PERIODIC POINTS ON VEECH SURFACES AND THE MORDELL-WEIL GROUP OVER A TEICHMÜLLER CURVE

MARTIN MÖLLER

ABSTRACT. Periodic points are points on Veech surfaces, whose orbit under the group of affine diffeomorphisms is finite. We characterize those points as being torsion points if the Veech surfaces is suitably mapped to its Jacobian or an appropriate factor thereof. For a primitive Veech surface in genus two we show that the only periodic points are the Weierstraß points and the singularities.
Our main tool is the Hodge-theoretic characterization of Teichmüller curves. We deduce from it a finiteness result for the Mordell-Weil group of the family of Jacobians over a Teichmüller curve. The link to the classification of periodic points is provided by interpreting them as sections of the family of curves over a covering of the Teichmüller curve.

INTRODUCTION

Let ΩM_g^* denote the tautological bundle minus the zero section over M_g, the moduli space of curves of genus g. Its points consist of pairs (X^0, ω^0) of a Riemann surface plus a non-zero holomorphic one-form on X^0. There is a natural $\mathrm{GL}_2^+(\mathbb{R})$-action on ΩM_g^* (see [Ve89] or [McM03]). In the rare cases where the projection of $\mathrm{GL}_2^+(\mathbb{R}) \cdot (X^0, \omega^0)$ to M_g is an algebraic curve C, this curve is called a *Teichmüller curve* and (X^0, ω^0) a *Veech surface*. These surfaces are also characterized by the following property:
Let $\mathrm{Aff}(X^0, \omega^0)$ denote the group of orientation-preserving diffeomorphisms on X^0 that are affine outside the zeros of ω^0 with respect to the charts determined by integrating ω^0. Then (X^0, ω^0) is a Veech surface if and only if $\mathrm{Aff}(X^0, \omega^0)$ is 'as big as possible', i.e. the matrix parts of these diffeomorphisms form a lattice Γ in $\mathrm{SL}_2(\mathbb{R})$.

A *periodic point* on a Veech surface is a point whose orbit under $\mathrm{Aff}(X^0, \omega^0)$ is finite. Examples of periodic points are the zeroes of ω^0 and Weierstraß points if $g = 2$ (see [GuHuSc03]).
The aim of this paper is to show:

Theorem 3.3: *The difference of two periodic points is torsion on an r-dimensional quotient of* $\mathrm{Jac}(X^0)$*, where* $r = [\mathbb{Q}(\mathrm{tr}(\Gamma)) : \mathbb{Q}]$.

This is a consequence of the description of the variation of Hodge structures over a Teichmüller curve ([Mo04]) and the interpretation of periodic points as elements of the Mordell-Weil group of the family of Jacobians over the Teichmüller curve, see below. We also show a converse to this statement in terms of torsion sections of the family of Veech surfaces over an unramified cover of the Teichmüller curve. We say for short that periodic points form a *torsion packet* ([Co85]) on X^0.
As a consequence we also obtain a different proof of the finiteness of the number of periodic

Date: December 30, 2005.
Supported by the DFG-Schwerpunkt "Komplexe Mannigfaltigkeiten".

points on a Teichmüller curve that does not arise via a torus covering. This was proved by Gutkin, Hubert and Schmidt (see [GuHuSc03]) using the flat geometry induced by ω^0.

We apply the characterization of periodic points via torsion points to show:

Theorem 5.1/5.2: *The only periodic points on a Veech surface in genus 2, which does not arise from a torus cover, are the Weierstraß points and the zeroes of ω^0.*

There is a natural $\mathrm{GL}_2^+(\mathbb{R})$-invariant stratification

$$\Omega M_g^* = \bigcup_{\sum k_i = 2g-2} \Omega M_g(k_1, \ldots, k_s)$$

according to the number and multiplicities of zeroes of ω^0. Since 'large' torsion packets are 'rare' (see e.g. [BoGr00]), we expect that for large g the strata of ΩM_g^* with 'many' (in particular $2g-2$ simple) zeroes should contain 'few' algebraically primitive (see below) Teichmüller curves.

In fact, for $g = 2$ McMullen shows ([McM04b] and [McM04c]), using the above characterization of periodic points, that in the stratum $\Omega M_2(1,1)$ there is only one Teichmüller curve not coming from genus 1. It is generated by the regular decagon. In contrast to that, the stratum $\Omega M_2(2)$ contains infinitely many Teichmüller curves that are not obtained via torus coverings (see [McM04a]).

Contents.

In Section 1 we start by recalling the language of translation surfaces in which periodic points were first studied in [GuHuSc03]. We translate this into the following setting:
To a Veech surface we may associate (via its $\mathrm{GL}_2^+(\mathbb{R})$-orbit) a Teichmüller curve $C \to M_g$. A Teichmüller curve defines (by pulling back the universal family over a finite cover of M_g) a family of curves (or a *fibered surface*) $f : X_1 \to C_1$, after passing to some finite unramified covering C_1 of C. Finally each fiber of f is a Veech surface X^0 and the 1-form ω^0 can be recovered from the variation of Hodge structures of f.
We show that periodic points correspond via this dictionary to sections of f.

In Section 2 we recall some material from [Mo04] in order to define the r-dimensional factor A_1/C_1 of $\mathrm{Jac}(X_1/C_1)$ we referred to above.
We will call a Veech surface *geometrically primitive* if it does not arise via coverings from smaller genus. We will call a Veech surface *algebraically primitive* if the relative Jacobian of an associated fibered surface is simple. The indefinite article here and whenever we talk of fibered surfaces refers only to the possibility of passing to an unramified covering.
We give an example that (unlike in genus two) these notions do not coincide in higher genus. In Theorem 2.6 we show that nevertheless a Veech surface stems from a unique geometrically primitive Veech surface.

The *Mordell-Weil group* $\mathrm{MW}(A/C)$ of a family of abelian varieties $g : A \to C$ is the group of rational sections of g or equivalently the group of $\mathbb{C}(C)$-valued points of A. In Section 3 we study the Mordell-Weil group for the factor A_1/C_1 of $\mathrm{Jac}(X_1/C_1)$ for a Teichmüller curve $C \to M_g$:

Theorem 3.1: *For each Teichmüller curve and any given unramified covering $C_1 \to C$ the group $\mathrm{MW}(A_1/C_1)$ is finite.*

We apply this to the characterization of periodic points via torsion sections (Thm. 3.3 and the converse Prop. 3.5).

The last two sections contain a degeneration argument similar to the one in [McM04c] for the explicit analysis of periodic points in genus two. While in loc. cit. ratios of sines appear, we need for our purposes that only finitely many ratios $\tan(a)/\tan(b)$ for $a, b \in \pi\mathbb{Q}$ lie in a quadratic number field. We give an explicit list of them, that is used in the last section.

