Maximally irregularly fibred surfaces of general type

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Abstract. We generalise a method of Xiao Gang to construct 'prototypes' of fibred surfaces with maximal irregularity without being a product. This enables us, in the case of fibre genus $g = 3$ to describe the possible singular fibres and to calculate the invariants of these surfaces. We also prove structure theorems on the moduli space for fibred surfaces with fibre genus $g = 2$ and $g = 3$.

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Introduction

Complex fibred surfaces $f : X \to B$ of small fibre genus may be studied by different techniques according to their irregularity $q(X) = H^1(X, \mathcal{O}_X)$. If b denotes the genus of the base B and g the genus of a fibre, the irregularity of a fibred surface is subject to $b \leq q \leq g + b$. In case $q = g + b$ the fibration is trivial, i.e. X is birational to a product.
Xiao ([Xi85]) and Seiler ([Sei95]) examined surfaces X with $g = 2$ and irregularity $q(X) = b$ using the fact that these surfaces are double coverings of ruled surfaces. Xiao also studies ([Xi85]) surfaces with $g = 2$ and $q = b+1$ using the fixed part of the Jacobian fibration. We extend this technique to surfaces with $q = b + g - 1$, which we call *maximally irregularly fibred*. Maximal irregularity implies $g \leq 7$ by ([Xi87a]) and maximally irregular fibrations with $g \leq 4$ are known to exist ([Pi89]).
We construct a *'prototype'* for maximally irregularly fibred surfaces with $g = 3$, i.e. a fibred surface such that any other surface with the same invariants arises via pullback by covering of the base curves (see Def. 1.3 for the precise definition). Compared to Xiao's case additional difficulties arise at the hyperelliptic locus due to the failure of infinitesimal Torelli. The prototype enables us to determine the degenerate fibres and invariants of maximally irregularly fibred surfaces with $g = 3$.
The techniques apply in principle also for $g = 4$ (and, if surfaces exist, also for $g \geq 5$), but these cases additionally need an answer to a Schottky type problem, as explained at the end of the paper.
Finally we show that these constructions glue together in families. We thus obtain structure results for components of the Gieseker moduli space of surfaces of general type. The surfaces admitting a maximally irregular fibration form connected components of the

moduli space. These components fibre over a moduli space of abelian varieties. The fibres are moduli spaces of stable mappings.

The paper is organised as follows: In §1 we recall some facts on the fixed part of the Jacobian fibration. We are then able to give the precise definition of prototype. Omitting some technical conditions on the abelian variety A one of the main theorems can then be roughly stated as follows:

Theorem 1.4' *Suppose $d \geq 3$ and let A be a $(1,d)$-polarised abelian variety of dimension 2. Then there is a fibred surface $S(A,d) \to B(A,d)$, such that any maximally irregular fibration $X \to B$ with fixed part A and fibre genus 3 is obtained via base change $B \to B(A,d)$ from $S(A,d)$.*
The base curve $B(A,d)$ is a double covering of the modular curve $X(d)$.

Its proof relies on a parametrisation of the Jacobians of curves into which the abelian variety A injects. This result is stated as Theorem 1.6 and its proof is presented in §2. §3 contains the proof of Theorem 1.4 together with the computation of the invariants of maximally irregular fibrations. Finally in §4 we use these results to derive some structure results for the corresponding components of the moduli space.
Most of the results can be found in the author's thesis ([Mo02]).

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Notation

We work over the complex numbers throughout. For a fibred surface $f : X \to B$ we denote $b = g(B)$ the base genus and $g = g(F)$ the fibre genus. The irregularity $q(X) = H^1(X, \mathcal{O}_X)$ is subject to $b \leq q \leq b + g$. We say the fibration f is *of type* (g,b).
Let $\mathfrak{C}_g(\cdot)$ (resp. $\mathcal{A}_g(\cdot)$) denote the moduli functor for smooth curves of genus g (resp. abelian varieties of dimension g) and let M_g (resp. A_g)) denote the corresponding coarse moduli spaces.
When discussing surfaces of general type, we denote by X its *canonical model*, i.e. a normal surface with K_X ample and at most rational double points. If necessary, S denotes the corresponding smooth minimal model. A *relative canonical model* of a flat family of surfaces of general type over some base T has a canonical model in the fibre over each complex point of T. Each flat family of surfaces of general type is birational to a relative canonical model (see [Tv72]). We denote by $\mathfrak{S}(\cdot)$ the moduli functor which associates with a scheme T the set of flat families of relative canonical models of surfaces of general type over T.

1 Prototypes for fibred surfaces

A *fibration* of a surface $f : X \to B$ is a surjection onto a smooth curve B with connected fibres. The fibration is called *regular* if $q(X) = b$ and *irregular otherwise.* If $q(X) = b + g$, the surface X is birational to a product of the base curve and smooth fibre. If the moduli map $B \to M_g$ has image reduced to a point the fibration is said to be *isotrivial* or *of constant moduli.*
In the sequel we are interested in the case of non-trivial fibrations with maximal irregularity, i.e. with $q(X) = g + b - 1$. We call them *maximally irregularly fibred.* The Jacobians of the fibres of such a surface have a large abelian variety in common, the *fixed part* ('partie fixe' or 'L/K-trace' in [La59]). We recall some facts about the fixed part of a fibred family $X \to B \to T$ of surfaces. The reader less interested in statements on the moduli space may think of $T = \operatorname{Spec}\mathbb{C}$ in sections $1 - 3$.

Let $A = \operatorname{Pic}^0_{X/T}/\operatorname{Jac}_{B/T}$. Over the locus B' where $X \to B$ is smooth (which is dense in every fibre over T) $\operatorname{Jac}_{X'/B'}$ exists. The Picard functor applied to the B'-morphism $X' \to X \times_T B'$ induces

$$i'_A : \operatorname{Pic}^0_{X \times_T B'/B'} \to \operatorname{Jac}_{X'/B'},$$

which factors through an injection

$$i_A : A \times_T B' \hookrightarrow \operatorname{Jac}_{X'/B'}.$$

We want to provide A with a polarisation. To that purpose we restrict the principal polarisation Θ_J of $\operatorname{Jac}_{X'/B'}$ to $A \times_T B'$. We consider polarisations as (relatively) ample line bundles, but keep the classical divisor notation. By the rigidity theorem the polarisation comes from a relatively ample line bundle Θ_A on A, i.e. $i_A^*\Theta_J = \operatorname{pr}_A^*(\Theta_A)$, where $\operatorname{pr}_A : A \times_T B' \to A$ is the projection.

Definition 1.1. *The polarised abelian scheme* $(A/T, \Theta_A)$ *is called the* fixed part *of the fibration* $X \to B \to T$. *If the degree of isogeny* $\lambda(\Theta_A)$ *is* d^2, *we call* d *the* associated degree *of the fibred family. We also say that* $X \to B \to T$ *is* of type $(A/T, \Theta_A)$.

Remark 1.2. i) In [Xi92], Xiao calls the degree of $\alpha : X \to \operatorname{Alb}_X$ (if generically finite) 'associated degree'. We prefer to call $\deg(\alpha)$ the *Albanese degree* (denoted by γ).
ii) We can obviously extend the notion of associated degree to any injection $A \to J$ of abelian varieties, whenever J carries a principal polarisation.

Definition 1.3. *Let* (A, Θ_A) *be a polarised abelian variety. A relatively minimal model of a fibred surface* $S(A, d) \to B(A, d)$ *is called a* prototype for fibred surfaces with fixed part (A, Θ_A), if each fibred family of surfaces $X \to B \to T \in \mathfrak{S}(T)$ of type (g, b) with associated degree d and fixed part $(A \times T, \operatorname{pr}_1^* \Theta_A)$ *is the relative canonical model of* $B \times_{B(A,d)} S(A, d) \to B$ *for a surjection* $\varphi : B \to B(A, d)$. *The latter is called a* prototype base change.

If conversely the pullback by any such φ *gives a fibred surface with fixed part A, we call* $S(A, d) \to B(A, d)$ *a* good prototype.

We do not demand φ to be unique. However φ turns out to be more or less unique in the cases studied below.

