VARIATIONS OF HODGE STRUCTURES OF A TEICHMÜLLER CURVE

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Abstract. Teichmüller curves are geodesic discs in Teichmüller space that project to an algebraic curve in the moduli space M_g. We show that for all $g \geq 2$ Teichmüller curves map to the locus of real multiplication in the moduli space of abelian varieties. Observe that McMullen has shown that precisely for $g = 2$ the locus of real multiplication is stable under the $\mathrm{SL}_2(\mathbb{R})$-action on the tautological bundle ΩM_g.

We also show that Teichmüller curves are defined over number fields and we provide a completely algebraic description of Teichmüller curves in terms of Higgs bundles. As a consequence we show that the absolute Galois group acts on the set of Teichmüller curves.

Introduction

The bundle ΩT_g^* over the Teichmüller space, whose points parameterize pairs (X, ω) of a Riemann surface X of genus g with a Teichmüller marking and a non-zero holomorphic 1-form ω has a natural $\mathrm{SL}_2(\mathbb{R})$-action. It can be described by post-composing the charts given by integration of ω with the linear map and providing the Riemann surface with the new complex structure. The linear action on the real and imaginary part of ω yields a one-form that is holomorphic for the new complex structure.

The orbit of $(X, \omega) \in \Omega T_g^*$ projected to T_g is a holomorphic embedding of the upper half-plane into Teichmüller space
$$\tilde{\jmath} : \mathbb{H} \to T_g,$$
which is totally geodesic for the Teichmüller metric. Only rarely do these geodesics project to algebraic curves $C = \mathbb{H}/\mathrm{Stab}(\tilde{\jmath})$ in the moduli space M_g. These curves are called *Teichmüller curves* and (X, ω) a *Veech surface*. Coverings of the torus ramified over only one point give Veech surfaces. Veech ([Ve89]) constructed a series of examples in all genera that do not arise via coverings. We will say more about $\mathrm{Stab}(\tilde{\jmath})$ in Section 1.

McMullen has shown in [McM1] that in genus 2 a Teichmüller curve always maps to the locus of real multiplication in the moduli space of abelian surfaces A_2 or to the locus of abelian surfaces that split up to isogeny. Moreover he has shown that the locus of eigenforms of real multiplication is $\mathrm{SL}_2(\mathbb{R})$-invariant. Since the property of having a double zero is also $\mathrm{SL}_2(\mathbb{R})$-invariant he constructs infinitely many examples in $g = 2$ which are not obtained via coverings of elliptic curves. McMullen also shows that for $g > 2$ the locus of eigenforms is no longer $\mathrm{SL}_2(\mathbb{R})$-invariant.

We investigate the variation of Hodge structures (VHS) of the family of Jacobians over a Teichmüller curve C or more precisely over an unramified cover of C where the universal family exists (see Section 1.4).
It turns out that the underlying local system has a $\mathbb{Q}$-factor which is the sum of r rank 2 local systems given by the action of $\mathrm{Stab}(\tilde{\jmath})$ and its Galois conjugates. Here $r \leq g$ equals the degree of the trace field of $\mathrm{Stab}(\tilde{\jmath})$ over $\mathbb{Q}$. As a consequence we obtain for all genera:

Theorem 2.7. The image of a Teichmüller curve $C \to A_g$ is contained in the locus of abelian varieties that split up to isogeny into $A_1 \times A_2$, where A_1 has dimension r and real multiplication by

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the trace field $K = \mathbb{Q}(\mathrm{tr}(\mathrm{Stab}(\tilde{\jmath})))$, where $r = [K : \mathbb{Q}]$. The generating differential $\omega \in H^0(X, \Omega^1_X)$ is an eigenform for the multiplication by K, i.e., $K \cdot \omega \subset \mathbb{C}\omega$.

For further results stated in Sections 4 and 5 it is convenient also to have the language of Higgs bundles available. The basic notions will be recalled in Section 3. Simpson's correspondence allows us to go back and forth from VHS to Higgs bundles.
Our study of Teichmüller curves is inspired by the work of Viehweg and Zuo on Shimura curves ([ViZu04]). They show that the VHS over a Shimura curve is build out of sub-VHS, whose Higgs field is "maximal" and unitary sub-VHS. Maximality here means that the Higgs field is an isomorphism. The notion is motivated since this happens if and only if a numerical upper bound (Arakelov inequality) is attained.

Teichmüller curves are in some sense similar to Shimura curves, since their VHS always contains a sub-VHS that is maximal Higgs; see Prop. 2.4 and its converse, Theorem 2.13.
In fact, we characterize Teichmüller curves algebraically using this notion:

Theorem 5.3. Suppose that the Higgs bundle of a family of curves $f : \mathcal{X} \to C = \mathbb{H}/\Gamma$ has a rank two Higgs-subbundle with maximal Higgs field. Then $C \to M_g$ is a finite covering of a Teichmüller curve.