Some notation.
Riemann surfaces are usually denoted by X^0, Y^0, etc., while X, X_1, Y, Y_1, etc. will be used for families of Riemann surfaces over base curves C, C_1, etc., whose fibers over $0 \in C(\mathbb{C})$ are X^0, Y^0, etc.
Overlines denote completions of curves or fibered surfaces.

Acknowledgements.
The author thanks P. Hubert, whose questions motivated this paper, and C. McMullen for useful discussion.

1. A dictionary

Translation surfaces.
A *translation surface* is a compact Riemann surface X^0 with an atlas $\{U_i, i \in I\}$, covering X^0 except for finitely many points (called *singularities* $\mathrm{Sing}(X^0)$) such that the transition functions between the charts are translations. A holomorphic one-form ω^0 on a Riemann surface X_0 induces the structure of a translation surface where $\mathrm{Sing}(X^0)$ can be any finite set containing the zeroes of ω^0. A *translation covering* between two translation surfaces $(X^0, \mathrm{Sing}(X^0))$ and $(Y^0, \mathrm{Sing}(Y^0))$ is a covering $\varphi : X^0 \to Y^0$ such that the charts of X^0 are pulled back from charts of Y^0 and such that $\varphi^{-1}(\mathrm{Sing}(Y^0)) = \mathrm{Sing}(X^0)$. These coverings are sometimes called *balanced* translation coverings. We deal exclusively with them.
If the translation structure on Y^0 is induced from ω^0_Y and φ is a translation covering then the translation structure on X^0 is induced from $\omega^0 = \varphi^*\omega^0_Y$.
Finally note that a (possibly ramified) covering $\varphi : X^0 \to Y^0$ plus differentials on X^0 and Y^0 satisfying $\omega^0 = \varphi^*\omega^0_Y$ induces a translation covering with $\mathrm{Sing}(Y^0)$ the zeros of ω^0_Y and $\mathrm{Sing}(X^0)$ their preimages. This set will in general properly contain the zeroes of ω^0. We will abbreviate a translation covering by $\pi : (X^0, \omega^0_X) \to (Y^0, \omega^0_Y)$ or $\pi : (X^0, \mathrm{Sing}(X^0)) \to (Y^0, \mathrm{Sing}(Y^0))$.

Affine diffeomorphisms, affine group.
On a translation surface we may consider diffeomorphisms that are orientation-preserving, affine with respect to the coordinate charts and that permute the singularities. They form the *group of affine diffeomorphisms* denoted by $\mathrm{Aff}(X^0, \mathrm{Sing}(X^0))$. The matrix part of such an affine diffeomorphism is well-defined and this yields a map D to a subgroup Γ in $\mathrm{SL}_2(\mathbb{R})$. We call Γ the *affine group* of $(X^0, \mathrm{Sing}(X^0))$. If $\mathrm{Sing}(X^0)$ consists of the zeroes of ω^0 we sometimes write $\mathrm{SL}(X^0, \omega^0)$ for Γ. The kernel of D consists of conformal automorphisms of X^0 preserving ω^0 and is hence finite.

Definition 1.1. *A point P on a translation surface is called* periodic *if its orbit under the group* $\mathrm{Aff}(X^0, \mathrm{Sing}(X^0))$ *is finite.*

Veech surfaces, Teichmüller curves.
If $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ is a lattice, the translation surface $(X^0, \mathrm{Sing}(X^0))$ is called a *Veech surface.* If $\mathrm{Sing}(X^0)$ consists of the zeroes of ω^0, this is the case if and only if the $\mathrm{GL}_2^+(\mathbb{R})$-orbit of (X^0, ω^0) in ΩM_g^* projects to an algebraic curve $\mathbb{H}/\Gamma = C \to M_g$, which is called a *Teichmüller curve.* The map $C \to M_g$ is injective up to finitely many normal crossings.

By abuse of notation we will also call Teichmüller curve a map $C_1 \to M_g$, which is the composition of an unramified cover $C_1 \to C$ composed by a Teichmüller curve $C \to M_g$ in the above sense.

If we restrict to the quotient $C_1 = \mathbb{H}/\Gamma_1$ for a sufficiently small subgroup $\Gamma_1 \subset \Gamma$ of finite index, C_1 will map to a finite cover of M_g over which the universal family exists. If we pull back this universal family to C_1 we obtain a *fibered surface* $f : X_1 \to C_1$ associated with the Teichmüller curve. See [Mo04] Section 1.4 for more details. We will also need a smooth semistable model $f : \overline{X_1} \to \overline{C_1}$ over the completion of C_1.
A Veech surface $(X^0, \mathrm{Sing}(X^0))$ is called *square-tiled* (or *arises as torus cover*, or *origami*) if it admits a translation covering to a torus with one singular point.

From now on we will exclusively deal with Teichmüller curves and translation surfaces that are Veech surfaces.

Note that 'fibered surface' refers to an object of complex dimension two. It contains as one of its fibers the translation surface X^0, an object of real dimension two.

Lemma 1.2. *A point P on a Veech surface (X^0, ω^0) is periodic if and only if there is an (algebraic) section of some fibered surface $f : X_1 \to C_1$ associated to the Teichmüller curve, which passes through P on the fiber X^0 of f.*

Proof: A section s of f over $C_1 = \mathbb{H}/\Gamma_1$ hits X^0 in one point P. The Γ-orbit of P consists of at most $[\Gamma : \Gamma_1]$ points, hence is finite by the choice of Γ_1.
Conversely given a periodic point P on X^0 we may take a double cover $\pi : Y^0 \to X^0$ branched at P and some zero of ω^0. The translation surface $(Y^0, \omega_Y^0 := \pi^*\omega^0)$ is still a Veech surface: Indeed let $\mathrm{Sing}(X^0) = Z(\omega^0) \cup \{P\}$. The affine group of $(X^0, \mathrm{Sing}(X^0))$, is of finite index in Γ by the definition of a periodic point. Now π defines a translation covering $(Y^0, \pi^{-1}(\mathrm{Sing}(X^0))) \to (X^0, \mathrm{Sing}(X^0))$ and one can apply [GuJu00] Thm. 4.9 to show the Veech property of (Y^0, ω_Y^0).
Over some $C_1 = \mathbb{H}/\Gamma_1$ for a subgroup Γ_1 of finite index in Γ we have a covering $\pi : Y_1 \to X_1$ of fibered surfaces over C_1, such that the original π is the fiber over some point $0 \in C_1(\mathbb{C})$. By construction of Y_1 as $\mathrm{SL}_2(\mathbb{R})$-orbit of (Y^0, ω_Y^0) the differential ω_Y^0 extends to a section ω_{Y_1} of the relative canonical sheaf $\omega_{\overline{Y_1}/\overline{C_1}}$. Again by definition of the $\mathrm{SL}_2(\mathbb{R})$-action the multiplicities of the zeros of ω_{Y_1} remain constant over C_1. Hence passing to a subgroup of finite index in Γ_1 (we nevertheless keep the notation) we may assume that the zeros of ω_{Y_1} define sections s_i of $f_Y : Y_1 \to C_1$.
The images of s_i under π are sections of f. One of them passes through P, as π is ramified over P. □