1.1 Prototypes for maximally irregular fibrations with $d \geq 3$ and $g = 2$ or $g = 3$

Let $X(d)$ be the completion of the modular curve $X'(d) = \mathbb{H}/\Gamma(d)$, where $\mathbb{H}$ is the upper half plane and $\Gamma(d)$ the principal congruence subgroup of level d. Then we have:

Theorem 1.4. *Let (A, Θ_A) be a $(1,d)$-polarised abelian variety of dimension 2 without a nontrivial principally polarised abelian subvariety and suppose $d \geq 3$.*
Then there is a fibred surface $S(A,d) \to B(A,d)$, which is a good prototype for maximally irregular fibrations with fixed part (A,Θ_A) and fibre genus 3. The base curve $B(A,d)$ is a double covering of the modular curve $X(d)$.
Moreover, the prototype base change φ is unique up to composition with the involution σ of $B(A,d)$ over $X(d)$.

The restrictions on A are necessary: If A had a nontrivial principally polarised abelian subvariety, the Jacobian of each fibre would be reducible, a contradiction. And in Section 2 we will see that the polarisation of the fixed part in case of maximal irregularity is always of type $(1,\ldots,1,d)$. Here and everywhere in the sequel A should be considered as a polarised variety. In particular the fibred surface $S(A,d)$ depends on the polarisation. To avoid overloading notations we frequently omit the polarisation.

This theorem should be compared with the following theorem, which is basically due to Xiao.

Theorem 1.5. *For each one-dimensional abelian variety A and each $d \geq 3$ there is a fibred surface $S(A,d) \to B(A,d)$, which is a good prototype for maximally irregular fibrations of type $(2,b)$ with fixed part A and associated degree d. The base curve $B(A,d)$ is the modular curve $X(d)$, it does not depend on A.*
Moreover the prototype base change φ is unique.

The proofs of these theorems will rely on the following parametrisation of injections of abelian varieties. The reader should think of the injection of the fixed part of the relative Jacobian of a fibred surface into the Jacobian of a fibre. $\mathrm{pr}_A : A \times X \to A$ always denotes the first projection.

Theorem 1.6. *Let (A,Θ_A) be a $(g-1)$ dimensional abelian variety with a polarisation of type $\delta = (1,\ldots,1,d)$. If $d \geq 3$ there is a family*

$$(j'(A,d) : J'(A,d) \to X'(d), \quad \Theta_{J'(A,d)})$$

of g-dimensional, principally polarised abelian varieties with an injection

$$i_{X'(d)} : A \times X'(d) \to J'(A,d)$$

over $X'(d)$, such that $i^*_{X'(d)}\Theta_{J'(A,d)} = pr^*_A\Theta_A$. The family $j'(A,d)$ is universal for principally polarised abelian varieties $(J/T,\Theta_J)$ with an injection $i_T : A \times T \to J$ over T, such that $i_T^*(\Theta_J) = \mathrm{pr}_A^*\,\Theta_A$.

In section 3 we will obtain the prototypes by applying a Torelli theorem to this universal family. As the Torelli map has no longer dense image in A_g for $g > 3$, our strategy is limited to $g \leq 3$. A relative version of Thm. 1.6 also holds true and will be needed in section 4.

1.2 Maximally irregular fibrations with $d = 2$ and $g = 2$ or $g = 3$, Isotriviality

The case $d = 2$ needs a separated treatment because (-1) does not act freely on $\mathbb{H}$ and hence Thm. 1.6 is false for $d = 2$. For $g = 2$ and $d = 2$ Xiao showed ([Xi85] Example 3.1) that these surfaces are double coverings of principal homogeneous spaces for $E \times B$, where E is the fixed part i.e. an elliptic curve.
We analyse now the case $g = 3$ and the possibility of a maximally irregular fibration to be isotrivial or to have a hyperelliptic generic fibre.

Proposition 1.7. *Let $f : X \to B$ be a maximally irregular fibration of type (g, b) with $g \geq 3$. If the generic fibre is hyperelliptic or if $X \to B$ is isotrivial, we have $g = 3$, $d = 2$ and the Albanese degree γ equals 2.*
Conversely if $f : X \to B$ has $g = 3$ and $\gamma = 2$, then f is isotrivial.

Proof: From [Pi89] or [Xi92] it follows that the image of the Albanese map of a hyperelliptic fibration is a product of the base curve and a curve C of genus $g-1$. Riemann-Hurwitz implies that $g = 3$ and that each fibre is an unramified double cover of C, hence $\gamma = 2$. By [LB92] Theorem 12.3.3 we have $d = 2$. If the fibration is isotrivial, Prop. 2.2 of [Ser96] implies that the fibre of the Albanese image has dimension $g - 1$ and we conclude as above.
For the converse note that $\gamma = 2$ implies that each fibre of f is an unramified double cover of the same curve of genus 2. $\square$

See [Ser96] or [Ca00] for more on isotrivial fibrations, which we exclude from now on. For section 4 we note that isotrivial fibrations of maximal irregularity form components of the moduli space, because γ is constant on connected components.
We will fix the 4 complements $\Gamma(2)_S$ of ± 1 in $\Gamma(2)$. Thereby the index S denotes the set of irregular cusps (see [Sh71]) of $\mathbb{H}/\Gamma(2)_S$, i.e. S is a subset of $\{0, 1, \infty\}$ of order 1 or 3.
Let (A, Θ_A) be a $(1, 2)$-polarised abelian surface without non-trivial principal polarised abelian subvariety. We thus obtain a (certainly not good) prototype also in this case:

Theorem 1.8. *For each S as above there is a fibred surface*

$$h(A, 2)_S : S(A, 2)_S \to \mathbb{P}^1$$

which is a prototype in the following sense:
For each family of surfaces $X \to B \to T$ of type $(3, b)$ with maximal irregularity, $d = 2$, $\gamma = 1$ and of type (A, Θ_A), there is an index set S and a unique morphism $\varphi : B \to \mathbb{P}^1$, such that X is the canonical model of $S(A, 2)_S \times_{\mathbb{P}^1} B$.

2 Moduli spaces for abelian varieties with a fixed injection

We fix an injection of abelian varieties $i_A : A \to J$, where J is principally polarised and we consider the case $\dim J = g$, and $\dim A = g-1$. Then there is a complementary abelian

subvariety $E = \mathrm{Ker}(i^\vee : J \to A^\vee)$ of dimension one, where $A^\vee$ is the dual abelian variety. This kernel is indeed connected (see [LB92] Section 12.1) and we denote by $i_E : E \to J$ the inclusion. By loc. cit. Cor. 12.1.5 there is a d such that the polarisations $\Theta_A = i_A^*\Theta_J$ and $\Theta_E = i_E^*\Theta_J$ are of type $(1,\dots,1,d)$ and (d) respectively. If i_A comes from a fibred surface, d coincides with the associated degree in Def. 1.1.
Let $V = H^0(J,\Omega_J)^\vee$ and $U = H_1(J,\mathbb{Z})$ be the uniformisation of $J = V/U$. Further let $V_A = H^0(A,\Omega_A)^\vee$ and $V_E = H^0(E,\Omega_E)^\vee$ and consider them as subspaces of V via i_A and i_E. We then obtain $U_A := H_1(A,\mathbb{Z}) = V_A \cap U$ and $U_E := H_1(E,\mathbb{Z}) = V_E \cap U$.

Lemma 2.1. *In the above situation, there exists a basis of homology adapted to the injections i_A and i_E ('adapted basis' for short)*

$$B = \{u_1,\dots,u_g,u_{g+1},\dots,u_{2g-2},u_{2g+1},u_{2g+2}\}$$

of U, i.e. a basis, such that

$$u_1,\dots u_{g-1},u_{g+1},\dots,u_{2g-1} = du_{2g+1} - u_g$$

is a symplectic basis of U_A and

$$u_g, u_{2g} = du_{2g+2} - u_{g-1}$$

is a symplectic basis of U_E.