Suppose that we have a Teichmüller curve with $r = g$. This implies that it does not arise from lower genus via a covering construction. The above algebraic description implies that Teichmüller curves are defined over number fields and that $G_{\mathbb{Q}}$ acts on the set of Teichmüller curves with $r = g$ (see Thm. 5.1 and Cor. 5.4).

There are also properties that are significantly different between Teichmüller and Shimura curves. For example, consider the locus of real multiplication by some field in the moduli space of abelian threefolds. There are Shimura subvarieties properly contained in this locus that contain (lots of) Shimura curves. But for Teichmüller curves we show:

Theorem 4.2. Suppose that $C \to M_g$ is a Teichmüller curve with $r = g$. Then the Shimura variety parameterizing abelian varieties with real multiplication by K is the smallest Shimura subvariety of A_g that the Teichmüller curve C maps to.

The above restrictions to the case $r = g$ may be explained as follows: The VHS over a Teichmüller curve has a rank 2 sub-VHS and its Galois conjugates, which we have control of. For the "rest", which may be enlarged by covering constructions, we can hardly say anything. It would be interesting to characterize the local systems that can appear in that "rest" when $r < g$.

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1. Setup

1.1. **Flat structures.** Start with a Riemann surface X of genus $g \geq 1$ and a non-zero holomorphic quadratic differential $q \in H^0(X, \Omega_X^{\otimes 2})$. This pair determines a *flat structure* on X: Cover X minus the zeros of q by simply connected open sets U_i on which q admits a square root ω. Integration of ω gives charts $\varphi_i : U_i \to \mathbb{C}$ such that the transition functions $\varphi_j \circ \varphi_i^{-1}$ are translations composed with ± 1. Although the $\mathrm{SL}_2(\mathbb{R})$-action and Teichmüller geodesics (see below) are defined for all pairs (X, q), we consider here only the case where $q = \omega^2$ is globally the square of a holomorphic one-form $\omega \in H^0(X, \Omega_X)$. This restriction is not too serious since a pair (X, q) with q not a square admits a double covering $\pi : X' \to X$ branched over the zeros of q of odd order such that $\pi^* q$ is a square. On the other hand it seems necessary, since quadratic differentials are not directly reflected in the Hodge structure; see Section 2.

1.2. $\mathrm{SL}_2(\mathbb{R})$**-action.** Denote by ΩT_g the vector bundle over the Teichmüller space T_g whose fibers are global holomorphic 1-forms. Equivalently, ΩT_g is the total space of the pullback of the tautological bundle (Hodge bundle) over M_g to T_g. Let ΩT_g^* denote the bundle with the zero section removed. ΩT_g^* carries a natural $\mathrm{SL}_2(\mathbb{R})$-action: Given $(X,\omega) \in \Omega T_g^*$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ consider the 1-form

$$\eta = \begin{pmatrix} 1 & i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathrm{Re}\,\omega \\ \mathrm{Im}\,\omega \end{pmatrix}. \tag{1}$$

There is a unique complex structure on the topological surface underlying X such that η is holomorphic. Call this surface Y and let $A \cdot (X,\omega) = (Y,\eta)$. Equivalently we may post-compose the charts φ_i of the flat structure by the action of A on $\mathbb{C} \cong \mathbb{R}^2$. The new complex structures glue, since the transition functions are translations.

The fibers of the projection $\Omega T_g^* \to T_g$ are stabilized by $\mathrm{SO}_2(\mathbb{R})$. Consider the quotient mapping $\tilde{\jmath} : \mathbb{H} \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \to T_g$. This map is holomorphic and a geodesic embedding for the Teichmüller (equivalently: Kobayashi) metric on T_g. In fact we can identify $\mathbb{H}$ with the unit disk $t\overline{\omega}/\omega$ for $|t| < 1$ in the space of Beltrami differentials; see [McM1] Section 2.

1.3. **Teichmüller curves.** On some rare occasions the projection $\mathbb{H} \to T_g \to M_g$ to the moduli space of curves of genus g is closed for the complex topology. By [SmWe04] Prop. 8 this is equivalent to the setwise stabilizer $\mathrm{Stab}(\tilde{\jmath}) := \mathrm{Stab}_{\Gamma_g}(\tilde{\jmath})$ of $\mathbb{H}$ under the action of the mapping class group Γ_g being a lattice in $\mathrm{SL}_2(\mathbb{R})$. In this case we call $\tilde{\jmath}$ (or $C := \mathbb{H}/\mathrm{Stab}(\tilde{\jmath})$ or $j : C \to M_g$) a *Teichmüller curve*. We also say that (X,ω) *generates* a Teichmüller curve. If we change the basepoint, i.e., if we start with $A \cdot (X,\omega)$ instead of (X,ω), this amounts to replacing $\mathrm{Stab}(\tilde{\jmath})$ by $A^{-1} \cdot \mathrm{Stab}(\tilde{\jmath}) \cdot A$. Note that C is a complex algebraic curve, but it is never complete by [Ve89] Thm. 2.1.