2. Algebraic and geometric primitivity

Let $K = \mathbb{Q}(\mathrm{tr}(\Gamma))$ denote the trace field of the affine group of a Veech surface (X^0, ω^0). It remains unchanged if we replace Γ by a subgroup of finite index. Let $r := [K : \mathbb{Q}]$ and let $L/\mathbb{Q}$ be the Galois closure of $K/\mathbb{Q}$. We recall from [Mo04] the decomposition of the variation of Hodge structures (VHS) over a Teichmüller curve generated by (X^0, ω^0):
Let $\mathbb{V} = R^1 f_* \mathbb{Z}$. In [Mo04] Prop. 2.4 we have shown that there is a decomposition as polarized VHS

$$\mathbb{V}_{\mathbb{Q}} = \mathbb{W}_{\mathbb{Q}} \oplus \mathbb{M}_{\mathbb{Q}}, \quad \mathbb{W}_L = \bigoplus_{\sigma \in \mathrm{Gal}(L/\mathbb{Q})/\mathrm{Gal}(K/\mathbb{Q})} \mathbb{L}^\sigma \tag{1}$$

Here $\mathbb{L}^\sigma$ are pairwise non-isomorphic local systems of rank two over K (loc. cit. Cor. 2.10) and none of the irreducible factors of $\mathbb{M}_\mathbb{C}$ is isomorphic to any of the $\mathbb{L}^\sigma_\mathbb{C}$. This yields (see loc. cit. Thm. 2.7) a decomposition of the Jacobian

$$\mathrm{Jac}(X_1/C_1) \sim A_1 \times B_1 \tag{2}$$

up to isogeny, where A_1 has dimension r and real multiplication by K.
Recall furthermore that the graded quotients of a VHS together with the Gauss-Manin connection form a Higgs bundle $(\mathcal{E}, \Theta)$. The summands $\mathbb{L}^\sigma$ of $\mathbb{V}_K$ give rank-two sub-Higgs bundles $(\mathcal{L}^\sigma \oplus (\mathcal{L}^\sigma)^{-1}), \tau^\sigma)$, where $S = \overline{C} \smallsetminus C$ and

$$\tau^\sigma : \mathcal{L}^\sigma \to (\mathcal{L}^\sigma)^{-1} \otimes \Omega^1_{\overline{C}}(\log S).$$

The subbundle $\mathcal{L}^{\mathrm{id}}$ of $f_*\omega_{\overline{X_1}/\overline{C_1}}$ is distinguished by the property that its restriction to the fiber X^0 gives $\mathbb{C} \cdot \omega^0$.
Teichmüller curves are characterized (see [Mo04] Thm. 5.3) by a decomposition of the VHS as above plus the property that $\mathbb{L}^{\mathrm{id}}$ is *maximal Higgs*, i.e. that τ^{id} is an isomorphism. We need only two properties of this notion here: It is stable under replacing C_1 by a finite unramified cover and the VHS over a Teichmüller curve has precisely one rank-two subbundle that is maximal Higgs.
The last property is stated explicitely in [Mo04] Lemma 3.1 for the set of local systems $\{\mathbb{L}^\sigma, \sigma \in \mathrm{Gal}(L/\mathbb{Q})/\mathrm{Gal}(K/\mathbb{Q})\}$. But any two maximal Higgs subbundles become isomorphic after replacing C_1 by a finite étale cover and none of the irreducible summands of $\mathbb{M}_\mathbb{C}$ is isomorphic to any of the $\mathbb{L}^\sigma_\mathbb{C}$.

Definition 2.1. *A Teichmüller curve is called* algebraically primitive *if its relative Jacobian* $\mathrm{Jac}(X_1/C_1)$ *is simple as abelian variety over the algebraic closure of the function field of* C_1.

Note that irreducibility of $\mathrm{Jac}(X_1/C_1)$ does not depend on replacing C_1 by unramified covers. Furthermore, simplicity of $\mathrm{Jac}(X_1/C_1)$ does not exclude that special fibers of $f : X \to C$ may have reducible Jacobians.

We will also say that a Veech surface is algebraically primitive, if the corresponding Teichmüller curve is algebraically primitive.

Lemma 2.2. *A Teichmüller curve is algebraically primitive if and only if* $r = g$.

Proof: We only have to show that $\mathbb{W}$ is irreducible over $\mathbb{Q}$. This follows immediately from the irreducibility of the local systems $\mathbb{L}^\sigma$ and the fact that for $\sigma, \tau \in \mathrm{Gal}(L/\mathbb{Q})/\mathrm{Gal}(K/\mathbb{Q})$ with $\sigma \neq \tau$ the local systems $\mathbb{L}^\sigma$ and $\mathbb{L}^\tau$ are not isomorphic (see [Mo04] Lemma 2.3). $\square$

There is also a natural geometric notion of primitivity for translation surfaces and one for Teichmüller curves without explicitly referring to any differential. We show that these two geometric definitions coincide.

Definition 2.3. *A translation surface* (X^0, ω^0_X) *is* geometrically imprimitive *if there exists a translation surface* (Y^0, ω^0_Y) *of smaller genus and a covering* $\pi : X^0 \to Y^0$ *such that* $\pi^*\omega^0_Y = \omega^0_X$.

Definition 2.4. *A Teichmüller curve* $C \to M_g$ *is* geometrically imprimitive *if there is an unramified cover* $C_1 \to C$*, a fibered surface* $f_Y : Y_1 \to C_1$ *coming from a Teichmüller curve* $C_1 \to M_h$ *with* $h < g$ *and a (possibly ramified) covering* $\pi : X_1 \to Y_1$ *over* C_1*. It is called* geometrically primitive *otherwise.*

Proposition 2.5. *A Teichmüller curve is geometrically primitive if and only if a corresponding Veech surface is geometrically primitive.*

Proof: Suppose that $C \to M_g$ is geometrically imprimitive and let $\pi : X_1 \to Y_1$ be the covering of fibered surfaces associated with the Teichmüller curves. We restrict the covering π to some fiber $X_1^0 \to Y_1^0$ over $0 \in C_1(\mathbb{C})$. We have to show that the differentials $\omega_{X_1}^0$ on X_1^0 and $\omega_{Y_1}^0$ on Y_1^0 that generate the Teichmüller curves define a translation covering. We let $\mathrm{Sing}(Y_1^0)$ be the zeroes of $\omega_{Y_1}^0$ and $\mathrm{Sing}(X_1^0)$ be their preimages via π. It suffices to see that (up to a multiplicative constant) we have $\pi^*\omega_{Y_1}^0 = \omega_{X_1}^0$. By definition $\omega_{X_1}^0$ and $\omega_{Y_1}^0$ are obtained from $\mathcal{L}_{X_1}^{\mathrm{id}}$ resp. $\mathcal{L}_{Y_1}^{\mathrm{id}}$ by restriction to 0. By the properties of maximal Higgs subbundles listed above, we have $\mathcal{L}_{X_1}^{\mathrm{id}} = \pi^*\mathcal{L}_{Y_1}^{\mathrm{id}}$. This completes the 'if'-part.

