Goethe-Universität Frankfurt
Institut für Mathematik
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Seminar
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Seminar

Noether-Lefschetz loci in $\mathcal{A}_g$ and modularity

Winter term 2025/26

Place: Alternating between Frankfurt and Darmstadt.
– Frankfurt: Room 309, Robert-Mayer-Straße 6-8
(not R-M 10!! This year the room is right above our coffee room!!)
– Darmstadt: Room 244 - S2|15, Schlossgartenstraße 7.

Format: Each meeting is made of two talks of *60 minutes*.

Time: Every meeting is on Monday from 15.15 to 17.45 with a coffee break of 30 minutes in between the two talks. Precise dates below.

Shimura varieties are quotients of symmetric spaces that often have other interpretations as moduli spaces of certain algebraic varieties. Rather than giving a general definition this seminar mainly focusses on the prime example, the moduli space of abelian varieties $\mathcal{A}_g = \mathrm{Sp}_{2g}(\mathbb{Z})\backslash \mathrm{Sp}_{2g}(\mathbb{R})/K$ and uses for comparison occasionally orthogonal Shimura varieties $\Gamma\backslash \mathrm{SO}(2,n)(\mathbb{R})/K$ where in both cases K is a maximal compact subgroup of the ambient Lie group and where Γ is a lattice in $\mathrm{SO}(2,n)(\mathbb{R})$. For appropriate choices of n and Γ these are moduli spaces of K3 surfaces or moduli space of Enriques surfaces.

These Shimura varieties naturally contain subvarieties, that appear under many names depending on dimension and context, such as special cycles, modular curves or Hirzebruch-Zagier cycles for Hilbert modular surfaces (the case $\mathrm{SO}(2,2)$), or Heegner divisors. The ones we consider here are intrinsically described in the Shimura variety as being simultaneously complex subvarieties and (totally) geodesic submanifolds and they are Shimura varieties themselves. Using the modular interpretations they can be described as loci where the abelian varieties have extra endomorphisms or more generally where the rank of the Picard group of the paramterized objects jumps up in rank. For this reason they are also called Noether-Lefschetz loci.

Starting with the works of Hirzebruch-Zagier on Hilbert modular surfaces a recurrent fruitful insight is to combine those special cycles into a generating series, sorted by 'degree', and to observe that in many cases the resulting object is a modular form (with values in some cohomology ring or Chow ring) of some sort (classical, Siegel modular form, vector valued modular form, or its metaplectic variant). Alternatively, one may fix a cycle of complementary dimension and retain that the generating series of intersection numbers with the fixed cycle is a modular form. The work of Kudla–Millson gives an explanation of many modularity results of this sort. Whenever the Lie group of the Shimura variety fits into a (Howe) dual pair, Kudla–Millson construct a theta function that can be used to prove the cycle-valued modularity.

The goal of this seminar is to study in more detail $\mathcal{A}_g$, its subvarieties, its Noether-Lefschetz loci, and their generating series. This leads to two modularity statements: one is a proven version (by Greer-Lian [GL25]) but considering the Noethe-Lefschetz loci in an auxiliary space. The other one, for $\mathcal{A}_g$ is a open conjecture (made also by Greer-Lian). The last two talks investigate recent progress that relates the tautological ring of $\mathcal{A}_g$ with the tautological ring of the compact type partial compactification of the moduli space of curves and that provides evidence for this conjecture.

What does $\star$, $\star\star$, and $\star\star\star$ mean?

$\star$: Suitable for Ph.D. students without much specific background.

$\star\star$: Suitable for Ph.D. students and postdocs; usually requires some background and knowledge of almost all previous talks.

$\star\star\star$: Suitable for ambitious Ph.D. students and postdocs, as well as for Professors. Requires solid background and/or the willingness to engage with the material in significant depth.

1. **Introduction to the modularity conjecture.** (27.10.25, Darmstadt) R. Zuffetti

   This talk gives an overall motivation, the above introduction and some more details.

2. **The moduli space of abelian varieties.** $\star$ (27.10.25, Darmstadt) M. Zhang

   Introduce very briefly principally polarized abelian varieties and show that their moduli space is $\mathcal{A}_g = \mathrm{Sp}_{2g}(\mathbb{Z})\backslash \mathrm{Sp}_{2g}(\mathbb{R})/U(g)$ where $\mathbb{H}_g = \mathrm{Sp}_{2g}(\mathbb{R})/U(g)$ is Siegel space.

   Introduce the space of Lagrangian Grassmanians Y, the compact dual of Siegel space and describe the embedding $\mathbb{H}_g \to Y_g$.

   Sketch the properties of the cell decomposition and deduce that its cohomology ring is the ring R_g generated by the u_i with relations $ch_{2k}(E) = 0$, as in [Gee15, Proposition 2.2].

   Coordinate with the speaker of Talk 3 to see where to draw the line between the two talks.

   Literatur: [BL04], [Gee99; Gee15].

3. **The tautological ring of $\mathcal{A}_g$.** $\star\star$ (10.11.25, Frankfurt) N. MÜLLER

   Define the tautological ring of $\mathcal{A}_g$ and show that it is isomorphic of R_g, using the GRR-calculations in [Gee15, Section 4] or [Gee99].