1.4. **Universal family.** We want to consider a Teichmüller curve $C \to M_g$ as a family of curves parameterized by C. All the results in the sequel will not depend on taking a subgroup Γ of finite index in $\mathrm{Stab}(\tilde{\jmath})$. There is a subgroup $\widetilde{\Gamma_g}$ of finite index in the mapping class group which is torsion free. Hence there is a universal family of curves over $T_g/\widetilde{\Gamma_g}$. We may take for this purpose $T_g/\widetilde{\Gamma_g} = M_g^{[3]}$, the moduli space of curves with level-n-structure for $n = 3$. We lift the Teichmüller curve to this covering: Let $\Gamma_1 := \mathrm{Stab}_{\widetilde{\Gamma_g}}(\tilde{\jmath})$ and let $C^{[3]} = \mathbb{H}/\Gamma_1 \to M_g^{[3]}$ be the induced quotient of $\tilde{\jmath}$. We let $f^{[3]} : \mathcal{X}^{[3]}_{\mathrm{univ}} \to M_g^{[3]}$ be this universal family and we pull it back to $C^{[3]}$. This is almost the family of curves we want. Its monodromies around the cusps are quasi-unipotent (see e.g. [Sd73] Thm. 6.1). For technical purposes we pass to a finite index subgroup Γ of Γ_1 such that there the family of curves admits a stable model over $\overline{\mathbb{H}/\Gamma}$, and hence monodromies around the cusps are unipotent. To ease notation we denote the pullback of the universal family to this covering by $f : \mathcal{X} \to C$ and call it a family of curves *coming from a Teichmüller curve*. We hope that no confusion with the original definition of C lying inside M_g will arise.

The whole situation is summarized in the following diagram:

$$\begin{tikzcd}
\overline{\mathcal{X}} \arrow[d, "f"] & \mathcal{X} \arrow[l, hook'] \arrow[r] \arrow[d, "f"] & \mathcal{X}^{[3]}_C \arrow[r] \arrow[dd, "f^{[3]}"] & \mathcal{X}^{[3]}_{\mathrm{Univ}} \arrow[dd, "f^{[3]}"] \\
\overline{C} & C := \mathbb{H}/\Gamma \arrow[l, hook'] \arrow[dr] & & \\
& & C^{[3]} = \mathbb{H}/\Gamma_1 \arrow[r] \arrow[d] & M_g^{[3]} \arrow[d] \\
& & C = \mathbb{H}/\mathrm{Stab}(\tilde{\jmath}) \arrow[r] & M_g
\end{tikzcd}$$

Here $\overline{C}$ is a smooth completion of C and $\overline{\mathcal{X}}$ is a family of stable curves over $\overline{C}$. We will denote throughout by $g : \mathrm{Jac}(\mathcal{X}/C) \to C$ the family of Jacobians of $f : \mathcal{X} \to C$ and abbreviate $A := \mathrm{Jac}(\mathcal{X}/C)$.

1.5. **The affine group.** Let $\mathrm{Aff}^+(X, \omega)$ denote the group of orientation-preserving diffeomorphisms, which are affine with respect to the flat structure defined by ω. The "derivative" $D : \mathrm{Aff}^+(X, \omega) \to \mathrm{SL}_2(\mathbb{R})$ associates with such a diffeomorphism its matrix part. We denote the image of D by $\mathrm{SL}(X, \omega)$. This group is related (see [McM1] Prop. 3.2) to the stabilizer by

$$\mathrm{Stab}(\tilde{\jmath}) = \mathrm{diag}(1, -1) \cdot \mathrm{SL}(X, \omega) \cdot \mathrm{diag}(1, -1). \tag{2}$$

The kernel of D is the group of conformal automorphisms of X preserving ω and corresponds to the pointwise stabilizer of $\mathbb{H}$ in T_g.

1.6. **Real multiplication.** Let $(\pi : A_1 \to S, \lambda : A_1 \to A_1^\vee)$ be a family of polarized abelian varieties of dimension r over some base S over the complex numbers. This family is said to have *real multiplication by a totally real number field* K with $[K : \mathbb{Q}] = r$ if there is an inclusion of $\mathbb{Q}$-algebras

$$K \hookrightarrow \mathrm{End}(A_1/S) \otimes \mathbb{Q}.$$

Since we are working in characteristic zero, we do not need to impose supplementary conditions on the tangent space in order to have a well-behaved moduli functor; see [vG87] Lemma X.1.2. Two families of abelian varieties with real multiplication are said to be isomorphic if there is an isomorphism respecting the polarization and the K-action.

