Exploring Marked Poset Polytopes

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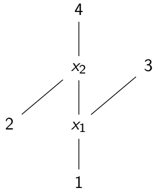
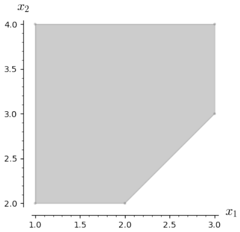


March 18, 2026



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- Certain marked order polytopes are integral equivalent to flow polytopes [Liu, Meszaros, Dizier 19]
- The polytope of q-rank functions are Minkowski summands of marked order polytopes
- They provide toric degenerations of type A flag variety [Fujita 21]

Definition

A **marked poset** (P, λ) is a finite poset P together with an induced subposet $P^* \subseteq P$ of **marked elements** and an order-preserving **marking** $\lambda : P^* \to \mathbb{R}$.

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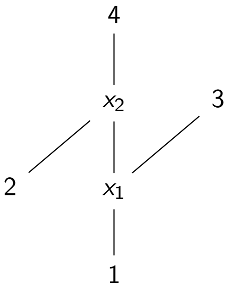

Definition

Let (P, λ) be a marked poset with $\min(P) \cup \max(P) \subseteq P^*$. The **marked order polytope** $\mathcal{O}(P, \lambda)$ associated to (P, λ) is the set of all $x \in \mathbb{R}^{P \backslash P^*}$ satisfying the following conditions:

(1) $\lambda(a) \leq x_p \leq \lambda(b)$, for all $p \in P \backslash P^*$ and $a, b \in P^*$ with $a < p < b$ and

(2) $x_p \leq x_q$, for the $p, q \in P \backslash P^*$ with $p \leq q$.

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(1) $\lambda(a) \leq x_p \leq \lambda(b)$, for all $p \in P\setminus P^*$ and $a, b \in P^*$ with $a \prec p \prec b$ and

(2) $x_p \leq x_q$, for the $p, q \in P\setminus P^*$ with $p \preceq q$.

Definition

Let (P,λ) be a marked poset with $\min(P) \cup \max(P) \subseteq P^*$. The **marked chain polytope** $\mathcal{C}(P,\lambda)$ associated to (P,λ) is the set of all $x \in \mathbb{R}_{\geq 0}^{P \setminus P^*}$ satisfying the following condition:

(1) $x_{p_1} + x_{p_2} + \ldots + x_{p_k} \leq \lambda(b) - \lambda(a)$, for each (maximal) chain $a \prec p_1 \prec p_2 \prec \ldots \prec p_k \prec b$ in P with $a, b \in P^*$.

Definition

Let $P = P^* \cup C \cup O$ be a partition of a poset P with $\min(P) \cup \max(P) \subseteq P^*$ and λ a marking. The elements of C and O are called **chain elements** and **order elements**, respectively. The **marked chain-order polytope** $\mathcal{O}_{C,O}(P,\lambda) \subseteq \mathbb{R}^P$ is the set of all $x = (x_p)_{p \in P} \in \mathbb{R}^P$ satisfying the following conditions:

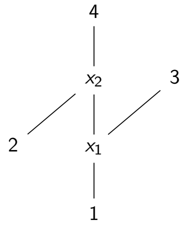
(1) for any $a \in P^*$, $x_a = \lambda(a)$;

(2) for $p \in C, x_p \geq 0$;

(3) for each saturated chain of chain elements $a < p_1 < \cdots < p_r < b$ between $a, b \in P^* \cup O, p_i \in C, r \geq 0$, we have

$$x_{p_1} + \cdots + x_{p_r} \leq x_b - x_a.$$

Examples



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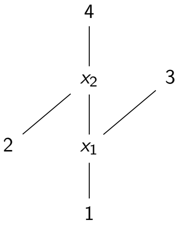
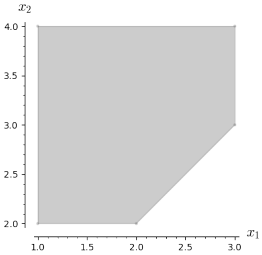
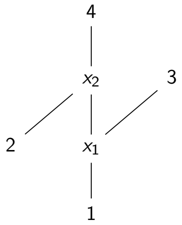
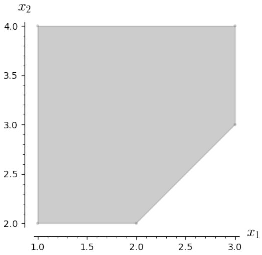