Proof: We denote the alternating form on U induced by Θ_J by $(\cdot,\cdot)$. The polarisations Θ_A and Θ_E induce alternating forms, which are by construction the restriction of $(\cdot,\cdot)$ to U_A and U_E. Denote the elements of a symplectic basis of $U_A \oplus U_E$ of type $(1,\dots,1,d,d)$ by $u_1,\dots,u_{g-2},a_1,a_2,u_{g+1},\dots,u_{2g-2},b_1,b_2$. The projection $p : V \to V/V_A$ induces an isomorphism $U/(U_A \oplus U_E) \to p(U)/p(U_E) \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z}$. Let $v_1, v_2 \in U$ be generators of this quotient. The type of the restricted alternating form implies that d divides all of (d_1v_1,a_i), (d_1v_1,b_i), (d_2v_2,a_i) and (d_2v_2,b_i) for $i = 1,2$. As neither v_1 nor v_2 is orthogonal to $\langle a_1,a_2,b_1,b_2\rangle$ this is only possible if $d = d_1 = d_2$.
Hence we find $a, a' \in U_A$ and $b, b' \in U_E$ such that $v_1 = (a+b)/d$ and $v_2 = (a'+b')/d$. The images $p(v_1), p(v_2)$ generate $p(U)/p(U_E)$ and thus $\{b,b'\}$ is, changing the order if necessary, a symplectic basis of U_E. Now U_A and U_E are orthogonal with respect to $(\cdot,\cdot)$ and $\{u_1,\dots,u_{g-2},u_{g+1},\dots,u_{2g-2},a,a',b,b'\}$ is also a basis of $U_A \oplus U_E$. The determinant of $(\cdot,\cdot)$ restricted to $U_A \oplus U_E$ is d^2 and together this implies $(a,a') = d$. Hence we can take $u_{g-1} = a'$, $u_g = b$, $u_{2g+1} = v_1$ and $u_{2g+2} = v_2$. □

A similarly constructed basis for $g = 2$ is called 'base homologuée' in [Xi85].

We now check how far this basis of homology is from being unique.

Lemma 2.2. *Let B be an adapted basis of U and let*

$$(u'_g, u'_{2g})^T = M(u_g,u_{2g})^T, \qquad M \in SL_2(\mathbb{Z}).$$

If we let $u'_{2g+1} = \frac{1}{d}(u'_g + u_{2g-1})$ and $u'_{2g+2} = \frac{1}{d}(u'_{2g} + u_{g-1})$, then

$$B' = \{u_1,\dots,u_{g-1},u'_g,u_{g+1},\dots,u_{2g-2},u'_{2g+1},u'_{2g+2}\}$$

is another adapted basis U, if and only if

$$M \in \Gamma(d) = \{M \in SL_2(\mathbb{Z}), \quad M \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod d\}.$$

Proof: This change of basis is independent of the elements $u_1, \ldots, u_{g-2}, u_{g+1}, \ldots u_{2g-2}$. We can hence copy the proof of Lemme 3.7 in [Xi85] literally. □

We now start the construction the family $j'(A,d)$ of Thm. 1.6. With respect to a symplectic basis the period matrix of A is of the form

$$\begin{pmatrix} z_{11} & \ldots & z_{1\,g-1} & 1 & \ldots & 0 & 0 \\ \vdots & & \vdots & & \ddots & & \\ z_{g-1\,1} & \ldots & z_{g-1\,g-1} & 0 & \ldots & 0 & d \end{pmatrix}^t = (Z, \mathrm{diag}(\delta))^{\mathrm{t}},$$

where $\delta = (1, \ldots, 1, d)$, for some Z in the Siegel half space $\mathbb{H}_{g-1}$. Let

$$p\colon\ \mathcal{V}\colon = \mathbb{H} \times \mathbb{C}^g \to \mathbb{H}$$

be the trivial vector bundle over the upper half plane $\mathbb{H}$. For $z \in \mathbb{H}$ the sections

$$\begin{array}{lclllcll} u_r(z) & = & (z_{r1}, \ldots, z_{r\,g-1}, 0) & u_{g+r}(z) & = & (e_r, 0) & \text{for}\, r = 1, \ldots, g-1 \\ u_g(z) & = & (0, \ldots, 0, 0, z) & u_{2g}(z) & = & (0_{g-1}, 1) & \\ u_{2g+1}(z) & = & \frac{1}{d}(0, \ldots, 0, d, z) & u_{2g+2}(z) & = & \frac{1}{d}(z_{g-1}, d) & \end{array}$$

define a lattice $U(z)$ in $\mathcal{V}$, where e_r are the rows of $\mathrm{diag}(\delta)$. The quotients $J(z) = \mathcal{V}/U(z)$ are a family $J(z) \to \mathbb{H}$ of complex tori. They admit an alternating form of type $\delta' = (1, \ldots, 1, d, d)$ with respect to $u_1, \ldots, u_{2g}$. This defines a principal polarisation on $J(z)$, which is thus a family of abelian varieties. The injection of $V_A = \mathbb{C}^{g-1}$ into the first $g-1$ components and the injection of the lattices

$$\langle u_1(z), \ldots, u_{g-1}(z), u_{g+1}(z), \ldots, u_{2g-1}(z)\rangle \hookrightarrow U(z)$$

defines an injection of the trivial family $A \times \mathbb{H}$ into $J(z)$ and by construction the restriction of the principal polarisation to this lattice is of the right type. Due to Lemma 2.2 we can define a $\Gamma(d)$-action on this family, which respects this injection: $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma(d)$ acts on $\mathbb{H}$ as usual and we let

$$\begin{array}{lcll} Mu_i & = & u_i & \text{for}\, i = 1, \ldots, g-1, g+1, \ldots, 2g-1 \\ Mu_g(z) & = & \delta u_g(Mz) - \beta u_{2g}(Mz) & = (0_{g-1}, z/\gamma z + \delta) \\ Mu_{2g}(z) & = & -\gamma u_g(Mz) + \alpha u_{2g}(Mz) & = (0_{g-1}, 1/\gamma z + \delta). \end{array}$$

By letting $u_{2g+1} = \frac{1}{d}(u_{2g-1} + u_g)$ and $u_{2g+2} = \frac{1}{d}(u_{g-1} + u_{2g})$ we have defined a $\Gamma(d)$-action on U such that $M : J(z) \to J(Mz)$ is an isomorphism over A.

Proof of Thm. 1.6: For $d \geq 3$ the group $\Gamma(d)$ acts freely on $\mathbb{H}$, so by taking quotients we obtain the desired family

$$j'(A,d) : J'(A,d) \to X'(d)$$

of principally polarised abelian varieties. It remains to show the universal property.
Let $(j : J \to T, \Theta_J)$ and i_A be as in the statement of the theorem and let $\pi : E \to T$ with $i_E : E \to J$ be the family of complementary abelian varieties. Let $R_1\pi_*\mathbb{Z}$ denote the local system in the complex topology with fibres $H_1(E_t, \mathbb{Z})$. Over a sufficiently small complex chart T_i the $R_1\pi_*\mathbb{Z}$ and $R_1j_*\mathbb{Z}$ are free. Hence we can find sections u_g^i and u_{2g}^i in $(R_1\pi_*\mathbb{Z})(T_i)$ that form a symplectic basis which can be completed in $(R_1j_*\mathbb{Z})(T_i)$ to sections $u_1^i, \ldots u_{2g+2}^i$ with the above properties. We can map these sections to $(j_*\Omega_J)^\vee$ and project them modulo $i_A((\mathrm{pr}_{i*}\Omega_A)^\vee)$ (where $\mathrm{pr}_i : A \times T_i \to T_i$ is the projection). The quotient of the two projected sections defines a map $\varphi_i : T_i \to \mathbb{H}$. Given another chart T_k we can find sections u_g^k and u_{2g}^k which define $\varphi_k : T_k \to \mathbb{H}$. By the above lemma the morphisms $T_i \to \mathbb{H}/\Gamma(d)$ glue to the desired morphism $\varphi : T \to \mathbb{H}/\Gamma(d)$.
We finally remark, that all these complex spaces are in fact algebraic, see [LB92] Section 10.8. □

We will now calculate the monodromy of $j'(A, d)$. For that purpose we fix the following symplectic basis of $H_1(J, \mathbb{Z})$ (where J is a fibre of $j'(A, d)$):

$$\alpha_r = u_r \ (r = 1, \ldots, g-2),\ \alpha_{g-1} = u_{2g+2},\ \alpha_g = u_{2g+1},\ \beta_r = u_{g+r} \ (r = 1, \ldots, g)$$

The period matrix $T(z)$ of J with respect to this basis is

$$T(z) = \begin{pmatrix} Z & Z_{12} \\ Z_{21} & z \end{pmatrix}, \quad \text{where } Z_{21} = (0, \ldots, 0, 1),\ Z_{12} = Z_{21}^T.$$

Lemma 2.3. *The monodromy of $j'(A, d)$ along a path γ around ∞ is*

$$M(\gamma) = I_{2g} + E_{g,2g}$$

Proof: The path γ lifts to a path from z to $z + d$ in $\mathbb{H}$. The result follows from noting that $M(\gamma)$ satisfies $T(z + d) = M(\gamma) \cdot T(z)$ (see [Na74]). □

Remark 2.4. As the monodromy is unipotent, [FC90] Th. V.6.7 shows that there is a unique extension of $j'(A, d)$ to a semi-abelian scheme over $X(d)$, which we denote by $j(A, d) : J(A, d) \to X(d)$.