Suppose S is a point. The action of K on $H^0(A_1, \Omega^1_{A_1})$ is diagonalizable, and we may choose an eigenbasis $\{\omega_1, \ldots, \omega_r\}$. We call ω_i an eigenform for real multiplication. Furthermore we may choose a symplectic basis $\{a_i, b_i\} \in H_1(A_1, \mathbb{R})$ adapted to $\{\omega_i\}$. By definition adapted means that $\int_{a_i} \omega_j = 0$ and $\int_{b_i} \omega_j = 0$ for $i \neq j$ and that $\int_{a_i} \omega_i / \int_{b_i} \omega_i \in \mathbb{H}$.

If S is simply connected this holds in the relative setting: we find sections ω_i of $\pi_* \Omega^1_{A_1}$ of eigenforms and sections a_i, b_i of the local system $R^1 \pi_* \mathbb{R}_{A_1}$ that are fiberwise adapted in the above sense.

This shows that the moduli functor (in the complex category) for abelian varieties with real multiplication by K with an adapted symplectic basis is represented by $\mathbb{H}^r$.

If we fix a level structure instead of a symplectic basis and if we fix the precise type of the real multiplication, i.e. an embedding of an order $\mathfrak{o} \subset K$ into the endomorphism ring of the abelian variety, then two elements of $\mathbb{H}^r$ become identified if they lie in the same orbit of a group Δ commensurable with $\mathrm{SL}_2(\mathfrak{o})$ depending on the level structure chosen. Here the r embeddings $K \hookrightarrow \mathbb{R}$ induce a map $\mathrm{SL}_2(\mathfrak{o}) \hookrightarrow \mathrm{SL}_2(\mathbb{R})^r$ and $\mathrm{SL}_2(\mathfrak{o})$ acts on $\mathbb{H}^r$ via this map. As above we do not care about specifying the level structure or the order $\mathfrak{o}$ since our results will not depend on passing to unramified covers.

For more details see [LaBi92], [vG87] and [McM1], [McM3].

2. Splitting of the local system

We start with a review on variations of Hodge structures (VHS) for a family $f : \mathcal{X} \to C$ of curves over a base curve C and we suppose that it extends to a family of stable curves $\overline{f} : \overline{\mathcal{X}} \to \overline{C}$. Fix a base point $c \in C(\mathbb{C})$. The cohomology $H^1(X, \mathbb{Z})$ of the fiber X of f over c is acted upon by $\pi_1(C, c)$. This data is equivalent to having a $\mathbb{Z}$-local system $R^1 f_* \mathbb{Z}_{\mathcal{X}}$ on C, where $\mathbb{Z}_{\mathcal{X}}$ is the constant sheaf $\mathbb{Z}$ on $\mathcal{X}$. The associated rank $2g$ vector bundle $R^1 f_* \mathbb{Z}_{\mathcal{X}} \otimes_{\mathbb{Z}} \mathcal{O}_C$ has a distinguished extension to $\overline{C}$ due to Deligne (see [De70]), which we denote by $E := (R^1 f_* \mathbb{Z}_{\mathcal{X}})_{\mathrm{ext}}$. The holomorphic one-forms form a subspace of $H^1(X, \mathbb{C})$. In the relative situation this fits together to the bundle $f_* \omega_{\overline{\mathcal{X}}/\overline{C}} \subset E$. Hodge theory plus Serre duality imply that the quotient

$$E / f_* \omega_{\overline{\mathcal{X}}/\overline{C}} \cong (f_* \omega_{\overline{\mathcal{X}}/\overline{C}})^\vee \cong R^1 f_* \mathcal{O}_{\mathcal{X}}.$$

Furthermore the fiberwise symplectic pairing $(\cdot,\cdot)$ on $H^1(X,\mathbb{R})$ gives a hermitian form

$$H(v,w) := i(v,\overline{w}).$$

The data $(R^1 f_* \mathbb{Z}_{\mathcal{X}},\ f_*\omega_{\overline{\mathcal{X}}/\overline{C}} \subset E,\ H)$ form a polarized VHS (pVHS) of weight 1. We are also interested in sub-VHS. For this purpose we have to allow the underlying local system to be only an F-local system for some field $F \subset \mathbb{C}$. In this case we talk of an F-VHS.

