Hamiltonian Cycles in Supersolvable and Simplicial Hyperplane Arrangements

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Jan Stricker, Jasmin Walizadeh

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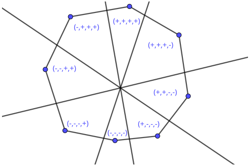
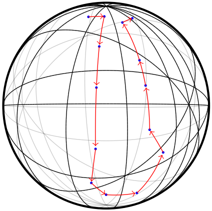


March 31, 2026



combinatorial synergies

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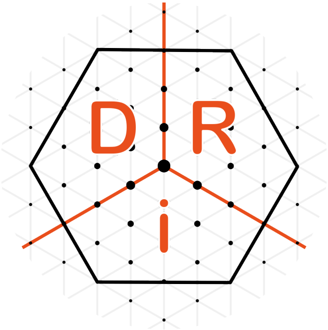
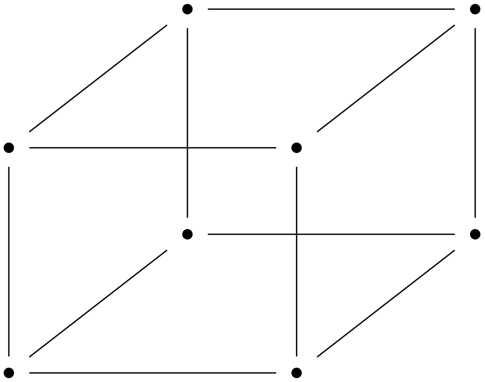
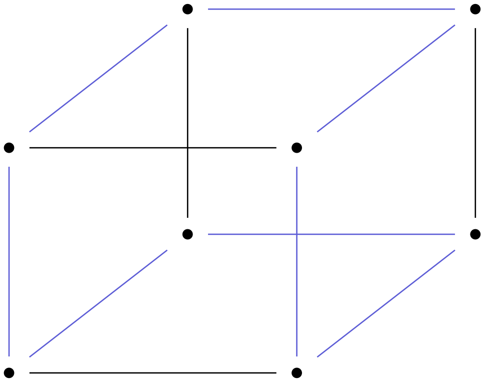
$\Rightarrow$

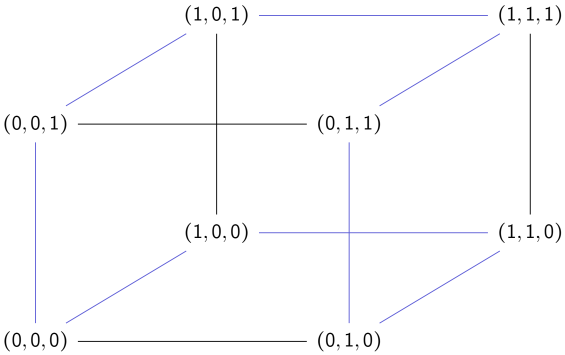


combinatorial synergies ⇒ DRi





Hamiltonian cycles



Hamiltonian cycles

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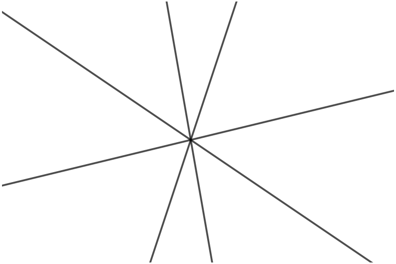
Gray Codes

Basics

- **Hyperplane arrangement**
  $\mathcal{A}$ set of hyperplanes



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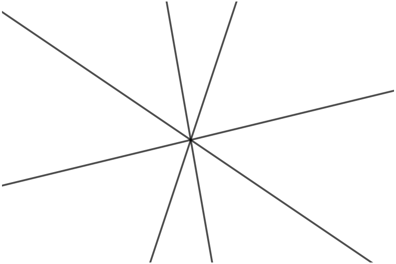
- **Hyperplane arrangement** $\mathcal{A}$ set of hyperplanes
- $\mathcal{A}$ is **central** $\mathcal{A}$ all hyperplanes contain



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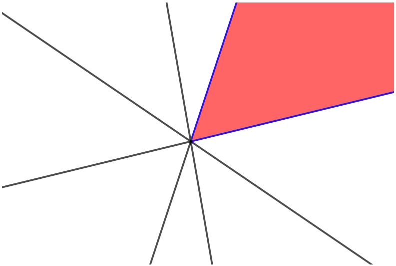
- $\mathbb{R}^n \setminus \bigcup_{H \in \mathcal{A}} H$ closure of connected components are **regions** $\mathcal{R}(\mathcal{A})$