(a) Marked order

Examples





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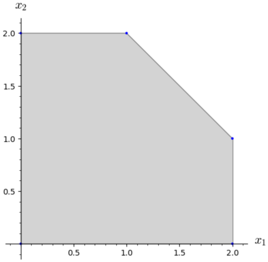
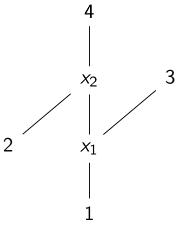
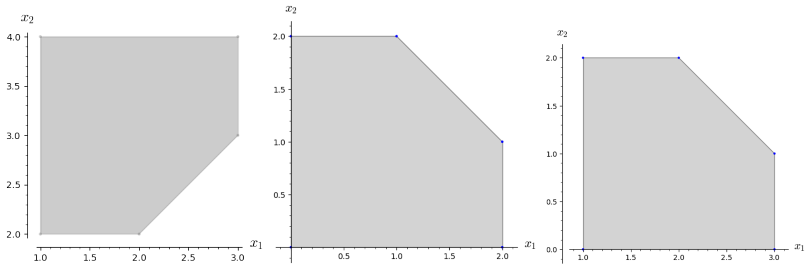


(b) Marked chain

Examples





(a) Marked order (b) Marked chain (c) Marked chain-order

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- We have nice descriptions of faces

Theorem (Jochemko,Sanyal 14; Pegel 18)

A partition π of a marked poset (P, λ) is a face partition if and only if it is (P, λ)-compatible, connected and the induced marking on $(P/\pi, \lambda/\pi)$ is strict.

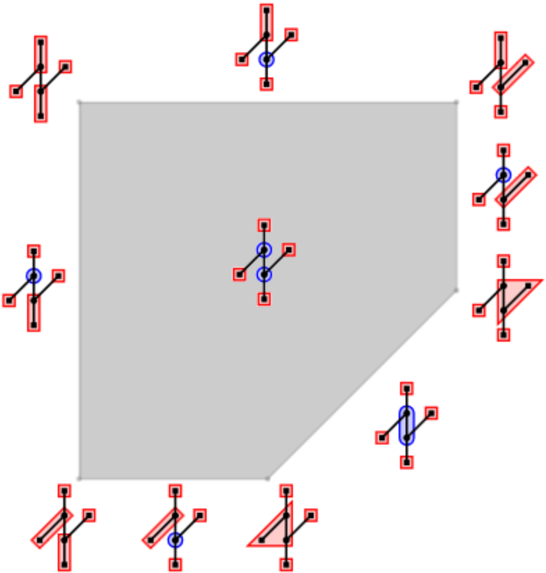

Figure: Faces of the marked order polytope

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Theorem (Fang, Fourier, Litza, Pegel 20)

The marked order, marked chain and marked chain-order polytopes with the same underlying marked poset are Ehrhart equivalent.

This means, for the same marked poset all these polytopes have the same Ehrhart polynomial.

Theorem (Fang, Fourier, Pegel 18)

The marked chain-order polytope $\mathcal{O}_{C,O}(P,\lambda)$ decomposes as the weighed Minkowski sum.

$$\mathcal{O}_{C,O}(P,\lambda) = c_0\mathcal{O}_{C,O}(P,\omega_0) + (c_1 - c_0)\mathcal{O}_{C,O}(P,\omega_0) + \cdots + (c_k - c_{k-1})\mathcal{O}_{C,O}(P,\omega_k)$$

And if λ is integral, it holds

$$\mathcal{O}^{\mathbb{Z}}_{C,O}(P,\lambda) = c_0\mathcal{O}^{\mathbb{Z}}_{C,O}(P,\omega_0) + (c_1 - c_0)\mathcal{O}^{\mathbb{Z}}_{C,O}(P,\omega_0) + \cdots + (c_k - c_{k-1})\mathcal{O}^{\mathbb{Z}}_{C,O}(P,\omega_k)$$

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Theorem (Fang, Fourier, Pegel 18)

For a marked poset (P,λ) the marked order polytope $\mathcal{O}_{C,O}(P,\lambda)$ has a unqiue interior point and the canonically translated polytope is reflexive if and only if P is a ranked poset and λ is arising from a rank function.

Definition and Theorem (Aprile 21; Gouveia, Parrilo, Thomas 10; Sullviant 06)

Let P be a polytope is a 2-level polytope if and only if

1. for any facet $F \subseteq P$ there exists $t \in \mathbb{R}^d$, such that all vertices $v \in V(P) \backslash V(F)$ lies in $t + \text{aff}(F)$.

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- Stanley's order and chain polytopes are 2-level polytopes.

Theorem

Let (P, λ) be a regular marked poset. The following conditions are equivalent:

1. *$\mathcal{O}(P, \lambda)$ is a 2-level polytope.*
2. *Each connected component of the poset P has one unique maximal and one unique minimal marked element.*
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Theorem

Let (P, λ) be a regular marked poset. The following conditions are equivalent:

1. *$\mathcal{C}(P, \lambda)$ is a 2-level polytope.*
2. *The difference of markings for each saturated chain in P differs by a constant value.*
3. *$\mathcal{C}(P, \lambda)$ is affinely isomorphic to a chain polytope.*

Theorem

Let (P, λ) be a regular marked poset and $P = P^* \cup C \cup O$ a decomposition into marked, chain and order elements. The marked chain-order polytope $\mathcal{O}_{C,O}(P, \lambda)$ is a 2-level polytope if and only if

1. *$\mathcal{O}(P \backslash C, \lambda)$ is a 2-level polytope, and*
2. *The higher and lower bounds of the chain elements C differ by a constant.*

Let $\mathcal{L}(P)$ be the linear extensions of a poset P and P_π the poset defined by the linear extension π.