Weight one will be sufficient for a reader mainly interested in understanding the results on Teichmüller curves in this section. In general and for Sections 3 and 4 we recall that a polarized L-VHS of weight n over a curve C consists of

• an L-local system $\mathbb{V}_L$ over C for a field $L \subset \mathbb{C}$ which gives rise to a connection ∇ on $E := (\mathbb{V}_L \otimes_L \mathcal{O}_C)_{\mathrm{ext}}$ and

• a decomposition $E = \bigoplus_{p\in\mathbb{Z}} E^{p,n-p}$ into C^∞-bundles, such that

i) $E^p := \bigoplus_{i\geq p} E^{i,n-i}$ are holomorphic subbundles and $\overline{E^q} := \bigoplus_{i\leq n-q} E^{i,n-i}$ are antiholomorphic subbundles and

ii) $\nabla(E^p) \subset \Omega^1_C \otimes E^{p-1}$ and $\nabla(\overline{E^q}) \subset \Omega^1_C \otimes \overline{E^{q-1}}$.

The VHS is *polarized* if there exists a hermitian form $H(\cdot,\cdot)$ on E such that the decomposition $E = \bigoplus E^{p,n-p}$ is orthogonal and such that $(-1)^{p-q} H(v,\overline{v}) > 0$ for all $v \in E^{p,q}$. A homomorphism of VHS $\varphi : E \to F$ of bidegree (r,s) is a homomorphism of the underlying local systems such that $\varphi(E^{p,q}) \subset F^{p+r,q+s}$.

We now specialize to the case of a family f coming from a Teichmüller curve as in Section 1.4. Consider the cohomology of the fiber X over $c \in C(\mathbb{C})$ with $\mathbb{R}$ coefficients. Since $C = \mathbb{H}/\Gamma$, the subspace $\mathbb{L}_c := \langle \mathrm{Re}\,\omega, \mathrm{Im}\,\omega\rangle_{\mathbb{R}} \subset H^1(X,\mathbb{R})$ is invariant under the action of

$$\pi_1(C,c) = \Gamma \subset \mathrm{SL}_2(\mathbb{R})$$

by the defining equation (1).
We thus obtain a rank 2 linear subsystem

$$\mathbb{L}_{\mathbb{R}} \subset \mathbb{V}_{\mathbb{R}} = (R^1\pi_*\mathbb{Q}_{\mathcal{X}}) \otimes_{\mathbb{Q}} \mathbb{R},$$

whose fiber over c is $\mathbb{L}_c$. We now want to decompose the VHS and we recall Deligne's semisimplicity for convenience. For our purpose the more well-known result for $\mathbb{Q}$-VHS is not sufficient; we need it for $\mathbb{C}$-VHS.

Theorem 2.1 (Deligne, [De87] Prop. 1.13)**.** *The local system $\mathbb{V}_{\mathbb{C}}$ on the algebraic curve C decomposes as*

$$\mathbb{V}_{\mathbb{C}} = \bigoplus_{i=1}^{n} (\mathbb{L}_i \otimes W_i),$$

where $\mathbb{L}_i$ are pairwise non-isomorphic irreducible $\mathbb{C}$-local systems and W_i are non-zero $\mathbb{C}$-vector spaces.
Moreover the $\mathbb{L}_i$ and the W_i carry polarized VHS, whose tensor product and sum gives back the Hodge structure on $\mathbb{V}_{\mathbb{C}}$. The Hodge structure on the $\mathbb{L}_i$ (and W_i is unique up to a shift of the bigrading.

Deligne states the above proposition for weight zero, but the proof carries over to any other weight without changes.

A few words on the proof: The semisimplicity of $\mathbb{V}_{\mathbb{C}}$ is due to the existence of a polarization; see [De71] Thm. 4.2.6. Given a local system $\mathbb{V}$ on C we denote by $\mathrm{End}(\mathbb{V}_{\mathbb{C}}) = H^0(C, \mathbb{E}\mathrm{nd}(\mathbb{V}_{\mathbb{C}}))$ the global sections of the local system of endomorphisms. Then, due to the irreducibility of the $\mathbb{L}_i$, we have

$$\mathrm{End}(\mathbb{V}_{\mathbb{C}}) = \prod_{i=1}^{n} \mathrm{End}(W_i).$$

Now the components of the Hodge decomposition of $\mathrm{End}(\mathbb{V}_{\mathbb{C}})$ are again flat global sections. This is the point where one needs that C is an algebraic curve or at least that any subharmonic function on C bounded above is constant. We thus obtain a Hodge structure on $\prod_{i=1}^{n} \mathrm{End}(W_i)$. The remaining steps consist of showing that this Hodge structure comes from one on W_i and using suitable projections $\mathbb{V}_{\mathbb{C}} \to \mathbb{L}_i$ to provide the $\mathbb{L}_i$ with a VHS.