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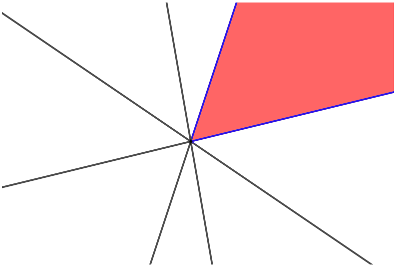
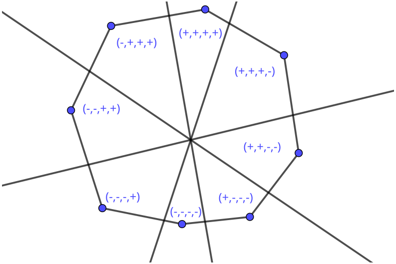
- The blue is a wall



Tope Graphs

Definition

Given a hyperplane arrangement $\mathcal{A} = \{H_1, \ldots, H_m\}$, we can define the **tope of a region** R as a vector in $\tau(R) \in \{-,+\}^m$ giving by the orientation of hyperplanes. The set of topes together with edges describing adjacency form the **tope graph** $\mathcal{T}(\mathcal{A})$.



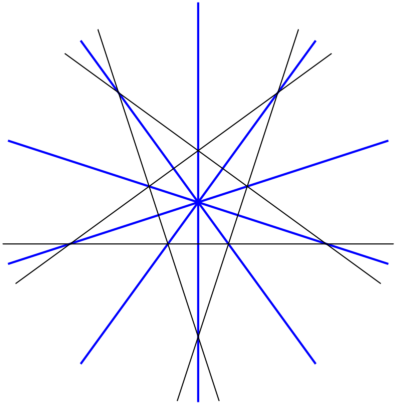
Reducible

Proposition

If a hyperplane arrangement $\mathcal{A}$ is the product of hyperplane arrangements with have a Hamiltonian cycles, then $\mathcal{A}$ has a Hamiltonian cycle.

We can focus only on Hamiltonian cycles in **irreducible** hyperplane arrangements.

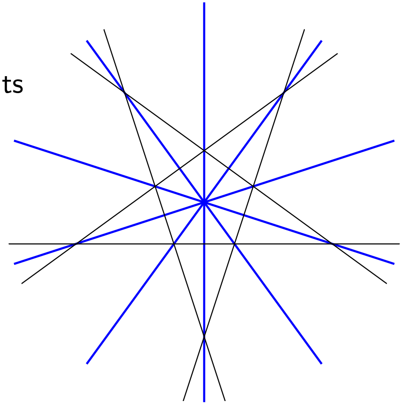
Supersolvable hyperplane arrangement



Supersolvable hyperplane arrangement

[Björner, Edelmann, Ziegler 1990]

- All rank-2 hyperplane arrangements are **supersolvable**



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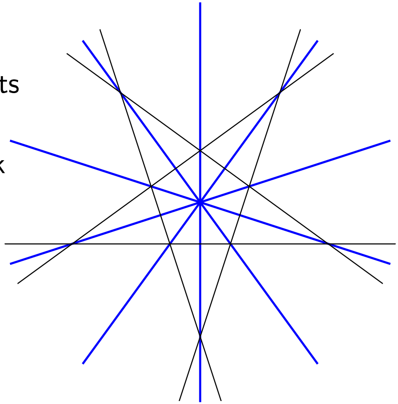

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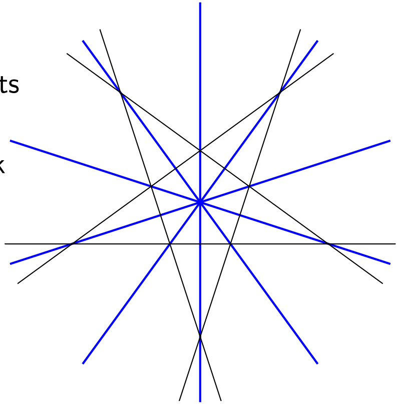

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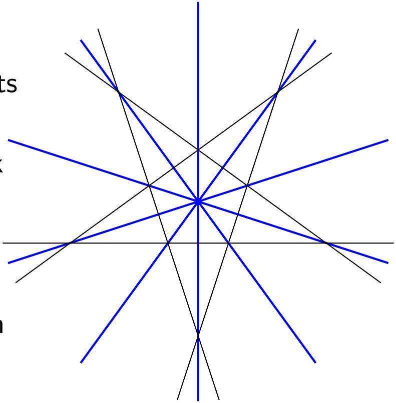
- $\mathcal{A}_0$ supersolvable arrangement of rank $n-1$
- For $\underline{H', H'' \in \mathcal{A}_1}$ exist $\underline{H \in \mathcal{A}_0}$ such that $\underline{H' \cap H'' \subseteq H}$



Definition

Let $\mathcal{A}$ be a hyperplane arrangement with partition $\mathcal{A} = \mathcal{A}_0 \uplus \mathcal{A}_1$. We define the **fiber** $\mathcal{F}(R)$ of a region $R \in \mathcal{R}(\mathcal{A})$ as the set of regions containing all regions that lie in the same region of $\mathcal{A}_0$.