   We are not mainly interested in the positive characteristic results in those papers. However, the existence of a large-dimensional compact subvariety in positive characteristic is a useful tool for a non-vanishing result.

   Literatur: [Gee99; Gee15].

4. **The toroidal compactification of $\mathcal{A}_g$ and its tautological ring.** $\star\star\star$ (10.11.25, Frankfurt) M. PRADO

   The tautological ring of the compactification and in Chow is still just R_g, which is the main result of [EV02]. In one talk, we aim to mainly (only) state the ingredients. That is: state the properties of the toroidal compactification as the wish list given in [EV02, Theorem 3.1], explainining the notions there (starting notably with the sheaf of logarithmic forms).

   Literatur: [EV02].

5. **Modular forms vs generating series.** $\star$ (1.12.25, Darmstadt) P. MÜLLER

   Introduce (holomorphic) modular forms with respect to $\mathrm{SL}_2(\mathbb{Z})$ and show that they admit a Fourier expansion [BGHZ08, Part 1, Section 1.1]. Introduce the notion of modular form *with level* [DS05, Section 1.2], explaining that the Fourier expansion may depend on cusps different from ∞ [DS05, p. 16]. Define the theta series associated to positive definite even lattices [BGHZ08, Part I, Section 3.2]. (Here one can choose to restrict to the unimodular case [KK25, V, Sections 2.5 and 2.6].) These theta series are *generating series* of representation numbers. Explain that the idea of studying arithmetic properties of numbers by means of modular properties of generating series has been vastly generalized: Introduce the notion of a dual reductive pair (e.g. [Gan23, Section 2.1]) and of the (geometric) theta correspondence [Li25, Introduction]. Details for the pair $(\mathrm{O}(p,q), \mathrm{SL}_2(\mathbb{R}))$ will be given in the next talk.

6. **The Kudla–Millson theta function.** $\star\star$ (1.12.25, Darmstadt) N. LUDWIG

   Recall how to construct locally symmetric spaces of orthogonal type from an even indefinite lattice L of signature (p,q). Define special cycles of codimension q. Although the theory works in general signature, restrict to the case of interest for the AG, namely $p = q$. Recall the Weil representation ρ_L of $\mathrm{SL}_2(\mathbb{Z})$ arising from L and the notion of modular form w.r.t. ρ_L. Show that the b-entry of such series, with $b \in L'/L$, is a scalar-valued modular form of level equal to the level of L. [BF10, Section 1].

   State the main result of Kudla and Millson: The generating series of special cycles is modular w.r.t. ρ_L. Provide an idea of the proof, a suggestion is as follows. Describe the main properties of the KM Schwartz function φ_{KM} (definition as a black box) [BF10, Section 3]. Construct the KM theta function using φ_{KM} as in [BF10, p. 14]. Show its modularity following the lines of [BF04, Section 2]. The theta series is *non-holomorphic* but it has a Fourier expansion. The coefficients of positive index are Poincaré dual to special cycles [KM88, (S) and Section 5]. The image of the KM series under $\overline{\partial}_\tau$, where τ is the symplectic variable, is exact in the orthogonal variable [KM90, p. 161] [BF04, p. 49]. Hence, the cohomology class of the series is a *holomorphic* modular form and coincides with the generating series of special cycles.

7. **Noether-Lefschetz loci.** $\star$ (8.12.25, Frankfurt) L. Schneider

The Noether-Lefschetz loci in $\mathcal{A}_g$ are the loci where the Picard number of the abelian variety is larger than the generic value one. The loci of Picard rank two have been classified in [DL90] to be loci, where the abelian variety splits (up to isogeny) in a product of two simple factors and the loci with real multiplication.

To understand the brief note [DL90] recall the relevant material about decomposition of abelian varieties from [BL04]. The case of abelian varieties that split off one elliptic curve factor is the most important for the sequel. Consult also [GL24].

Literatur: [DL90], [BL04], [GL24].

8. **Noether-Lefschetz modularity in $\mathcal{A}_1 \times \mathcal{A}_g$.** $\star\star$ (8.12.25, Frankfurt) T. Driscoll–Spittler

Show the main result [GL25, Theorem 1.2] that the generating series of Noether-Lefschetz loci (from now on this refers to the loci of abelian varieties that split off an elliptic curve up to isogeny) is modular in $\mathcal{A}_1 \times \mathcal{A}_g$ and also its version with level structures.

State the modularity conjecture [GL25, Conjecture 1].

Literatur: [GL25],[GL24].

9. **Tautological projections.** $\star\star\star$ (19.1.26, Darmstadt) J. Horn

Define the tautological projection following [CMOP24]. To start, this requires talking about Moduli space of compact curves and λ_g-pairing. State the formula for tautological projections of loci of product abelian varieties and sketch its proof.

The speaker should have a look into [COP25] to provide broader context and outlook

Literatur: [CMOP24], [COP25].

10. **Modularity conjecture and Hecke orbits.** $\star\star\star$ (19.1.26, Darmstadt) M. Möller

Recall the definition of Hecke operators and prove that λ_{g-1} together with the Noether-Lefschetz loci are a Hecke-stable submodule of the Chow ring of $\mathcal{A}_g$ [Lop24, Proposition 18]. Recall the modularity conjecture [GL25, Conjecture 1] and prove the tautological shadow of it [Lop24, Corollary 4].

Literatur: [Lop24].

Literatur

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