One more notation: Let L be a field. An L-local system $\mathbb{W}$ on C is *defined over* a subring $R \subset L$ if there is a local system of torsion-free R-modules $\mathbb{W}_R$ with $\mathbb{W} \cong \mathbb{W}_R \otimes_R L$. Equivalently the representation of $\pi_1(C, c)$ of the fiber $\mathbb{W}_c \cong L^{\mathrm{rank}\,\mathbb{W}}$ is conjugate to a representation that factors though $\mathrm{GL}(R^{\mathrm{rank}\,\mathbb{W}})$.

We come back to the case of Teichmüller curves: Since the Fuchsian group Γ defines a local system that is irreducible over $\mathbb{C}$, we may suppose that $\mathbb{L}_1 = \mathbb{L}_{\mathbb{C}}$. Let $K = \mathbb{Q}(\mathrm{tr}(\Gamma))$ denote the trace field of the Fuchsian group. It is a real number field of degree $r := [K : \mathbb{Q}] \leq g$ over $\mathbb{Q}$ ([McM1] Thm. 5.1).

Lemma 2.2. *$\mathbb{L}_{\mathbb{R}}$ is defined over a number field $K_1 \subset \mathbb{R}$, which has degree at most two over K.*

Proof: By Takeuchi ([Ta69]) it suffices to take $K_1 = K(\lambda)$, where λ is an eigenvalue of a hyperbolic element in Γ. $\square$

We denote by $\mathbb{L}$ the K_1-local system such that $\mathbb{L} \otimes_{K_1} \mathbb{R} = \mathbb{L}_{\mathbb{R}}$. Denote by L the Galois closure of $K_1/\mathbb{Q}$ and let $r = [K : \mathbb{Q}]$. The $\mathrm{Gal}(L/\mathbb{Q})$-conjugates $\mathbb{L}^\sigma$ of $\mathbb{L}$ are also rank 2 irreducible local subsystems of $\mathbb{V}_L$. Hence the $(\mathbb{L}^\sigma)_{\mathbb{C}}$ appear among the $\mathbb{L}_i$ in the semisimplicity theorem.

Lemma 2.3. *For $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ the local subsystems $\mathbb{L}$ and $\mathbb{L}^\sigma$ in $\mathbb{V}_L$ are isomorphic if and only if σ fixes K. Moreover W_1 is one-dimensional.*

Proof: If $\mathbb{L}$ and $\mathbb{L}^\sigma$ are isomorphic, the traces of Γ are invariant under σ: hence σ has to fix K. For the converse let $\phi \in \mathrm{Aff}(X, \omega)$ be a diffeomorphism whose image $D(\phi)$ lies in the subgroup Γ of $\mathrm{SL}(X, \omega)$ and which is hyperbolic, i.e., whose trace t has absolute value greater than 2. We let ϕ^* denoted the induced morphism on $H^1(X, \mathbb{Q})$. By [McM2] Thm. 9.5 the sub-vectorspace $\mathrm{Ker}((\phi^* + (\phi^*)^{-1}) - t \cdot \mathrm{id})$ of $H^1(X, \mathbb{R})$ is two-dimensional and contains $\mathbb{L}_c$. If $\mathbb{L} \not\cong \mathbb{L}^\sigma$ for $\sigma \in \mathrm{Gal}(L/K)$ or if $\dim W_1 \geq 2$, then this subspace would have to be of dimension greater than two. $\square$

We denote the conjugate L-local systems of $\mathbb{L}$ under a fixed system of representatives of the group $\mathrm{Gal}(L/\mathbb{Q})/\mathrm{Gal}(L/K)$ by $\mathbb{L} = \mathbb{L}_1, \ldots, \mathbb{L}_r$ with the slight abuse of notation that they appear as $\mathbb{C}$-local systems in the semisimplicity theorem.
We can now sum up:

Proposition 2.4. *The local system $\mathbb{V} = R^1 f_* \mathbb{Z}$ splits over $\mathbb{Q}$ as*

$$\mathbb{V}_{\mathbb{Q}} = \mathbb{W}_{\mathbb{Q}} \oplus \mathbb{M}_{\mathbb{Q}}, \quad \text{where} \quad \mathbb{W}_L = \mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_r,$$

such that each of the $\mathbb{L}_i$ carries a polarized L-VHS of weight one and $\mathbb{M}_{\mathbb{Q}}$ carries a polarized $\mathbb{Q}$-VHS of weight one whose sum is the VHS on $\mathbb{V}$. Moreover none of the $(\mathbb{L}_i)_{\mathbb{C}}$ is contained in $\mathbb{M}_{\mathbb{C}}$.