Remark 2.5. The choice of a symplectic basis of $H_1(J, \mathbb{Z})$ as above defines an injection $\phi : \Gamma(d) \to Sp(g, \mathbb{Z})$ into the symplectic group and an injection $\mathbb{H} \to \mathbb{H}_g$ equivariant with respect to ϕ and the natural actions of these groups. The moduli map of the family $j'(A, d)$ is hence an immersion

$$m' : X'(d) \to A_g.$$

To prove the prototype theorems, we need a version of this map using fine moduli spaces. Fix a level-$[n]$-structure on the abelian varieties parametrised by $X(d)$. Writing down ϕ explicitely, we see that a level-$[n]$-structure on $J(z) \to \mathbb{H}$ is invariant under $M \in \Gamma(d)$, if M is actually in $\Gamma(nd)$. Let $X_n'(d)$ denote $\mathbb{H}/\Gamma(nd)$. This curve equals of course $X'(nd)$, we just want to emphasize its different role here. Let $j_n'(A, d) : J_n'(A, d) \to X_n'(d)$ be the pullback of $J'(A,d)$. $X'_n(d)$ is the moduli space of abelian threefolds with an injection by A plus the level-$[n]$-structure. Now the moduli map of $j'_n(A,d)$ is an immersion

$$m'_n : X'_n(d) \to A_g^{[n]}$$

we were heading for. Note that both m' and m'_n depend on A.

Lemma 2.6. *For $g=3$ and A without non-trivial principally polarised subvariety almost all fibres of $j'(A,d)$ are not reducible as abelian varieties with polarisation.*
If moreover A does not contain any elliptic curve then none of the fibres of $j'(A,d)$ is reducible as abelian variety with polarisation.

Proof: We first prove the second statement: Suppose the contrary is the case for $t \in X(d)$, i.e.

$$(J_t, \Theta_{J_t}) \cong (A' \times E', p_1^*\Theta_{A'} + p_2\Theta_{E'}).$$

By hypothesis $p_2 \circ i_A : A \to E'$ has to be the zero map. Hence the injectivity of i_A implies that it is actually an isomorphism. But then $i_A^*\Theta_A$ is a principal polarisation, not of type $(1,d)$.

Suppose under the hypothesis of the first statement that we have for $t \in X(d)$ a splitting as above. Consider the inclusion $i_{E_t} : E_t \to J_t$ of the complementary abelian variety. If $p_2 \circ i_{E_t} : E_t \to E'$ was zero the image of $i_A^\vee|_{E'}$ would be a principally polarized subvariety of A. Otherwise $i_{E_t}^\vee p_2 \circ i_A$ gives an isogeny of an elliptic curve in A to E_t. This cannot happen for all $t \in X(d)$. Hence it happens only a finite number of times. $\square$

Remark 2.7. If A contains an elliptic curve E there may be fibres of $j'(A,d)$, that are reducible with polarisation.
To see this take (A, Θ_0) principally polarised, containing an elliptic curve E but irreducible with polarisation. Let $d' = \deg(\Theta_0|_E)$. Provide E with a principal polarisation Θ_E and let $i^\vee : A \to E$ be the dual of the inclusion. Then

$$(id_A \times i^\vee) : A \to J := A \times E$$

is injective and the pullback of the principal polarisation $p_1^*\Theta_0 \otimes p_2^*\Theta_E$ is a polarization of type $(1, d'+1)$ on A.

3 Proof of the prototype theorems

To prove Thm. 1.4 we want to apply a Torelli theorem to the family constructed in Thm. 1.6. The fact that the map $i : M_g \to A_g$ from the moduli space of curves to the moduli space of abelian varieties is an isomorphism (see [OS80]) is not sufficient here, because we have to pull back the universal families. We therefore use auxiliary level-$[n]$-structures. Let $g \geq 3$, $n \geq 3$ and Σ be the involution on $M_g^{[n]}$, sending the level structure

α to $-\alpha$. We call $V_g^{[n]}$ the quotient by this involution.

$$\begin{tikzcd} M_g^{[n]} \arrow[d, "q"] \arrow[dr, "i^{[n]}"] & \\ V_g^{[n]} \arrow[r, "i_V^{[n]}"] \arrow[d] & A_g^{[n]} \arrow[d] \\ M_g \arrow[r, "i"] & A_g \end{tikzcd}$$

Let $H \subset M_g$ be the hyperelliptic locus and H_V its preimage in $V_g^{[n]}$. The map q is ramified exactly over H_V; over $V_g^{[n]} \smallsetminus H_V$ there is a universal family of curves.

Theorem 3.1. *([OS80] Thm. 3.1) The map $i_V^{[n]}$ is an embedding.*

Proposition 3.2. *Let $h_i : C_i \to T$ for $i = 1, 2$ be families of smooth curves of genus $g \geq 2$ without hyperelliptic fibres, with sections s_i and induced embeddings $f_i : C_i \to \mathrm{Jac}_{C_i/T}$. If $\beta : \mathrm{Jac}_{C_1/T} \to \mathrm{Jac}_{C_2/T}$ is an isomorphism, there exists a unique isomorphism $\alpha : C_1 \to C_2$ and a translation t_c, such that*

$$f_2 \circ \alpha = t_c \circ \beta \circ f_1.$$

Proof: For $T = \operatorname{Spec} \mathbb{C}$ this is [Mi86] Theorem 12.1. If T is the spectrum a local Artinian ring, [OS80] Proposition 2.5 implies the existence of the isomorphism, which is unique on the special fibre and therefore unique, because curves of genus ≥ 2 do not have infinitesimal automorphisms. The general statement now follows thanks to the uniqueness by descent. $\square$

Corollary 3.3. *Let $j : J \to T$ be a principally polarised abelian scheme of dimension $g \geq 3$, such that the image of the induced mapping $\varphi : T \to A_g$ is contained in $i(M_g \smallsetminus H)$. Then there is a unique family of curves $h : C \to T$, whose Jacobian is, after an appropriate faithfully flat base change, isomorphic to J.*

Proof: We make a base change $\tilde{T} \to T$ such that $\tilde{j} : \tilde{J} = J \times_T \tilde{T} \to \tilde{T}$ admits a level-$[n]$-structure in order to use the above results. Fixing a level structure and letting $\tilde{\varphi} : \tilde{T} \to A_g^{[n]}$, we obtain a family of curves $\tilde{h} : \tilde{C} \to \tilde{T}$ by pulling back the universal family over $V_g^{[n]} \smallsetminus H_V$ via $(i_V^{[n]})^{-1} \circ \tilde{\varphi}$. Fix an isomorphism $\mathrm{Jac}_{\tilde{C}/\tilde{T}} \to \tilde{J}$ (which is unique only up to ± 1) and suppose $\tilde{h}$ admits a section $\tilde{s}$. We can assure this by making another base change. $\tilde{s}$ induces an embedding $\tilde{f} : \tilde{C} \to \mathrm{Jac}_{\tilde{C}/\tilde{T}}$. The natural descent data on $\tilde{J}$ give an isomorphism $\beta_{12} : \mathrm{pr}_1^* \mathrm{Jac}_{\tilde{C}/\tilde{T}} \to \mathrm{pr}_2^* \mathrm{Jac}_{\tilde{C}/\tilde{T}}$, where as usual pr_i denotes the projections $\tilde{T} \times_T \tilde{T} \to \tilde{T}$. By the above proposition we obtain an isomorphism α_{12} between the pullbacks of the families of curves, commuting with β_{12} up to translation. It satisfies the cocycle condition, because the descent data on $\tilde{J}$ do so. As curves of genus ≥ 2 come along with an ample canonical sheaf, the descent data are effective (see e.g. [BLR90] Theorem 6.7).