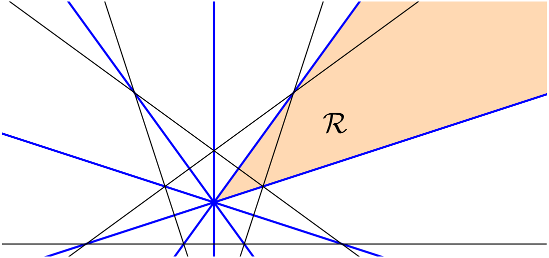


Figure: The blue hyperplanes are the hyperplanes of $\mathcal{A}_0$. The shaded area is the fiber $\mathcal{F}(R)$ of a region R.

Theorem (Körber, Schnieders, Walizadeh, S. 2025)

The tope graphs of supersolvable oriented matroids and supersolvable hyperplane arrangements admit a Hamiltonian cycle.

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- The induced subgraph of every fiber is a path of length $|\mathcal{A}_1|$
- Each hyperplane occurs exactly once in the path [Björner, Edelmann, Ziegler 1990]

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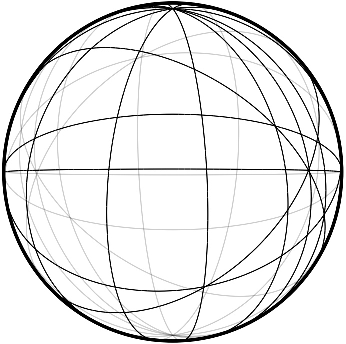

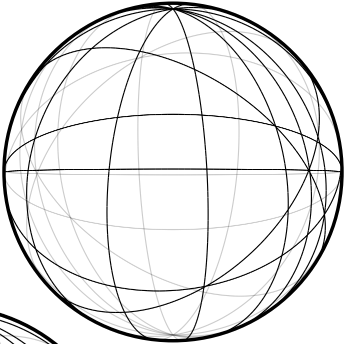

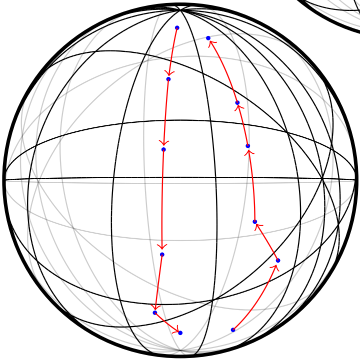

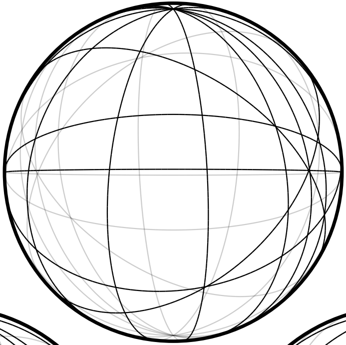

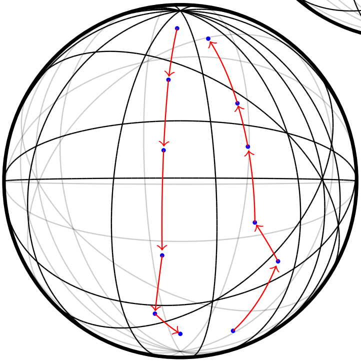

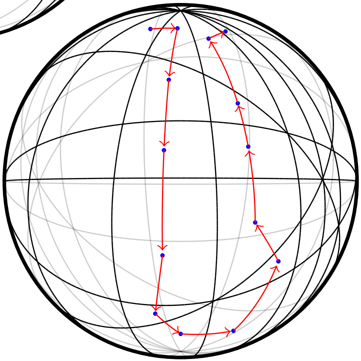

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- Coxeter types: $\mathcal{A}_n$ $(n \geq 1)$, $\mathcal{B}_n$ $(n \geq 2)$, $\mathcal{D}_n$ $(n \geq 4)$, E_6, E_7, E_8, F_4, H_3, H_4, and $\mathcal{I}_m$ $(m = 5$ or $m > 7)$