Proof: Apply Thm. 2.1 and let

$$\mathbb{M}_{\mathbb{C}} := \sum_{i=r+1}^{n} (\mathbb{L}_i \otimes W_i).$$

The vector spaces W_i for $i = 2, \ldots, r$ are one-dimensional, since otherwise by Galois conjugation we would have $\dim W_1 \geq 2$ contradicting Lemma 2.3. The local system $\mathbb{W}_L$ is defined over $\mathbb{Q}$, since $\mathrm{Gal}(L/\mathbb{Q})$ acts by permuting its summands. Hence also the complement $\mathbb{M}_{\mathbb{C}}$ is defined over $\mathbb{Q}$. $\square$.

Remark 2.5. The argument, which implies that $\dim W_1 = 1$ in the above Lemma 2.3 may be explained as follows: The hyperbolic element ϕ is a pseudo-Anosov diffeomorphism of X. It has unique contracting and expanding foliations and unique maximal and minimal eigenvalues when acting on $H^1(X, \mathbb{R})$. The subspace singled out by $\mathbb{L}_1$ is the sum of the corresponding eigenspaces; compare [McM1] Thm. 5.3. This lemma is the only point, besides the definition of a Teichmüller curve that gives rise to $\mathbb{L}_1$, where actually Teichmüller theory is used. A corresponding algebraic argument seems not to be available. It would permit, e.g., extending some of the results in [ViZu05] to higher dimensions.

There is another argument proving $\dim W_1 = 1$ also using the Teichmüller metric, based on [McM1] Thm. 4.2. Suppose the dimension was $s \geq 2$. Choose sections $\omega_i(\tau)$ for $i = 1, \ldots, s$ of the pullback of the $(1,0)$-part of $(\mathbb{L}_1 \otimes W_1) \otimes_{\mathbb{C}} \mathcal{O}_C$ to the universal covering $\mathbb{H}$. We may suppose that ω_1 generates the Teichmüller curve. Choose a symplectic basis $\{a_i, b_i\}$ for $i = 1, \ldots, s$ of sections of $R_1 f_* \mathbb{Z}_{\mathcal{X}}$ such that fiberwise $\int_{a_i} \omega_j = \delta_{ij}$ and $\int_{b_i} \omega_j = 0$ for $i \neq j$. By construction the map

$$\mathbb{H} \ni \tau \mapsto \int_{b_1} \omega_1(\tau) \in \mathbb{H} \tag{3}$$

is an isometry. Since the local system $\mathbb{L}_1$ appears several times the same is true if we replace ω_1 by ω_j for $j = 2, \ldots, s$ in equation (3). This contradicts the theorem just cited, which states that these maps are strict contractions for differentials not proportional to ω_1.

We now study the implications of the decomposition in Prop. 2.4 on the endomorphism ring of the family of Jacobians A/C. Given a local system $\mathbb{V}$ on C we denote by $\mathrm{End}(\mathbb{V}) = H^0(C, \mathbb{E}\mathrm{nd}(\mathbb{V}))$ the global sections of the local system of endomorphisms of $\mathbb{V}$. We have

$$\mathbb{E}\mathrm{nd}(\mathbb{V}_L) \supset \mathbb{E}\mathrm{nd}(\mathbb{W}_L) = \bigoplus_{i=1}^{r} \mathbb{E}\mathrm{nd}(\mathbb{L}_i) \oplus \bigoplus_{1 \leq i,j \leq r, i \neq j} \mathbb{H}om(\mathbb{L}_i, \mathbb{L}_j).$$

For $a \in K$ consider the following element of $\mathrm{End}(\mathbb{V}_L)$: We define its action on $\mathbb{L}_i$ as $a^{\sigma_i} \cdot \mathrm{id}$ (where $\sigma_i \in \mathrm{Gal}(L/\mathbb{Q})$ maps $\mathbb{L}_1$ to $\mathbb{L}_i$). This endomorphism is $\mathrm{Gal}(L/\mathbb{Q})$-invariant, hence in $\mathrm{End}(\mathbb{V}_{\mathbb{Q}})$, and of bidegree $(0,0)$. By Deligne's description of endomorphisms of abelian varieties ([De71] Rem. 4.4.6) this means that the field K is contained in $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. The decomposition of $\mathbb{V}_{\mathbb{Q}}$ over $\mathbb{Q}$ translates into an isogeny of abelian schemes over C,

$$A \longrightarrow A_1 \times_C A_2,$$

where $\dim A_1 = r$.