It remains to show uniqueness: Let $C_i \to T$ for $i = 1, 2$ be two families of curves with the desired property. Again after base change $\tilde{T} \to T$ we have isomorphisms $\lambda_i : \mathrm{Jac}_{\tilde{C}_i/\tilde{T}} \to \tilde{J}$, unique up to ± 1. Fixing them, the cocycle condition on $\mathrm{Jac}_{\tilde{C}_i/\tilde{T}}$ and $\tilde{J}$ implies that $\lambda_2^{-1} \circ \lambda_1$ respects the descent data. By the above proposition this gives a unique isomorphism between $\tilde{C}_1$ and $\tilde{C}_2$, commuting with the embeddings and $\lambda_2^{-1} \circ \lambda_1$ up to translation. Hence this isomorphism descends to the one between $C_1 \to C_2$ we sought. $\square$

Remark 3.4. Letting $p : \tilde{T} \times_T \tilde{T} \to T$ the morphism β_{12} in the above proof is induced by the morphism $p^*C \to p^*C$ coming from the exchange of factors only up to ± 1 . Lemma 3.6 gives an example where β_{12} does indeed not coincide with this morphism. And this Lemma also reveals that we cannot replace 'faithfully flat' by 'étale' in the above corollary.

Proof of Thm. 1.4: We check (Step 1) that the image of the moduli map m' lies generically in the image of $i(M_g \setminus H)$. We then (Step 2) apply Cor. 3.3 to pullback the family $j(d)$ via i and complete it to a prototype $\mathcal{S}(A,d) \to X(d)$. We show that the prototype base change φ is unique. In Step 3 we provide all families with auxiliary level-$[n]$-structures. The double covering $M_g^{[n]} \to V_g^{[n]}$ will induces a double covering of $X_n'(d)$. When we finally divide out the level-structures we obtain the double covering $B(A,d) \to X(d)$ and a good prototype $S(A,d) \to B(A,d)$. It will turn out in Cor. 3.7 that the double covering is ramified, in particular connected, hence that $\mathcal{S}(A,d) \to X(d)$ was indeed not a good prototype. For simplification we omit the dependence on (A,d) from notation.

Step 1: The image of the moduli map $m_n' : X_n'(d) \to A_g^{[n]}$ lies generically in the image of $i_V^{[n]}$ by Lemma 2.6. Let $B_n' \to X_n(d)' \times_{V_3^{[n]}} M_3^{[n]}$ be the normalisation of the fibre product, in which the first map is $(i_V^{[n]})^{-1} \circ m_n'$ and let B_n be the smooth completion of B_n'. If $m'(X'(d))$ was contained in the image $i(H)$ of the hyperelliptic locus, the pullback of the universal family over $M_3^{[n]}$ to B_n' would have hyperelliptic generic fibre. By construction its completion to a fibred surface over B_n has maximal irregularity and $d \geq 3$. This contradicts Prop. 1.7.

Step 2: We now apply Cor. 3.3 and obtain a family of curves over the non-hyperelliptic locus of $X'(d)$. The relatively minimal model of a completion is denoted by $h : \mathcal{S} \to X(d)$. To prove the universal property of the prototype, let $X \to C \to T$ be a fibred family of surfaces with fixed part $A \times T$. Let C' the locus with smooth fibres. As explained in Section 1 we obtain an injection

$$i_A : A \times_T C' \hookrightarrow \mathrm{Jac}_{X'/C'}.$$

By Theorem 1.6 we obtain a morphism $\varphi' : C' \to X'(d)$ which we can extend to $\varphi : C \to X(d)$, because C is smooth. φ is onto, since otherwise for any $t \in T$ the Jacobian $\mathrm{Jac}_{X_t'/C_t'}$ would be a product and $X_t \to C_t$ isotrivial. By Prop. 1.7 this contradicts $d \geq 3$. The birational equivalence of X and $C \times_{X(d)} S(A,d)$ is now nothing but the uniqueness assertion of Corollary 3.3.
To prove the uniqueness of φ, let $P \in C(\mathbb{C})$ be such that the fibre F_P is smooth. The composition $m' \circ \varphi$ maps P to the point in A_3 corresponding to the isomorphism class of Jac_{F_P}. The uniqueness now follows from the injectivity of m (Remark 2.5).

Step 3: Consider again the curve B'_n defined in the first step and let $h'_n : S'_n \to B'_n$ be the pullback of the universal family over $M_3^{[n]}$

Via ϕ (see Remark 2.5) the factor group $G = \Gamma(d)/\Gamma(nd)$ acts without fixed points on $X'_n(d)$, on $M_3^{[n]}$ and thus on B'_n. G also acts on the universal family and hence on its pullback to B'_n. The quotient of the pullback by G, denoted by $h' : S' \to B'$ does no longer depend on any choices (level structure, symplectic basis) made above. We now check that the completion to a relatively minimal model $h(A,d) : S(A,d) \to B(A,d)$ has the desired properties.

By construction $\mathrm{Jac}_{S'_n/B'_n}$ is globally isomorphic to $J'_n \times_{X'_n(d)} B'_n$. This isomorphism is G-invariant and so $\mathrm{Jac}_{S'/B'} \cong J' \times_{X'(d)} B'$. This says that $h : X \to B$ has maximal irregularity. Hence it will be a good prototype, once we have shown that given a fibred family of surfaces $X \to C \to T$, the morphism $\varphi : C \to X(d)$ constructed in step 2 factors via $B(A,d)$.

After a suitable étale base change $\tilde{C} \to C$ the family $\tilde{X} \to \tilde{C}$ admits a level-$[n]$-structure and the morphisms to $\tilde{C} \to \overline{M_3}^{[n]}$ and $\tilde{C} \to X_n(d)$ together with the smoothness of $\tilde{C}$ give a morphism $\varphi_n : \tilde{C} \to B_n(A,d)$. The composition of φ_n with the projection $B_n(A,d) \to B(A,d)$ is independent of the choice of the level structure and hence gives the desired factorisation.

The uniqueness of φ up to σ follows from the uniqueness statement for the family over $X(d)$. □

We now come back to the case $d = 2$ and $\gamma = 1$.

Proof of Thm. 1.8: For each set of irregular cusps S we can proceed as in section 2 to construct families of principally polarised abelian varieties of dimension 3

$$j'(A,2)_S : J(A,d)_S \to X'(2)_S,$$

where of course $X'(2)_S \cong X'(2) \cong \mathbb{P}^1 \smallsetminus \{0,1,\infty\}$ (as schemes, not as quotient stacks). We claim that over a connected base T each principally polarised abelian scheme $(J/T, \Theta_J)$ with an injection $A \times T \to J$ over T satisfying $i_A^*(\Theta_J) = \mathrm{pr}_A^*(\Theta_A)$ is the pullback of $j'(A,2)_S$ for a suitable S.

In fact we can construct locally on charts T_i as in the proof of Thm. 1.6 sections u_g^i and u_{2g}^i, whose quotient in V/V_A gives a morphism $T_i \to \mathbb{H}$. Similar sections u_g^k and u_{2g}^k on another chart T_k differ on $T_i \cap T_k$ form these by the action of $M \in \Gamma(2)$, according to Lemma 2.2. But the normalisation to the upper half plane implies that M belongs to one of the complements of ± 1.

To prove the existence of the prototype we proceed as in Thm. 1.4: we take an auxiliary level structure and apply Corollary 3.3 to obtain $h(A,2)_S$.

For the prototype property we can also conclude as above, once we have shown that we can distinguish the abelian schemes $j(A,2)_S$ by its monodromy around the cusps. Take w.l.o.g. the cusp at ∞. In the regular and irregular case we have respectively

$$\mathrm{Stab}_{\Gamma(2)_S}(\infty) = \langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \rangle, \qquad \text{hence} \quad M(\gamma) = I_6 + E_{3,6}$$

$$\mathrm{Stab}_{\Gamma(2)_S}(\infty) = \langle \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \rangle, \qquad \text{hence} \quad M(\gamma) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

These monodromy matrices are not conjugate in the symplectic group $Sp(3, \mathbb{Z})$ and this proves the claim. □

Invariants of prototypes

To calculate the invariants of these prototypes for $d \geq 3$ and for irreducible A, we need to know, where $B(A, d) \to X(d)$ is actually ramified.

We say that a point $P \in X(d)$ is *hyperelliptic*, if the image of the morphism $m : X(d) \to \overline{M_3}$, which extends $i^{-1} \circ m'$, is in the closure of the hyperelliptic locus. This depends of course on the abelian variety A. We use the same terminology for the morphism $m_n : X_n(d) \to \overline{V_3^{[n]}}$. We first give a proof of Proposition 3.13 in [Na74], which we have not been able to find in the literature.