Conversely a translation covering defines two Teichmüller curves with commensurable affine groups ([Vo96] Thm. 5.4 or [GuJu00] Thm. 4.9). We pass to a subgroup Γ_1 contained in both with finite index and small enough to have universal families. Then the $\mathrm{SL}_2(\mathbb{R})$-images of the translation covering patch together to a covering of fibered surfaces over $\mathbb{H}/\Gamma_1$. □

Examples: i) The two notions of primitivity coincide for genus 2 and will therefore be abbreviated just by 'primitive': If $r = 1$ the factor A of the Jacobian is one-dimensional, hence a family of elliptic curves and the projection onto this factor implies geometric imprimitivity ([GuJu00] Thm. 5.9).
ii) The Riemann surface $y^2 = x^7 - 1$ with $\omega_0 = dx/y$ is a Veech surface (see [Ve89]) and $\Gamma = \Delta(2, 7, \infty)$ has a trace field of degree 3 over $\mathbb{Q}$. The corresponding Teichmüller curve is algebraically primitive, hence geometrically primitive.
iii) For genus $g = 3$ the notions of geometric and algebraic primitivity no longer coincide: The Riemann surface $y^{12} = x^3(x-1)^4$ with $\omega^0 = ydx/[x(x-1)]$ is studied in [HuSc01] Thm. 5. It is obtained from unfolding the billiard in the triangle with angles $3\pi/12$, $4\pi/12$ and $5\pi/12$. It is a Veech surface with $\Gamma = \Delta(6, \infty, \infty)$. The trace field is of degree only 2 over $\mathbb{Q}$. Hence the Teichmüller curve is not algebraically primitive. But it is geometrically primitive, as remarked in loc. cit.: Since $r = 2$ it cannot arise as a torus cover. If it arose as a genus two cover, this cover would have to be unramified by Riemann-Hurwitz. Hence ω^0 would have zeros of order at most two. But ω^0 has indeed a zero of order four.

In [HuSc01] the authors analyze the properties that are preserved if one goes up and down a tree of translation coverings. We show that the situation is simple, if the singularities of the translation surfaces are chosen suitably, i.e. each tree has a root.

Theorem 2.6. *A translation surface (X^0, ω^0) is obtained as a translation covering from a geometrically primitive translation surface $(X^0_{\mathrm{prim}}, \omega^0_{\mathrm{prim}})$. If the genus X^0_{prim} is greater than one, this primitive surface is unique.*
Moreover, the construction of the primitive surface is equivariant with respect to the $\mathrm{SL}_2(\mathbb{R})$-action, if the genus of X^0_{prim} is greater than one.
If (X^0, ω^0) generates a Teichmüller curve, then so does $(X^0_{\mathrm{prim}}, \omega^0_{\mathrm{prim}})$. In this case $\pi : X^0 \to X^0_{\mathrm{prim}}$ is branched over periodic points.

Proof: We first prove existence and uniqueness of the primitive surface and drop the superscripts that indicate a special fiber.
Let $\mathrm{Jac}(X) \to A_X^{\max}$ be the maximal abelian quotient such that ω_X pulls back from $A_X^{\max}$.

Equivalently $A_X^{\max}$ is the quotient of $\mathrm{Jac}(X)$ by all connected abelian subvarieties B' such that the pullback of ω_X to B' vanishes. If $\dim A_X^{\max} = 1$ this is the primitive surface, we are done.

Otherwise, embed X into its Jacobian and consider the normalization X_A of the images of $X \to \mathrm{Jac}(X) \to A_X^{\max} \to A$ for all isogenies $A_X^{\max} \to A$. Since X_A generates A as a group, we have $g(X_A) \geq \dim A \geq 2$. The curves X_A form an inductive system which eventually stabilizes since the genera are bounded below. We let X_{prim} be the limit of the X_A and claim that it has the required properties.

First, by construction of A and since X_{prim} generates A as a group, there is a differential ω_{prim} on X_{prim} such that $\varphi_X^* \omega_{\mathrm{prim}} = \omega_X$. Hence $\varphi_X : X \to X_{\mathrm{prim}}$ defines a translation cover once the singularities are suitably chosen.

Second, $(X_{\mathrm{prim}}, \omega_{\mathrm{prim}})$ is indeed primitive: Suppose there is a covering $X_{\mathrm{prim}} \to Y$ with $\pi^* \omega_Y = \omega_{\mathrm{prim}}$ for some differential on Y. Let $\mathrm{Jac}(\pi)$ be the induced morphism on the Jacobians, commuting with π and suitable embeddings of the curves into their Jacobians. Consider the following diagram:

$$\begin{tikzcd}
X_{\mathrm{prim}} \arrow[r] \arrow[d, "\pi"] \arrow[drr, dashed, "\cong"] & \mathrm{Jac}(X) \arrow[rr] \arrow[dd, "\mathrm{Jac}(\pi)"] & & A_X^{\max} \arrow[dd, dashed, "\exists" ', "\overline{\mathrm{Jac}(\pi)}"] \\
Y \arrow[rr] \arrow[dr] & & Z \arrow[dr] & \\
& \mathrm{Jac}(Y) \arrow[rr] & & A_Y^{\max}
\end{tikzcd}$$

By definition ω_{prim} vanishes on $K := \mathrm{Ker}(\mathrm{Jac}(X) \to A_X^{\max})$. Since $\pi^*\omega_Y = \omega_{\mathrm{prim}}$ the differential ω_Y vanishes on $\mathrm{Jac}(\pi)(K)$. This means that $\mathrm{Jac}(\pi)(K)$ is in the kernel of $\mathrm{Jac}(Y) \to A(Y)$ and hence $\mathrm{Jac}(\pi)$ descends to a homomorphism on the quotients $\overline{\mathrm{Jac}(\pi)} : A_X^{\max} \to A_Y^{\max}$. This is an isogeny by construction of $A_X^{\max}$. The map $X_{\mathrm{prim}} \to Z$, which is defined as the normalization of the image in $A_Y^{\max}$, is an isomorphism by construction and so is π.

Third, for the uniqueness we have to show that for a translation covering $\pi : (X, \omega_X) \to (Y, \omega_Y)$ there is a morphism $\pi_{\mathrm{prim}} : X_{\mathrm{prim}} \to Y_{\mathrm{prim}}$. As in the second step we have an induced map $\overline{\mathrm{Jac}(\pi)} : A_X^{\max} \to A_Y^{\max}$. The curve X_{prim} was obtained as the normalization of the image of X is some quotient $q_X : A_X^{\max} \to A_X$. It hence maps to the normalization Z of the image of Y in $A_X^{\max} / \langle \mathrm{Ker}(q_X), \mathrm{Ker}(\overline{\mathrm{Jac}(\pi)}) \rangle$. Since Z maps to Y_{prim} by construction we are done.