Definition

For a marked poset (P,λ) with $\pi \in \mathcal{L}(P)$, a **marked chain** I in P_π is a maximal chain I in P_π consisting of elements from $P \backslash P^*$ that lies between two elements $a, b \in P^*$. We these bounds with a_I and b_I.

Theorem

Let (P,λ) be a regular marked poset and $\mathcal{L}(P)$ the set of all linear extensions of P where the markings are ordered in increasing order. The Ehrhart polynomial of $\mathcal{O}(P,\lambda)$ is

$$Ehr_{\mathcal{O}(P,\lambda)}(m) = \sum_{\pi \in \mathcal{L}(P)} \prod_{\substack{I \subseteq P_\pi \\ \text{marked chain}}} \binom{m(\lambda(b_I) - \lambda(a_I)) - d + |I|}{|I|}.$$

Here, d is the number of descents of π between a_I and b_I.

Question

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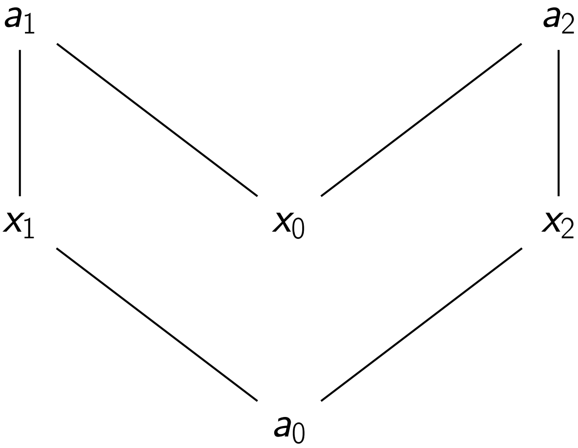
- Do we have nice combinatorial interpretations of invariants for 2-level marked chain-order polytopes?
- Investigating the Ehrhart polynomial for special marked posets.

Thank you!



[ABS11] Federico Ardila, Thomas Bliem, and Dido Salazar. "Gelfand—Tsetlin Polytopes and Feigin—Fourier—Littelmann—Vinberg Polytopes as Marked Poset Polytopes". In: *Discrete Mathematics & Theoretical Computer Science* Proceedings (2011).

[Apr+24] Manuel Aprile, Michele Conforti, Samuel Fiorini, Yuri Faenza, Tony Huynh, and Marco Macchia. "Slack matrices, k-products, and 2-level polytopes". In: *Discrete Applied Mathematics* 354 (2024), pp. 199–213.

[Fan+20] Xin Fang, Ghislain Fourier, Jan-Philipp Litza, and Christoph Pegel. "A continuous family of marked poset polytopes". In: *SIAM Journal on Discrete Mathematics* 34.1 (2020), pp. 611–639.

[FF16] Xin Fang and Ghislain Fourier. "Marked chain-order polytopes". In: *European Journal of Combinatorics* 58 (2016), pp. 267–282.

[FFP18] Xin Fang, Ghislain Fourier, and Christoph Pegel. "The Minkowski property and reflexivity of marked poset polytopes". In: *arXiv preprint arXiv:1807.03970* (2018).

[Fuj21] Naoki Fujita. "Semi-toric degenerations of Richardson varieties arising from cluster structures on flag varieties". In: *arXiv preprint arXiv:2110.12731* (2021).

[GPT10] Joao Gouveia, Pablo A Parrilo, and Rekha R Thomas. "Theta bodies for polynomial ideals". In: *SIAM Journal on Optimization* 20.4 (2010), pp. 2097–2118.

[JS14] Katharina Jochemko and Raman Sanyal. "Arithmetic of marked order polytopes, monotone triangle reciprocity, and partial colorings". In: *SIAM Journal on Discrete Mathematics* 28.3 (2014), pp. 1540–1558.

[LMD19] Ricky I Liu, Karola Mészáros, and Avery St Dizier. "Gelfand–Tsetlin Polytopes: A Story of Flow and Order Polytopes". In: *SIAM Journal on Discrete Mathematics* 33.4 (2019), pp. 2394–2415.

[Peg18] Christoph Pegel. "The face structure and geometry of marked order polyhedra". In: *Order* 35.3 (2018), pp. 467–488.

[Str24] Jan Stricker. "2-levelness of Marked Poset Polytopes". In: *arXiv preprint arXiv:2401.07492* (2024).

[Sul06] Seth Sullivant. "Compressed polytopes and statistical disclosure limitation". In: *Tohoku Mathematical Journal, Second Series* 58.3 (2006), pp. 433–445.