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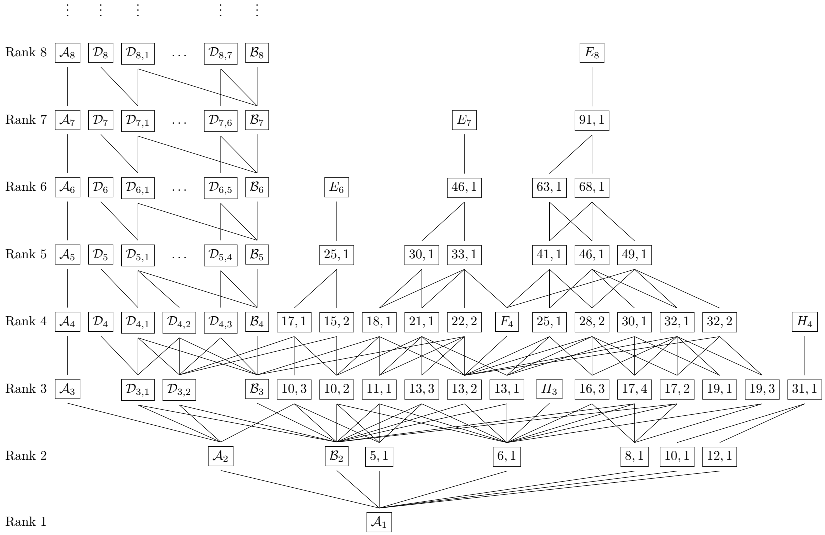
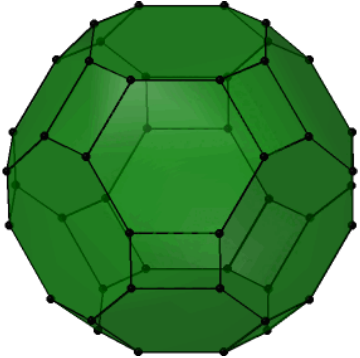


Figure: A Hasse diagram showing how restrictions of reflection arrangements relate to each other. Here, $\boxed{m, k}$ refers to the kth arrangement with m hyperplanes.

Theorem (Körber, Schnieders, S., Walizadeh 2025)

All sporadic restrictions admit a Hamiltonian cycle.



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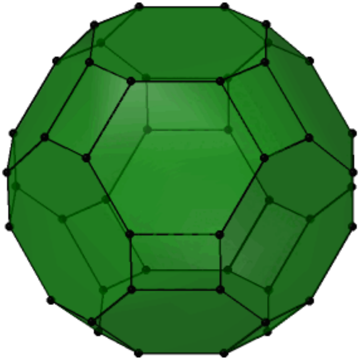

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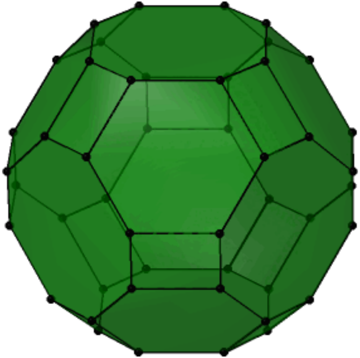

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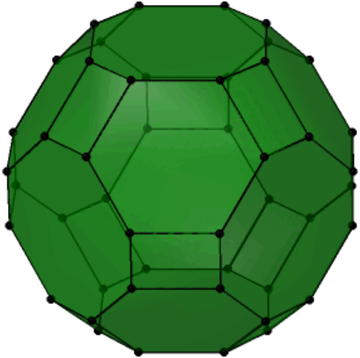
All arrangements of type $\mathcal{D}_{n,s}$ for $n \geq 2$ admit a Hamiltonian cycle.



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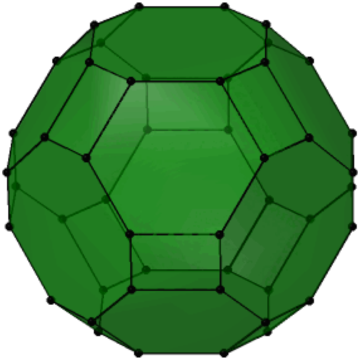

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- 95 sporadic arrangements + 3 infinite families

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- There are many Hamiltonian cycles

The three infinite families

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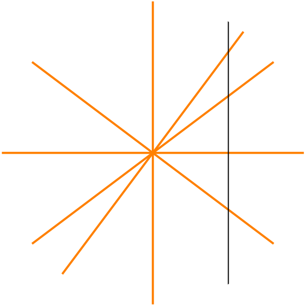
The family $\mathcal{R}(0)$