For the $\mathrm{SL}_2(\mathbb{R})$-equivariance let $(X^1, \omega^1) = A \cdot (X^0, \omega^0)$ for some $A \in \mathrm{SL}_2(\mathbb{R})$. Primitivity implies the existence of a translation coverings $\pi : A \cdot X^0_{\mathrm{prim}} \to X^1_{\mathrm{prim}}$ and $\pi' : A^{-1} \cdot X^1_{\mathrm{prim}} \to X^0_{\mathrm{prim}}$. Hence either both primitive curves have genus 1 or both have bigger genus greater than one and we are done by uniqueness.

In case of a torus cover the statement that $(X^0_{\mathrm{prim}}, \omega^0_{\mathrm{prim}})$ generates a Teichmüller curve is trivial. In the other cases the previous argument implies that the affine group of $(X^0_{\mathrm{prim}}, \omega^0_{\mathrm{prim}})$ contains the one of (X^0, ω^0).

If both translation surfaces generate Teichmüller curves, there is a subgroup of finite index of $\mathrm{Aff}(X^0, \omega^0)$ that descends to X^0_{prim}. This group has to fix branch points and hence the whole group can generate only finite orbits of branch points. $\square$

3. The Mordell-Weil group

Theorem 3.1. *Let $f : X_1 \to C_1$ be a fibered surface associated with a Teichmüller curve. Then the Mordell-Weil group of A_1/C_1 is finite. Here A_1 is the factor of* $\mathrm{Jac}(X_1/C_1)$ *with real multiplication by K.*
In particular if the Teichmüller curve is algebraically primitive then $\mathrm{MW}(\mathrm{Jac}(X_1/C_1))$ *is finite.*

The finiteness of the Mordell-Weil group is invariant under isogenies. Thus there is no need to specify A_1 in its isogeny class. In particular we may suppose that the $\mathbb{Q}$-local system $\mathbb{W}$ comes from a $\mathbb{Z}$-local system $\mathbb{W}_{\mathbb{Z}}$.
Furthermore the statement of the theorem becomes stronger the smaller the subgroup Γ_1 with $C_1 = \mathbb{H}/\Gamma_1$ is. Thus we may replace C_1 by an unramified cover and suppose that we have unipotent monodromies. To simplify notation we will call this cover C, which should not be confused with the notation for the original Teichmüller curve.
Let $\overline{g} : \overline{A} \to \overline{C}$ be an extension of $g : A \to C$ to a semiabelian scheme. A unique such extension exists due to the unipotent monodromies. We denote by $H^0(C, \mathcal{O}_C(A/C))$ (resp. by $H^0(\overline{C}, \mathcal{O}^{\mathrm{an}}_{\overline{C}}(\overline{A}))$) the group of algebraic sections of A/C (resp. the group of analytic sections of $\overline{A}/\overline{C}$).
Two remarks: First, the analytic sections of $\overline{A}/\overline{C}$ are necessarily algebraic. Nevertheless we have to use the analytic category, because we want to use uniformization in the sequel. Second, by properness of g any rational section of g extends to the whole curve C, hence $\mathrm{MW}(A/C) = H^0(C, \mathcal{O}_C(A/C))$.

Lemma 3.2. *The restriction map*
$$r : H^0(\overline{C}, \mathcal{O}^{\mathrm{an}}_{\overline{C}}(\overline{A})) \to H^0(C, \mathcal{O}_C(A/C))$$
is injective with finite cokernel.

Proof: The proof is from Prop. 6.8 in [Sa93]. We reproduce a sketch for convenience of the reader:
There exists a group scheme $N \to \overline{C}$ (called the Néron model of $\overline{A}/\overline{C}$) containing $\overline{A}/\overline{C}$ as a connected component, with the following property: For smooth $Y \to \overline{C}$ a rational map $Y - - > N$ over $\overline{C}$ extends to a morphism $Y \to N$.
We apply this property to sections $s : C \to A$. They extend to rational maps from $\overline{C}$ a priori to a projective completion of N, but in fact to a morphism $\overline{C} \to N$. Hence
$$H^0(C, \mathcal{O}_C(A/C)) \to H^0(\overline{C}, \mathcal{O}_{\overline{C}}(N/\overline{C}))$$
is an isomorphism. This shows that the cokernel of r consists of sections of the finite group scheme $N/\overline{A}$ and is hence finite. □

If the local system $\mathbb{W}$ on C carries a polarized VHS of weight m (in our case $m = 1$) then $H^i(\overline{C}, j_*\mathbb{W}_{\mathbb{C}})$ is known to carry a Hodge structure of weight $m + i$ (see [Zu79]). Here $j : C \to \overline{C}$ is the inclusion. Indeed let $\Omega^\bullet(\mathbb{W})_{(2)}$ denote the deRham complex with L_2-growth conditions at the punctures. Zucker (see [Zu79]) shows (extending Deligne's results to the non-compact case) that we may identify $H^i(\overline{C}, j_*\mathbb{W}_{\mathbb{C}})$ with the hypercohomology groups $\mathbf{H}^i(\overline{C}, \Omega^\bullet(\mathbb{W})_{(2)})$. Then the Hodge structure comes from the filtration on $\Omega^\bullet(\mathbb{W})_{(2)}$ induced by the Hodge filtration on $\mathbb{W}$.
But in fact there is another complex more easily accessible and quasi-isomorphic to $\Omega^\bullet(\mathbb{W})_{(2)}$ ([Zu79] Prop. 9.1).

We describe how to calculate the Hodge structure on $H^i(\overline{C}, j_*\mathbb{W}_\mathbb{C})$ in our situation: The Hodge filtration on $\mathbb{W}$ is

$$0 = \mathcal{F}^2 \subset \mathcal{F}^1 = g_*\omega_{\overline{A}/\overline{C}} \subset \mathcal{F}^0 = (\mathbb{W} \otimes \mathcal{O}_C)_{\mathrm{ext}},$$

where the subscript denotes the Deligne extension to $\overline{C}$. As graded pieces we have

$$\mathcal{E}^{1,0} = \mathcal{F}^1/\mathcal{F}^2 = g_*\omega_{\overline{A}/\overline{C}}, \quad \mathcal{E}^{0,1} = \mathcal{F}^0/\mathcal{F}^1 = R^1 g_* \mathcal{O}_{\overline{A}}.$$

Combining [Zu79] Thm. 7.13 and Prop. 9.1 (see also the restatement after Lemma 12.14 in loc. cit.) we conclude for $p \in \{0, 1, 2\}$

$$H^1(\overline{C}, j_*\mathbb{W}_\mathbb{C})^{(p,2-p)} = \mathbf{H}^1(\overline{C}, \mathcal{E}^{p,1-p} \longrightarrow \mathcal{E}^{p-1,2-p} \otimes \Omega^1_{\overline{C}}(\log S)) \qquad (3)$$

where the mapping in the complex on the right is the graded quotient of the Gauss-Manin connection (equivalently: the Kodaira-Spencer mapping) and $S = \overline{C} \smallsetminus C$.