Proposition 3.5. *Let $X \to B$ be a family of curves over a one-dimensional base B, smooth outside $P \in B$. If the monodromy around P is unipotent, X admits a stable model over B (i.e. without base change).*

Proof: Let $B' = B \smallsetminus P$. By [FC90] Theorem V.6.7 the monodromy hypothesis implies that $\mathrm{Jac}_{X'/B'}$ has an extension to a semi-abelian scheme $J \to B$, which is unique by [FC90] Proposition I.2.7. Take a cyclic cover $\tilde{B} \to B$ totally ramified over P, with covering group generated by σ, such that $\tilde{h} : \tilde{X} = X \times_B \tilde{B} \to \tilde{B}$ admits a semistable model. Thus $\mathrm{Jac}_{\tilde{X}/\tilde{B}} = J \times_B \tilde{B}$, and σ acts trivially on the fibre over P of this abelian scheme. This means that σ acts trivially on the fibre of $\tilde{h}$ over P and the claim follows. □

Lemma 3.6. *m_n does not factor in any complex neighbourhood of a hyperelliptic point via $\overline{M_3}^{[n]}$. The image of $B_n(A, d) \to \overline{M_3}^{[n]}$ intersects the hyperelliptic locus transversally.*

Proof: Let $P \in B_n(A, d)$ lie over a hyperelliptic point of $X_n(d)$. Due to the injectivity of m_n both claims follow, once we have shown that the image of $\mathrm{pr}_2 : B_n(A, d) \to \overline{M_3}^{[n]}$ intersects the hyperelliptic locus transversally at $\mathrm{pr}_2(P)$ and that the tangent space of $\mathrm{pr}_2(B_n(A, d))$ at $\mathrm{pr}_2(P)$ is invariant under the involution Σ.

Take a punctured neighbourhood U' of P and let $h_{U'}$ be the pullback of $h(A, d)$ to U'. By construction the Jacobian of $h_{U'}$ is $J_{U'} = J(A, d) \times_{X(d)} U'$ and the monodromy around

P is unipotent by Lemma 2.3. By the above proposition we can extend $h_{U'}$ to a stable family of curves over $U = U' \cup \{P\}$. Let C be the fibre of h_U over P. We distinguish whether P maps to $X'_n(d)$ (first case) or to a cusp (second case).
Denote by $p : J_{U'} \to A^\vee \times U'$ the dual of the inclusion i_A of the fixed part. Consider how the curve $p(C)$ is deformed on $A^\vee$ instead of just considering abstract deformations. In the first case the geometric genus of $p(C)$ is 3, because otherwise $d = 2$. As the canonical bundle on $A^\vee$ is trivial, first order deformations of the normalisation of $p(C)$ (or equivalently: of the pair (C, p)) are parametrised by $H^0(C, K_C)$ (see [Ta84], section before Lemma 1.5). The Kodaira-Spencer-mapping hence induces the following commuting triangle, equivariant with respect to the action of the hyperelliptic involution:

$$\begin{tikzcd} T_{B_n(A,d),P} \arrow[rr, "\kappa_{h,P}"] \arrow[dr, "\kappa_P"'] & & H^1(C, T_C) \\ & H^0(C, K_C) \arrow[ur, "\delta"'] & \end{tikzcd}$$

δ stems from the connecting homomorphism of

$$0 \to T_C \to T_A|_C \to K_C \to 0.$$

The map κ_P is not zero because J_U is not a trivial deformation of Jac_C in any neighbourhood of P, as one can see already by using the family $J(z) \to \mathbb{H}$. The hyperelliptic involution acts on $H^0(C, K_C)$ as (-1), and so the image of $\kappa_{h,P}$ lies in the (1-dimensional) eigenspace of (-1) of $H^1(C, T_C)$.
The other case works essentially the same way: If $\pi : \tilde{C} \to C$ is the normalisation, first order deformations of the pair $(\tilde{C}, p \circ \pi)$ are parametrised by $H^0(\tilde{C}, K_{\tilde{C}})$ (see [Ta84] Lemma 1.5 and Remark 1.6). We thereby use the fact that C is stable and therefore the ramification divisor of π is trivial. On $H^0(\tilde{C}, K_{\tilde{C}})$ the hyperelliptic involution still acts as (-1) and we conclude as above. $\square$

Corollary 3.7. *The twofold covering $B(A, d) \to X(d)$ is ramified precisely over the hyperelliptic points of $X(d)$.*

Let $t(d)$ be the number of cusps of $X(d)$ and $s_g(A, d)$ the number of points where $J(A, d)$ is proper, but reducible for $g = 2$ or $g = 3$ respectively. Defining

$$\Delta_d = \frac{d^2}{24} \prod_{p|d} (1 - \frac{1}{p^2}),$$

we know from [Sh71]:

$$g(X(d)) = (d - 6)\Delta_d + 1$$
$$t(d) = 12\Delta_d.$$

We first sum up Xiao's results ([Xi85]) for $g = 2$:

Corollary 3.8. *For $g = 2$, $s_2(d) = s_2(A,d)$ does not depend on A. More precisely:*

$$\begin{array}{lcl} s(d) & = & (5d-6)\Delta_d \\ c_2(S(A,d)) & = & s(d)+t(d)+4g(X(d))-4 \;=\; (9d-18)\Delta_d \\ \chi(\mathcal{O}_{S(A,d)}) & = & 2g(X(d))-2+\frac{1}{2}t(d) \;=\; (2d-6)\Delta_d \\ K^2_{S(A,d)} & = & 6\chi(\mathcal{O}_{S(A,d)})+3g(X(d))-3 \;=\; (15d-54)\Delta_d \end{array}$$

The singular fibres of surfaces in $\mathfrak{S}^{\text{m.i.}}_{2,b}$ are two elliptic curves attached to each other at a node, if the base point maps in $X(d)$ to one of the $s_2(d)$ points corresponding to a proper but reducible Jacobian or an elliptic curve with a node, if the base point maps to a cusp of $X(d)$.

Sketch of Proof: First, in order to prove Theorem 1.5, the family of polarisation divisors of $j'(A,d)$ is already the family of curves $h'(A,d)$. Hence we have $\mathrm{Jac}_{S(A,d)/B(A,d)} \cong J(A,d)$ globally and no trouble with Torelli. The proof of the prototype property and the uniqueness of φ works as above.
The irreducible fibres can be recognised by calculating their monodromy as in Lemma 2.3 and comparing with the list in [NU73]. To calculate the invariants, the techniques are similar to the ones below. □

In the case $g = 3$ we obtain:

Corollary 3.9. *For $d \geq 3$ and A irreducible the genus of $B(A,d)$ and $s_3(d) = s_3(A,d)$ does not depend on A. Moreover we have:*

$$\begin{array}{lcl} s(d) & = & 0 \\ g(B(A,d)) & = & (20d-36)\Delta_d+1 \\ c_2(S(A,d)) & = & (160d-264)\Delta_d \\ \chi(\mathcal{O}_{S(A,d)}) & = & (42d-72)\Delta_d \\ K^2_{S(A,d)} & = & (344d-600)\Delta_d \end{array}$$

The only singular fibres of surfaces in $\mathfrak{S}^{\text{m.i.}}_{3,b}$ are curves of genus 2 with one node or a smooth curve of genus 2 together with a smooth $\mathbb{P}^1$, which have two transversal intersections. In particular $S(A,d) \to B(A,d)$ is semistable. The index of $S(A,d)$ is $\tau = (8d-24)\Delta_d$, hence positive for $d > 3$.

Proof: First, $s_3(A,d) = 0$ was shown in Lemma 2.6. For simplicity, we now drop (A,d) from the notation.
Secondly, we examine the fibres of $h : S \to B$. By Cor. 3.7 the map $B_n \to X_n(d)$ is of degree two with the hyperelliptic points as ramification locus. By Lemma 2.3 the monodromy around the cusps of $B(A,d)$ is $I_6 + 2E_{3,6}$ or $I_6 + E_{3,6}$, depending on whether the cusp is hyperelliptic or not. Thus by Proposition 3.5 h is semi-stable and the non-smooth fibres lie over the cusps. The period matrix calculated before Lemma 2.3 implies that the Jacobian of these fibres is an extension of an abelian surface by a torus. To prove the assertion, we must exclude that these fibres consist of a genus 2 curve attached to a $\mathbb{P}^1$ with a self-intersection. But in this case, the period matrix would have a block structure.

