \widetilde{K}_{\mathbb{R}}(S^q) \xrightarrow{(J\beta)^*} \widetilde{K}_{\mathbb{R}}(S^{q+r}) \ .$$

For the Spin$(8n)$-bundle $\beta\colon E \to S^{r+1}$ we have the Thom isomorphism
$\varphi_K\colon K_{\mathbb{R}}(S^{r+1}) \to \widetilde{K}_{\mathbb{R}}(T(E))$. The Thom class $U_E \in \widetilde{K}_{\mathbb{R}}(T(E))$ is given as $\varphi_k(1)$. Using the isomorphism from Lemma 4.34 we consider $\varphi_K(1)$ as an element in $\widetilde{K}_{\mathbb{R}}(\mathbb{R}^{q+r+1} \cup_{J\beta} S^q)$. Since the sequence above is exact we get $(J\beta)^*(i^*(\varphi_K(1))) = 0$.
We now show that $i^*(\varphi_K(1))$ is a generator of $\widetilde{K}_{\mathbb{R}}(S^q)$. Then $(J\beta)^*$ is zero since it vanishes on a generator.
Running through the proof of Lemma 4.34 we see that under the identification
$\widetilde{K}_{\mathbb{R}}(\mathbb{R}^{q+r+1} \cup_{J\beta} S^q) \cong \widetilde{K}_{\mathbb{R}}(T(E))$ the map $\iota^*\colon \widetilde{K}_{\mathbb{R}}(\mathbb{R}^{q+r+1} \cup_{J\beta} S^q) \to \widetilde{K}_{\mathbb{R}}(S^q)$ corresponds to the restriction to a fibre of $T(E)$. This may be expressed by the commutative diagram

$$\begin{array}{ccc} \widetilde{K}_{\mathbb{R}}(\mathbb{R}^{q+r+1} \cup_{J\beta} S^q) & \xrightarrow{\iota^*} & \widetilde{K}_{\mathbb{R}}(S^q) \\ \wr\Vert & \nearrow & \\ \widetilde{K}_{\mathbb{R}}(T(E)) & & \end{array}$$

where the isomorphism on the left is the one from Lemma 4.34 and the map on the right is induced by the restriction to a fibre of $T(E)$.
By Remark 3.70 we get that $i^*(\varphi_K(1))$ is a generator of $\widetilde{K}_{\mathbb{R}}(S^q)$. □

Note that by Remark 4.20 we have shown that $e_{\mathbb{R}}$ is always defined on the image of the J-homomorpism.
Now we have the preliminary results to prove the main theorem.

Proof of Theorem 4.31. By the same reasoning as above let β be a real vector bundle of rank $q = 8n$ over S^{r+1} such that β carries a Spin(q)-structure and is a representative of a generator of $\widetilde{K}_{\mathbb{R}}(S^{r+1})$. We again obtain an element $\varphi_K(1)$ in $\widetilde{K}_{\mathbb{R}}(\mathbb{R}^{q+r+1} \cup_{J\beta} S^q)$.
Because of the previous Theorem $e_{\mathbb{R}}J\beta$ is defined. We now show that $e_{\mathbb{R}}J\beta$ is not zero.
The short exact sequence defining $e_{\mathbb{R}}J\beta$ is

$$0 \longrightarrow \widetilde{K}_{\mathbb{R}}(S^{r+q+1}) \xrightarrow{j^*} \widetilde{K}_{\mathbb{R}}(T(E)) \xrightarrow{i^*} \widetilde{K}_{\mathbb{R}}(S^q) \longrightarrow 0 \ .$$

Here we again used Lemma 4.34 to identified $\widetilde{K}_{\mathbb{R}}(\mathbb{R}^{q+r+1} \cup_{J\beta} S^q)$ with $\widetilde{K}_{\mathbb{R}}(T(E))$.
In the previous proof we have seen that $i^*(\varphi_K(1))$ is a generator of $\widetilde{K}_{\mathbb{R}}(S^q)$. Therefore $\varphi_K(1)$ qualifies as the element ξ from the description of the e-invariant in Section 4.3.
Let $\widetilde{\eta}$ be the non-zero element in $\widetilde{K}_{\mathbb{R}}(S^{q+r+1}) \cong \mathbb{Z}/2\mathbb{Z}$. Because of the injectivity of j^* the element $j^*(\widetilde{\eta})$ is not zero in $\widetilde{K}_{\mathbb{R}}(T(E))$.
We also have $\widetilde{K}_{\mathbb{R}}(T(E)) \cong K_{\mathbb{R}}(S^{r+1}) \cong \widetilde{K}_{\mathbb{R}}(S^{r+1}) \oplus \mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$. The element $\widetilde{\eta}$ has order two which implies $j^*(\widetilde{\eta}) = \varphi_K(\beta) \in \widetilde{K}_{\mathbb{R}}(T(E))$, since this is the only element of order 2 in $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$.
Therefore we choose $\varphi_K(\beta)$ as the generator η.
To show that this is not the trivial extension we need to determine the Adams operations in $\widetilde{K}_{\mathbb{R}}(\mathbb{R}^{q+r+1} \cup_{J\beta} S^q)$. Therefore we want to know $\psi^k\varphi_K(1)$. By the proof of Proposition 4.32 it suffices to know this term for odd k.
By [Ada65] Theorem 5.17 we have

$$\varphi_K^{-1}\psi^k\varphi_K(1) = \begin{Bmatrix} 1 & \text{if } k \equiv \pm 1 \bmod 8 \\ 1+\beta & \textit{if } k \equiv \pm 3 \bmod 8 \end{Bmatrix}$$

and therefore

$$\psi^k\varphi_K(1) = \begin{Bmatrix} \varphi_K(1) & \textit{if } k \equiv \pm 1 \bmod 8 \\ \varphi_K(1) + \varphi_K(\beta) & \textit{if } k \equiv \pm 3 \bmod 8 \end{Bmatrix}.$$

With our choices of ξ and η we get

$$\psi^k\xi = \begin{Bmatrix} \xi & \textit{if } k \equiv \pm 1 \bmod 8 \\ \xi + \eta & \textit{if } k \equiv \pm 3 \bmod 8 \end{Bmatrix}.$$

By Remark 4.33 this is the non zero element in $\mathrm{Ext}^1_s(\widetilde{K}_{\mathbb{R}}(S^q), \widetilde{K}_{\mathbb{R}}(S^{r+q+1}))$ and therefore $e_{\mathbb{R}}J\beta$ is non zero. □

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