Proof of the theorem: The uniformization of $\overline{A}/\overline{C}$ yields a short exact sequence

$$0 \to j_*\mathbb{W}_\mathbb{Z} \to \mathcal{E}^{0,1} \to \mathcal{O}^{\mathrm{an}}_{\overline{C}}(\overline{A}) \to 0.$$

We take cohomology and note that $H^0(\overline{C}, \mathcal{E}^{0,1})$ vanishes as $\overline{A}/\overline{C}$ has no fixed part. Hence

$$\begin{aligned} H^0(\overline{C}, \mathcal{O}^{\mathrm{an}}_{\overline{C}}(\overline{A})) &= \mathrm{Ker}(H^1(\overline{C}, j_*\mathbb{W}_\mathbb{Z}) \to H^1(\overline{C}, \mathcal{E}^{0,1})) \\ &= H^1(\overline{C}, j_*\mathbb{W}_\mathbb{Z}) \cap (\mathrm{Ker}(H^1(\overline{C}, j_*\mathbb{W}_\mathbb{C}) \to H^1(\overline{C}, \mathcal{E}^{0,1}))) \\ &= H^1(\overline{C}, j_*\mathbb{W}_\mathbb{Z}) \cap H^1(\overline{C}, j_*\mathbb{W}_\mathbb{C})^{1,1}. \end{aligned}$$

By (1) and (3) we deduce

$$H^1(\overline{C}, j_*\mathbb{W}_\mathbb{C})^{1,1} = \oplus_{\sigma_i \in \mathrm{Gal}(K/\mathbb{Q})} \mathbf{H}^1(\overline{C}, \mathcal{L}_i \to (\mathcal{L}_i)^{-1} \otimes \Omega^1_{\overline{C}}(\log S)),$$

where $\mathcal{L}_i$ is the $(1,0)$-part of $(\mathbb{L}^{\sigma_i} \otimes \mathcal{O}_C)_{\mathrm{ext}}$. As the Kodaira-Spencer map for $\sigma_1 := \mathrm{id}$ is an isomorphism (this is the definition of 'maximal Higgs'), the first summand vanishes. But the action of K permutes the summands transitively and hence $H^1(\overline{C}, j_*\mathbb{W}_\mathbb{Z}) \cap H^1(\overline{C}, j_*\mathbb{W}_\mathbb{C})^{1,1} = 0$. □

Theorem 3.3. *Let $\varphi : X^0 \to \mathrm{Jac}(X^0) \to A^0$ be the embedding of a Veech surface into its Jacobian (normalized such that one of the zeros of ω^0 maps to 0) composed by the projection to the factor A^0.*
The periodic points on a Veech surface map via φ to torsion points on A^0. In particular there is only a finite number of periodic points on a Veech surface if $r > 1$, i.e. if the surface is not square-tiled.

The finiteness result was obtained by Gutkin, Hubert and Schmidt by entirely different methods in [GuHuSc03]. Our method has the advantage of providing an effective bound, see Cor. 3.6 below.

Proof: A periodic point of (X^0, ω^0) gives a section of some fibered surface $X_1 \to C_1$ by Lemma 1.2 and via φ a section of $A_{C_1} \to C_1$. This section has finite order by Thm. 3.1 thus proving the first statement.
By Thm. 5.1 in [Mo04] the family of abelian varieties A/C and also the section are defined over some number field. We fix some fiber of $f : X \to C$ defined over some number field, say our original X^0. If $r > 1$ the image $\varphi(X^0)$ in A^0 is a curve, which generates an abelian variety of dimension r. Hence it cannot be (a translate of) an abelian subvariety.
In this situation the Manin-Mumford conjecture says that the $\overline{\mathbb{Q}}$-rational points of $\varphi(X^0)$ have finite intersection with A^0_{tors}. We have seen that all periodic points are contained in

this intersection.
Proofs of the Manin-Mumford conjecture were obtained in different generality by Raynaud ([Ra83]), Serre, Hindry, Vojta, Buium, Hrushovski, McQuillan. For what we need here the proof of Pink and Roessler ([PiRo02]) is sufficient and maybe the most easily accessible. □

Corollary 3.4. *If (X^0, ω^0) is a Veech surface that generates an algebraically primitive Teichmüller curve, then all periodic points form a* torsion packet, *i.e., for two periodic points P, Q the difference $P - Q$ is torsion (as a divisor class).*

There is a converse to the above theorem, if we look for torsion sections instead of looking fibrewise:

Proposition 3.5. *Let $\varphi_C : X \to \mathrm{Jac}(X/C) \to A$ be the family of maps considered in the previous theorem and let $\varphi_{C_1} : X_1 \to A_1$ be the map obtained by an unramified base change $C_1 \to C$.*
Periodic points on X are precisely the preimages via φ_{C_1} of sections of A_1/C_1 for an unramified covering $C_1 \to C$ of sufficiently large degree.

Proof: We may choose A as in the proof of Thm. 2.6, since the statement of the proposition is invariant under isogenies of A. Sections of $\varphi_{C_1}(X)$ extend to sections of its normalization. This normalization is the family of curves $f_{\mathrm{prim}} : X_{\mathrm{prim}} \to C_1$ attached to the primitive Veech surface X^0_{prim} of Thm. 2.6. Hence these sections give all the periodic points on each fiber of X_{prim} for $C_1 \to C$ large enough by the criterion for periodic points given in Lemma 1.2. Since under a translation cover $\pi : X^0 \to X^0_{\mathrm{prim}}$ of Veech surfaces the preimages of the periodic points on X^0_{prim} are the periodic points on X^0, the claim follows. □

Corollary 3.6. *If $r > 1$ there is a universal bound depending only on g for the number of periodic points on a Veech surface of genus g.*

In fact Buium ([Bu94]) gives a bound on the number of torsion sections of a family of curves in a family of abelian varieties. This bound only depends on the genus of the curve and the dimension of the abelian variety, but it grows very fast with g.

4. Ratios of tangents

For the next section we will need:

Theorem 4.1. *For each $d > 0$, there are only finitely many pairs of rational numbers $0 < \alpha < \beta < 1/2$ such that*

$$\mu = \tan(\pi\beta)/\tan(\pi\alpha)$$

is an algebraic number of degree d over $\mathbb{Q}$.

Proof: This follows from [McM04c] Thm. 2.1 and the addition formula

$$\frac{\tan(\frac{x+y}{2})}{\tan(\frac{x-y}{2})} = \frac{\frac{\sin(x)}{\sin(y)} + 1}{\frac{\sin(x)}{\sin(y)} - 1}.$$

□

For $d = 2$ a list of these quotients can be obtained easily from Table 3 in [McM04c]. For later use we list those quotients which are non-units. By Galois conjugation in the cyclotomic field containing μ we may furthermore suppose $\alpha = 1/s$ for some $s \in \mathbb{N}$.