Thirdly we calculate the number of hyperelliptic fibres of h. For this purpose we use the following equation ([HM98] (3.165)):

$$H = 18\lambda - 2\delta_0 - 3\delta_1.$$

H is number of hyperelliptic fibres, λ is the degree $h_*\omega_{S/B}$ and δ_i is the number of curves belonging to the boundary components Δ_i (both H and δ_i have to be counted with multiplicity). Calling $j_B : J_B \to B$ the pullback of $j(A,d)$ to B, we have

$$\deg h_*\omega_{S/B} = -\deg(\bigwedge^3 R^1 j_{B*}\mathcal{O}_{J_B}) = -2\deg(\bigwedge^3 R^1 j(A,d)_*\mathcal{O}_{J(A,d)}).$$

We can evaluate the right-hand side as in [Xi85] Thm. 3.10 using modular forms and we obtain

$$\deg(\bigwedge^3 R^1 j(A,d)_*\mathcal{O}_{J(A,d)}) = -(g(X(d)) - 1 + t(d)/2) = -d\Delta_d.$$

The monodromy implies $\delta_0 = 2t(d)$ and we have seen that $\delta_1 = s_3(d) = 0$. We hence conclude

$$H = (36d - 48)\Delta_d$$

and by Lemma 3.6 this is the number of hyperelliptic fibres of h. This enables us to compute the genus of $B = B(A,d)$.
To determine $c_2(S(A,d))$ we let F be a generic fibre of h and use the formula

$$\sum_{F'\text{singular}} (\chi_{\text{top}}(F') - \chi_{\text{top}}(F)) = c_2(S(A,d)) - \chi_{\text{top}}(F)\cdot\chi_{\text{top}}(B(A,d)).$$

Note that the non-hyperelliptic singular fibres contribute by one to $\chi_{\text{top}}(F)$, the hyperelliptic fibres by two.
Finally, to determine $\chi(\mathcal{O}_S)$ remember that (by the Leray spectral sequence and Riemann-Roch)

$$\chi(\mathcal{O}_S) = \chi(\mathcal{O}_B) - \chi(R^1h_*\mathcal{O}_S) = 2(g(B)-1) - \deg(\bigwedge^3 R^1h_*\mathcal{O}_S).$$

Using the above calculation, this degree is $-2d\Delta_d$ and the claim follows. $\square$

Remark 3.10. If $\tilde{X} \to \tilde{B}$ is the minimal model of a surface in $\mathfrak{S}^{\text{m.i.}}_{3,\tilde{b}}(\mathbb{C})$ with a prototype base change $\varphi : \tilde{B} \to B(A,d)$ of degree n, we have

$$K^2_{\tilde{X}} = nK^2_{S(A,d)} + 8(\tilde{b} - 1 + nb - n).$$

If $P \in \tilde{B}$ lies over a cusp of $B(A,d)$ and ramifies to the order e, topology implies that

$$\chi_{\text{top}}(F_P) - \chi_{\text{top}}(F) = e$$

and by the above formulae we can determine completely the possible invariants of surfaces in $\mathfrak{S}^{\text{m.i.}}_{3,\tilde{b}}(\mathbb{C})$.

Remark 3.11. Let S be a projective surface of general type. Assume that S is of maximal Albanese dimension and the canonical map Φ_S is composite with a pencil. Note that then $q(S) = 2$. Suppose the generic fibre of the fibration $f : S \to C$ associated with Φ_S has genus 3.
By Thm. 1.4 and Cor. 3.9 the associated degree of f is 2. Hence such surfaces are classified by Theorem 2.8. Explicit examples of such surfaces are given in [Be79] Ex. 2 and [Xi87b] Ex. 3.

4 The components of the moduli space for maximally irregularly fibred surfaces

Families of fibred surfaces and fibred families of surfaces

Theorems by Beauville and Siu (for the base curve, [Be91], [Siu87])) and Catanese (for the fibre, [Ca00]) state that the property of having a fibration of type (g, b) is deformation invariant, if both $b \geq 2$ and $g \geq 2$. We call this the fibre genus condition (FGC) and base genus condition (BGC) respectively, and *we restrict ourselves from now on to such surfaces.* Due to these theorems we can state:

Definition 4.1. *We denote by $\mathfrak{S}_{g,b}(\cdot)$ the (open and closed) subfunctor of $\mathfrak{S}(\cdot)$ parametrising families of surfaces of general type such that each fibre admits a fibration of type (g, b). We denote by $N_{g,b}$ the union of the corresponding components of the moduli space.*

For the analysis of the moduli space of surfaces, the question remains, whether a family $X \to T$ of fibred surfaces in $\mathfrak{S}_{g,b}(T)$ is, what we call a *fibred family of surfaces*, that is if there is a family of curves $B \to T$ such that $X \to B \to T$ induces the fibrations of type (g, b). For regular fibrations, the Albanese mapping gives an affirmative answer. We now give a sufficient condition for families of irregularly fibred surfaces to be fibred in families and a numerical criterion for testing the condition.

Definition 4.2. *A family of surfaces $X \to T$ is said to have* at most one fibration, *if given two fibrations $h_i : X \to B_i$ over T, there is a T-isomorphism $\psi : B_1 \to B_2$ satisfying $\psi \circ h_1 = h_2$.*
The subfunctor $\mathfrak{S}_{g,b}(\cdot)$ of $\mathfrak{S}(\cdot)$ is said to have this property, if it is fulfilled for each T and each surface in $\mathfrak{S}_{g,b}(T)$.

Theorem 4.3. *If $\mathfrak{S}_{g,b}(\cdot)$ parametrises irregularly fibred surfaces of type (g, b) with at most one fibration, then there is a natural transformation $\mathfrak{S}_{g,b} \to \mathfrak{C}_b$ which (on complex points) sends each surface to the base of the fibration. Hence we have a natural morphism*

$$N_{g,b} \to M_b$$

between the corresponding coarse moduli spaces.

Proof: It is well known ([Ca91] Theorem 4.9 or [Ser92] Claim 5.1) that, given a fibration $h : X \to B$, one has a surjection of deformation functors $\mathrm{Def}_h \to \mathrm{Def}_X$, provided that

the cohomology group $H^0(R^1 h_* \mathcal{O}_X \otimes T_X)$ vanishes. This vanishing property follows from relative duality and a theorem of Fujita (see loc. cit.). This solves the problem for a local Artinian base. If the base is a complete local ring R, Grothendieck's algebraisation theorem ([EGA] III théorème 5.4.5) implies that the deformations over the Artinian quotients stem from a deformation of B over R.
For the general case, we cover the base T by the spectra of the completions of its local rings. As we suppose the base to be Noetherian, finitely many will be sufficient and we call the union of these schemes $\tilde{T}$. What we need is an (fppf-) descent datum on the base curve $\tilde{B} \to \tilde{T}$ obtained by the above deformation techniques. By (BGC) $\tilde{B}$ comes along with an ample canonical sheaf and the descent datum will automatically be effective ([BLR90] Theorem 6.1.7).
$\tilde{X} = X \times_T \tilde{T}$ has a natural morphism $\varphi : \mathrm{pr}_1^* \tilde{X} \to \mathrm{pr}_2^* \tilde{X}$, where pr_i are the usual projections $\tilde{T} \times_T \tilde{T} \to \tilde{T}$. Given the fibration $\tilde{h} : \tilde{X} \to \tilde{B}$, we can apply the hypothesis to $\mathrm{pr}_1^*(\tilde{h})$ and $\mathrm{pr}_1^*(\tilde{h}) \circ \psi$ to obtain $\psi : \mathrm{pr}_1^* \tilde{B} \to \mathrm{pr}_2^* \tilde{B}$. The map ψ obviously satisfies the cocycle condition, because φ does and because $\tilde{h}$ is surjective. □

Criterion 4.4. *A family of surfaces $X \to T \in \mathfrak{S}_{g,b}(T)$, whose fibres X_t satisfy*

$$K_{X_t}^2 > 4(g-1)^2,$$

has at most one fibration of type (g, b).