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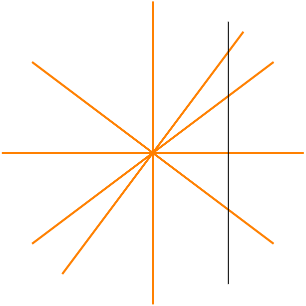

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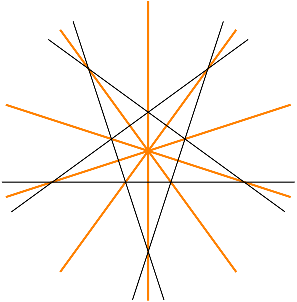

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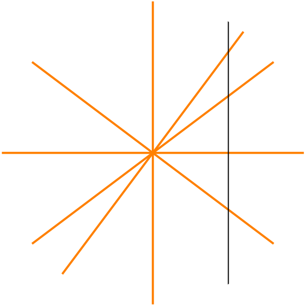

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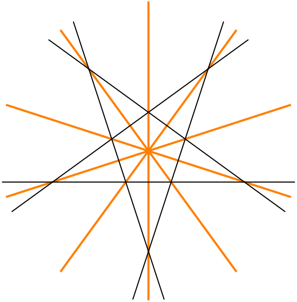

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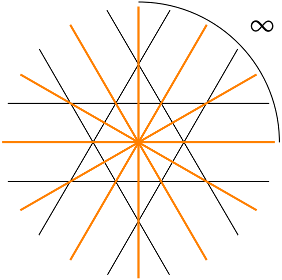

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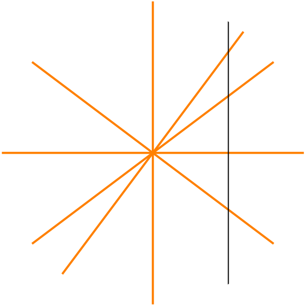

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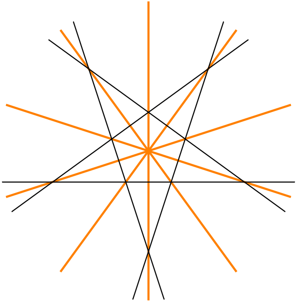
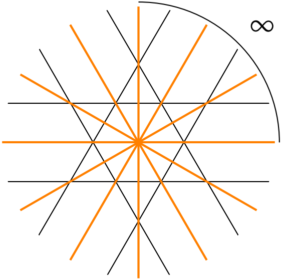


The family $\mathcal{R}(2)$



- These are families are supersolvable.

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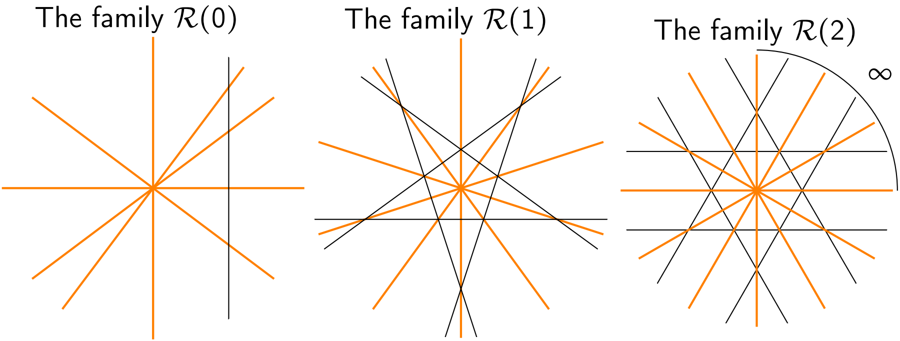

- These are families are supersolvable.
- With our Theorem for supersolvable we find Hamiltonian cycles.

Future beholds

- Working on proof without the computer

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- Going up in dimension/rank

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- Going up in dimension/rank
- Using the knowledge on tope graphs to understand more structure

Summary arXiv:2508.14538

Theorem

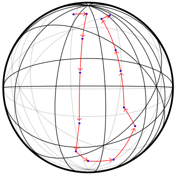
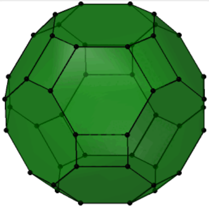
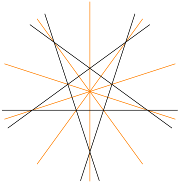
The tope graphs of supersolvable oriented matroids and supersolvable hyperplane arrangements admit a Hamiltonian cycle.

Theorem

All restrictions of reflection arrangements admit a Hamiltonian cycle.

Theorem

All simplicial hyperplane arrangements in the Grünbaum–Cuntz catalogue admit a Hamiltonian cycle.

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