α	β	μ	Trace	Norm
1/10	1/5	$\sqrt{5}$	0	-5
1/10	2/5	$5+2\sqrt{5}$	10	5
1/5	3/10	$1+2\sqrt{1/5}$	2	1/5
1/12	1/6	$(3+2\sqrt{3})/3$	2	$-1/3$
1/12	1/3	$3+2\sqrt{3}$	6	-3
1/6	1/4	$\sqrt{3}$	0	-3
1/6	5/12	$3+2\sqrt{3}$	6	-3
1/4	1/3	$\sqrt{3}$	0	-3
1/3	5/12	$(3+2\sqrt{3})/3$	2	$-1/3$

Table 1: Quadratic ratios of tangents that are non-units

5. Periodic points in genus two

The $\mathrm{GL}_2^+(\mathbb{R})$-action on ΩM_2^* respects the multiplicities of the zeroes of the differential. We denote the corresponding strata by $\Omega M_2(2)$ and $\Omega M_2(1,1)$.

5.1. **The stratum** $\Omega M_2(2)$**.** By [McM04a] each Veech surface in the stratum $\Omega M_2(2)$ contains an L-shaped surface in its $\mathrm{GL}_2^+(\mathbb{R})$-orbit.

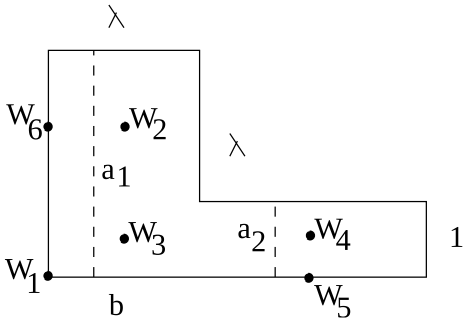

FIGURE 1. Admissible representative of a primitive Veech surface in $\Omega M_2(2)$

Here $\lambda = (e + \sqrt{(e^2+4b)})/2$, where $e \in \{-1,0,1\}$ and $b \in \mathbb{N}$ with the restriction that $e+1 < b$ and if $e = 1$ then b is even. A triple (b, e, λ) satisfying these conditions is called *admissible.*

In this section we show:

Theorem 5.1. *The only periodic points on a primitive Veech surface in* $\Omega M_2(2)$ *are the Weierstraß points.*

Proof: Let $f : X_1 \to C_1$ be a fibered surface (and $\overline{X_1} \to \overline{C_1}$ its stable completion) corresponding to a given primitive Veech surface in $\Omega M_2(2)$. Suppose the Veech surface contains a non-Weierstraß periodic point P. Passing to an unramified cover of C_1 we may suppose by Lemma 1.2 that the Weierstraß points and P extend to sections s_{W_i} for

$i=1,\ldots,6$ and s_P of f. We suppose that s_{W_1} passes through the zero of ω^0. By Thm. 3.3 and the primitivity assumption the section $s_P - s_{W_1}$ is torsion.

We start from an admissible representative (X^0,ω^0) and we let $(1\ \ 0)^T$ correspond to the horizontal and $(0\ \ 1)^T$ to the vertical direction. Observe what happens if the Veech surface degenerates in the vertical direction, i.e. along the path

$$(X^t,\omega^t) = \mathrm{diag}(e^t, e^{-t}) \cdot (X^0,\omega^0)$$

for $t \to \infty$. Note that the action of $\mathrm{diag}(e^t, e^{-t})$ does not change the ratio of the heights of the vertical cylinders.
The Riemann surfaces X^t are obtained by first cutting X^0 along the centers of the vertical cylinders and then glueing a pair of annuli (of some modulus increasing with t) along the cut circles. The limit curve is obtained by 'squeezing' the interior of each vertical cylinder to a point. See [Ma74] for more details.

Hence the stable model of the limit curve X^∞ is a rational curve with two pairs of points identified. By construction these nodes are fixed points of the hyperelliptic involution.

Normalizing suitably, we may suppose that X^∞ looks as follows: The Weierstraß section s_{W_1} intersects X^∞ in ∞, s_{W_2} intersects X^∞ in zero and the hyperelliptic involution becomes $z \mapsto -z$. By a linear transformation we may suppose that s_P intersects X^∞ in 1 and that the remaining Weierstraß sections are glued to pairs x with $-x$ and y with $-y$. for some $x, y \in \mathbb{C} \smallsetminus \{0, \pm 1\}$.

Furthermore ω^0 comes from a subbundle of $f_*\omega_{\overline{X_1}/\overline{C_1}}$ and specializes to a section ω^∞ of the dualizing sheaf on the singular fiber X^∞. Thus the differential ω^∞ has to vanish to the order two at ∞ and it has simple poles at x, $-x$, y and $-y$. Furthermore we must have

$$\mathrm{Res}_{\omega^\infty}(x) = \mathrm{Res}_{\omega^\infty}(-x) \quad \text{and} \quad \mathrm{Res}_{\omega^\infty}(y) = \mathrm{Res}_{\omega^\infty}(-y).$$

The differential

$$\omega^\infty = \left(\frac{y}{z-x} - \frac{y}{z+x} - \frac{x}{z-y} + \frac{x}{z+y}\right)dz$$

has this property and by Riemann-Roch it is unique up to scalar multiple.

The invariance of the height ratios implies

$$\frac{y}{x} = \frac{\mathrm{Res}_{\omega^\infty}(x)}{\mathrm{Res}_{\omega^\infty}(-y)} = \frac{\int_{a_1}\omega^0}{\int_{a_2}\omega^0} = \frac{\lambda+1}{1}$$

up to sign and interchanging the roles of x and y.

Due to the irreducibility of X^∞ and [McM04c] Thm. 3.4 the divisor $s_P - s_{W_1}$ remains torsion on the singular fiber. Thus $1-\infty$ is say N-torsion. In order to have a well-defined map $g: X^\infty \to \mathbb{P}^1$, $z \mapsto (z-1)^N$ we must have both

$$(x-1)^N = (-x-1)^N \quad \text{and} \quad (y-1)^N = (-y-1)^N.$$

This implies that $x, y \in i\mathbb{R}$, in fact we must have $x = i\tan(A\pi/N)$ and $y = i\tan(B\pi/N)$ for some $A, B \in \mathbb{Z}$.
We now use the list of tangent ratios of the previous section: If y/x is a unit then

$$|\mathrm{Norm}(\lambda+1)| = |e+1-b| = 1.$$

The only admissible triples that satisfy this condition are $e=0$ and $b=2$ (which gives the octagon) and $e=-1$ and $b=1$ (which gives the pentagon). For these two cases the theorem was proved in [GuHuSc03] Examples 3 and 4. Alternatively one might use that these Teichmüller curves pass through the Riemann surfaces $y^2 = x(x^4 - 1)$ resp. $y^2 = x^5 - 1$, whose torsion points are known ([BoGr00]) . Then one can conclude as in the proof of Thm. 5.2. If y/x is not a unit we can rule out each element of the list of table 1 by the conditions that $\mathrm{Norm}(\lambda + 1)$ has to be an integer and $\mathrm{Trace}(\lambda + 1) \in \{1, 2, 3\}$. □