Proof: In case $T = \operatorname{Spec} \mathbb{C}$ this is [Xi85] Proposition 6.4. In general, suppose there are two fibrations $X \to B_i \to T$ $(i = 1, 2)$. Consider the product morphism $h : X \to B_1 \times_T B_2$. The image Y of h is flat over T by the criterion in [EGA] IV, 11.3.11. Hence the image is a family of curves. Consider the projections $\mathrm{pr}_i : Y \to B_i$. They are finite and isomorphisms for each closed point $t \in T$. Hence pr_i is étale (this is an open condition!) and radicial (by the case $T = \operatorname{Spec} \mathbb{C}$ and [EGA] I, 3.7.1). Now [EGA] IV, 17.9.1 proves that pr_i is an isomorphism and the criterion follows. □

The families of fibred surfaces which we investigate here satisfy the hypothesis of this criterion: Let $\mathfrak{S}_{g,b}^{\text{m.i.}}(\cdot)$ be the subfunctor of $\mathfrak{S}_{g,b}$ parametrising fibred surfaces with maximally irregularity. Accordingly, $N_{g,b}^{\text{m.i.}}$ denotes the union of the corresponding components of the moduli space.

Lemma 4.5. *A family of fibred surfaces in $\mathfrak{S}_{g,b}^{\text{m.i.}}(T)$ with $g = 2$ or $g = 3$, which is not isotrivial, has at most one fibration and is hence a fibred family.*

Proof: To apply Criterion 4.4 to $X \in \mathfrak{S}_{g,b}^{\text{m.i.}}(\mathbb{C})$, the Arakelov inequality

$$K_X^2 \geq 8(b-1)(g-1)$$

(see [Be82]) is sufficient for $g = 2$ thanks to (BGC). For $g = 3$ we have to exclude the case $b = 2$ and $K_X^2 = 16$. Using the well known relation of $K_X^2, \chi(\mathcal{O}_X)$ and the slope λ (see [Xi87a])

$$K_X^2 = \lambda\chi(\mathcal{O}_X) + (8-\lambda)(b-1)(g-1),$$

we deduce $\chi(\mathcal{O}_X) = 2$ and this implies that X is a fibre bundle (see again [Be82]), hence isotrivial. □

Theorem 1.6 in families

We need a relative version of this theorem obtained by letting Z vary in $\mathbb{H}_{g-1}$. Dividing out the $\Gamma(d)$-action on the family $J(z,Z) \to \mathbb{H} \times \mathbb{H}_{g-1}$ constructed in Section 2 (there with fixed Z), we obtain a family of principally polarised abelian varieties over $X'(d) \times \mathbb{H}_{g-1}$. For technical reasons we do not take the quotient of $\mathbb{H}_{g-1}$ by the whole symplectic group but fix a subgroup G, such that $\mathbb{H}_{g-1}/G$ is a moduli space for abelian varieties with a polarisation of type δ and a level-$[n]$-structure. The quotient by G,

$$j'(d) : J'(d) \to X'(d) \times A_{g-1,\delta}^{[n]},$$

is a family of polarised abelian varieties over the product of the modular curve and the moduli space of abelian varieties with level-$[n]$-structure. The same proof of Thm. 1.6 yields:

Corollary 4.6. *$j'(d)$ is the universal family of g-dimensional, principally polarised abelian schemes with an injection of an $(g-1)$-dimensional abelian scheme A endowed with level-$[n]$-structure, provided that $d \geq 3$.*

Components of the Gieseker moduli space

The results in the previous section suggest to describe $N_{g,b}^{\mathrm{m.i.}}$ by the fixed part of the fibration plus the morphisms of the base curve to $B(A,d)$. For that purpose, let $\mathcal{A}_{g-1,\delta}(\cdot)^f$ denote the subfunctor parametrising abelian schemes occurring as fixed parts of maximally irregular fibrations of type (g,b). According to the Theorems 1.5 and 1.4 this means no restriction for $g=2$ and no principally polarised 1-dimensional abelian subvarieties for $g=3$. We denote by $A_{g-1,\delta}^f$ the corresponding subscheme of $A_{g-1,\delta}$.

For a fixed curve D we need the following functor for coverings:

$$\mathfrak{C}_b(D,m)(T) = \{(B/T, \varphi : B \to D) \,\text{where}\, B \in \mathfrak{C}_b(T),\, \deg(\varphi) = m\}.$$

This is an open subfunctor of a stable mappings functor, that (according to [FuPe95]) has a coarse moduli space $M_b(D,m) \subseteq \overline{M_b}(D,m)$.

We can now use the prototype to prove a structure theorem for the moduli spaces $N_{g,b}^{m.i.}$:

Theorem 4.7. *The moduli space for maximally irregularly fibred surfaces of type (g,b) ($g \in \{2,3\}$) with associated degree $d \geq 3$ decomposes into components according to d and the degree m of the morphism $\varphi : B \to B(A,d)$ induced by the fibration $X \to B$.*
In case $g=2$ each such component is isomorphic to

$$M_b(X(d),m) \times A_1$$

and has dimension $2b-2-m(2g(X(d))-2)+1$.
In case $g=3$ each such component admits a morphism to $A_{2,\delta}^f$ with the fibre over A isomorphic to $M_b(B(A,d),m)/\sigma$, where σ comes from the involution of $B(A,d) \to X(d)$. The dimension of the moduli space is $2b-m(2g(B(A,d))-2)+3$.

Proof: Recall that by Lemma 4.5 each family of surfaces $X \to T$ in question is a fibred family $X \to B \to T$. The fixed part of this fibration induces a morphism $\psi : T \to A^f_{g-1,\delta}$. After an étale base change $\tilde{T} \to T$ we may suppose that the fixed part has a level-$[n]$-structure inducing $\tilde{\psi} : \tilde{T} \to (A^{[n]}_{g-1,\delta})^f$. We use the level-structure for a moment to apply Corollary 4.6. The morphism $i_A : A_{B'} = A \times_T B' \to \mathrm{Jac}_{X'/B'}$ constructed at the beginning of Section 1 gives $\varphi' : B' \times_T \tilde{T} \to X'(d)$. As B is smooth φ' extends to $B \times_T \tilde{T}$. By the universal property of $j'(d)$ this morphism is unique and therefore glues with the descent data on $\tilde{T}$ to give $\varphi : B \to X(d)$. In view of Theorem 1.4 we have shown, that the moduli functor admits a natural transformation to the product $\mathfrak{C}_b(X(d), m) \times \mathcal{A}_1$ in case $g = 2$ and in case $g = 3$ that it has a natural transformation to $\mathcal{A}^f_{2,\delta}$, whose fibres over A are in $\mathfrak{C}_b(B(A, d), m)$.
Conversely, given $(A, \Theta_A) \in A^f_{g-1,\delta}(\mathbb{C})$ and $\varphi \in M_b(B(A, d), m)$, the property 'good prototype' gives a fibred surface with the prescribed A, d, m and φ. If two fibrations $h_i : X_i \to B_i$ associated to (ϕ_i, A_i) are isomorphic, the fixed parts A_i are necessarily isomorphic as polarised abelian schemes. As the fibrations are unique by Lemma 4.5, we obtain an isomorphism $\iota : B_1 \to B_2$ and it remains to show that $\phi_2 \circ \iota = \phi_1$, resp. up to σ in the case $g = 3$. If this was wrong at $P \in X'(d)$, compare the Jacobians over the preimage in B_i to obtain a contradiction to the injectivity of m' (see Rem. 2.5).
The maximality among the coarsely representing spaces in the case $g = 2$ is easily checked by redoing this argument with a family A/T instead of A . In case $g = 3$, we only claimed such a statement for fixed $A \in A^f_{2,\delta}$ and this was already done in the proof of theorem 1.4. □

The proof shows that for any $g \geq 2$ and any $d \geq 2$ we have a natural transformation

$$\eta : \mathfrak{S}_{g,b} \to \mathfrak{C}_b(X(d), m) \times \mathcal{A}^f_{g-1,\delta}$$

and therefore a morphism $\eta : N_{g,b} \to M_b(X(d), m) \times A^f_{g-1,\delta}$, which is injective for $d \geq 3$ and at most $4 : 1$ for $d = 2$ and $\gamma = 1$, provided that all families are fibred families (e.g. for b big enough).

Remark 4.8. For $g \geq 4$ the crux of the matters is to determine what $A^f_{g-1,\delta}$ in fact is, i.e. to determine the abelian varieties of dimension $g - 1$, that occur as fixed parts of one-dimensional families of Jacobians of dimension g.
Pirola showed in [Pi92] that $A^f_{g-1,\delta}$ is non-empty for $g = 4$ but it is an interesting question to decide whether it is empty or not for $5 \leq g \leq 7$.

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