5.2. **The stratum $\Omega M_2(1,1)$.** This stratum contains only one primitive Teichmüller curve, the one generated by the regular decagon (see [McM04b] and [McM04c]). The decagon corresponds to the Riemann surface $y^2 = x^5 - 1$ with the differential $x dx/y$ and its $\mathrm{GL}_2^+(\mathbb{R})$-orbit contains the following translation surface:

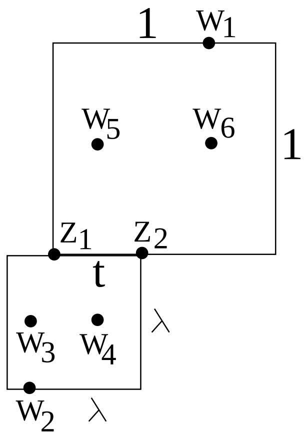

FIGURE 2. Representative in the $\mathrm{GL}_2^+(\mathbb{R})$-orbit of the decagon

Here the side lengths of the squares are 1 and $\lambda = (-1 + \sqrt{5})/2$ and $t = \sqrt{5}/5$. We denote this surface by (X^0, ω^0).

Theorem 5.2. *The only periodic points of the decagon are the Weierstraß points and the zeroes of ω^0.*

Proof: We consider the degeneration of the surface in Figure 2 by squeezing the horizontal direction. The singular fiber X^∞ in the limit is a $\mathbb{P}^1$ with two pairs of points identified. We suppose that the sections s_{Z_i} extending the two zeroes Z_1 and Z_2 of ω^0 intersect X^∞ in 0 and ∞ respectively. We may suppose that the hyperelliptic involution acts by $z \mapsto 1/z$. This forces the Weierstraß sections s_{W_1} and s_{W_2} to intersect in X^∞ in -1 and 1 respectively. Furthermore s_{W_3} and s_{W_4} intersect X^∞ in the pair of identified points x and $1/x$ while s_{W_5} and s_{W_6} degenerate to the pair of identified points y and $1/y$.

The differential ω^∞ has simple zeros in 0 and ∞ and simple poles at x, $1/x$, y and $1/y$ such that

$$\mathrm{Res}_{\omega^\infty}(x) = \mathrm{Res}_{\omega^\infty}(1/x) \quad \text{and} \quad \mathrm{Res}_{\omega^\infty}(y) = \mathrm{Res}_{\omega^\infty}(1/y).$$

This implies that

$$\omega^\infty = \left(\frac{y - 1/y}{z - x} - \frac{y - 1/y}{z - 1/x} + \frac{1/x - x}{z - y} - \frac{1/x - x}{z - 1/y}\right) dz$$

By Thm. 3.3 the difference $Z_1 - Z_2$ is torsion. Considering the surface $y^2 = x^5 - 1$ one notices (see e.g. [BoGr00]) that $Z_1 - Z_2$ is 5-torsion, that $Z_1 - W_1$ is 5-torsion and that whenever $R - W_1$ is torsion with R a non-Weierstraß point, then $R - W_1$ is 5-torsion. Hence $g_1(z) = z^5$ and $g_2(z) = (z+1)^5$ are well-defined on X^∞. This implies that (up to replacing x or y by its inverse or interchanging x and y) we have $x = \exp(2\pi i/5)$ and $y = \exp(4\pi i/5)$. If there was another periodic point R on the decagon, which becomes r on X^∞ then $g_3(z) = (z-r)^5$ would have to be well-defined on X^∞. This implies immediately that r has to be real. Let M_{10} be the set of complex numbers with argument a multiple of $2\pi/10$. Then $\{x - r, r \in \mathbb{R}\}$ intersects M_{10} only for $r = -1$ or $r \geq 0$ and $\{y - r, r \in \mathbb{R}\}$ intersects M_{10} only for $r \leq 0$. Hence there is no possible choice for $r \notin \{-1, 0, \infty\}$. $\square$

References

[BoGr00] Boxall, J., Grant D., *Examples of torsion points on genus two curves*, Trans. Amer. Math. Soc. 352 (2000), 4533–4555

[Bu94] Buium, A., *On a question of B. Mazur*, Duke Math. J. 75, No. 3 (1994), 639–644

[Co85] Coleman, R., *Torsion points on curves and p-adic Abelian integrals*, Annals of Math. 121 (1985), 111–168

[GuJu00] Gutkin, E., Judge, C., *Affine mappings of translation surfaces*, Duke Math. J. 103, No. 2 (2000), 191–212

[GuHuSc03] Gutkin, E., Hubert, P., Schmidt, T., *Affine diffeomorphisms of translation surfaces: periodic points, Fuchsian groups, and arithmeticity*, Ann. Sci. École Norm. Sup. 36 No. 6 (2003) 847–866

[HuSc01] Hubert, P., Schmidt, T.A., *Invariants of translation surfaces*, Ann. Inst. Fourier 51, No. 2 (2001) 461–495

[Ma74] Masur, H., *On a class of geodesics in Teichmüller space*, Annals of Math. 102 (1975), 205–221

[McM03] McMullen, C., *Billiards and Teichmüller curves on Hilbert modular sufaces*, J. Amer. Math. Soc. 16 (2003), 857–885

[McM04a] McMullen, C., *Teichmüller curves in genus two: Discriminant and spin*, Math. Ann. 333 (2005), 87–130

[McM04b] McMullen, C., *Teichmüller curves in genus two: The decagon and beyond*, J. reine angew. Math. 582 (2005), 173–199

[McM04c] McMullen, C., *Teichmüller curves in genus two: Torsion divisors and ratios of sines*, preprint (2004)

[Mo04] Möller, M., *Variations of Hodge structures of Teichmüller curves*, preprint (2004), to appear in J. Amer. Math. Soc.

[PiRo02] Pink, R., Rössler, D., *On Hrushovski's proof of the Manin-Mumford conjecture*, Proc. ICM Beijing, Vol. I, Higher Ed. Press (2002), 539–546

[Ra83] Raynaud, M., *Courbes sur une variété abélienne et points des torsion*, Invent. Math. 71 (1983), 207–233

[Sa93] Saito, M.-H., *Finiteness of the Mordell-Weil groups of Kuga fiber spaces of abelian varieties*, Publ. RIMS 29 (1993), 29–62

[Ve89] Veech, W., *Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards*, Invent. Math. 97 (1989), 533–583

[Vo96] Vorobets, Y., *Plane structures and billiards in rational polygons: the Veech alternative*, Russ. Math. Surv. 51 (1996), 779–817

[Zu79] Zucker, S., *Hodge theory with degenerating coefficients: L_2 cohomology in the Poincaré metric*, Annals of Math. 109 (1979), 415–476