By the classification of endomorphism rings of abelian varieties (see e.g. [De71] 4.4.5) a field $K \subset \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is totally real or an imaginary quadratic extension of a totally real field. Since K arose as a trace field, it is real and we have shown:

Proposition 2.6. *The trace field $K = \mathbb{Q}(\mathrm{tr}(\Gamma)) = \mathbb{Q}(\mathrm{tr}(\mathrm{SL}(X, \omega)))$ is totally real.*

Theorem 2.7. *The image of a Teichmüller curve $C \to A_g$ is contained in the locus of abelian varieties that split up to isogeny into $A_1 \times_{\mathbb{C}} A_2$, where A_1 has dimension r and real multiplication by the field $K = \mathbb{Q}(\mathrm{tr}(\mathrm{SL}(X, \omega)))$. The generating differential $\omega \in H^0(X, \Omega^1_X)$ is an eigenform for multiplication by K, i.e. $K \cdot \omega \subset \mathbb{C}\omega$.*

Remark 2.8. McMullen noted in [McM1] Thm. 7.5 that the eigenlocus for real multiplication is for $g \geq 3$ no longer invariant under the action of $\mathrm{SL}_2(\mathbb{R})$ on holomorphic differentials of Riemann surfaces of genus g. He starts with an eigendifferential $\omega = x dx/y$ for $\xi = \zeta_7 + \zeta_7^{-1}$ on the curve $X : y^2 = x^7 - 1$ and shows the following: The automorphisms $(x, y) \mapsto (\zeta_7 x, y)$ of X are elements of $\mathrm{Aff}^+(X, \omega)$. The group $\mathrm{SL}(X, \omega)$ contains an element γ with trace ξ, but nevertheless the $\mathrm{SL}_2(\mathbb{R})$-orbit of (X, ω) leaves the locus of real multiplication. This is no contradiction. The point is that the proof of Thm. 2.7 relies on Deligne's semisimplicity, which holds only if on the base of the family any subharmonic function bounded above is constant. The upper half-plane (also mod $\langle \gamma \rangle$) is not of this type.

Remark 2.9. On the curve $X_n : y^2 = x^n - 1$ a basis of holomorphic differentials is $\omega_i = x^{i-1}dx/y$ for $i = 1, \dots, g$. In [Ve89] Veech showed that the geodesics generated by $(\omega_1)^2$ are Teichmüller curves. Thm. 2.7 together with McMullen's theorem shows that, say for odd n, the geodesics generated by $(\omega_i)^2$ for $1 < i < g$ are not Teichmüller curves. In fact one can write $(\omega_i)^2 = \omega_{i-1}\omega_{i+1}$ and apply Ahlfors' variational formula to show that the $\mathrm{SL}_2(\mathbb{R})$-deformations leave the locus of real multiplication.

2.1. **The quaternion algebra associated with** $\mathrm{SL}(X, \omega)$**.** For a Fuchsian group Γ contained in $\mathrm{SL}_2(K_1)$ for some number field K_1 with trace field K, the vector space $K \cdot \Gamma$ naturally has the structure of a quaternion algebra Q over K. For each embedding $\sigma : K_1 \to \mathbb{R}$ the quaternion algebra $Q \times_{K_1,\sigma} \mathbb{R}$ is isomorphic to the matrix ring $M_2(\mathbb{R})$ or to the Hamiltonians. In the first case the place σ is said to be unramified; in the second case it is called ramified.
The analysis of the VHS of a Teichmüller curve yields:

Corollary 2.10. *The quaternion algebra Q associated with* $\mathrm{SL}(X, w)$ *is isomorphic to $M_2(K)$. In particular all infinite places are unramified and $\mathbb{L}$ is defined over K, not only over K_1.*

Proof: The result does not depend on passing from $\mathrm{SL}(X, \omega)$ to the subgroup of finite index Γ. The fiber of $\bigoplus_{i=1}^r \mathbb{L}_i$ over c is defined over $\mathbb{Q}$, say it equals $F \otimes_{\mathbb{Q}} L$. An element $\gamma \in \Gamma$ acts on this fiber as $\mathrm{diag}(\gamma, \gamma^{\sigma_2}, \dots, \gamma^{\sigma_r})$, where $\sigma_i \in \mathrm{Gal}(L/\mathbb{Q})/\mathrm{Gal}(L/K)$. This action commutes with the action of $k \in K$ on $F \otimes_{\mathbb{Q}} L$ as $\mathrm{diag}(k, k, k^{\sigma_2}, k^{\sigma_2}, \dots, k^{\sigma_r}, k^{\sigma_r})$. Hence K indeed acts on F and we obtain a map $\psi : \Gamma \to \mathrm{SL}_2(K)$. The action on $F \otimes_{\mathbb{Q}} \mathbb{R}$ corresponds to post-composing ψ with the r maps $\mathrm{SL}_2(K) \to \mathrm{SL}_2(\mathbb{R})$ from the embeddings $K \hookrightarrow \mathbb{R}$. The first embedding corresponds to the action on the subspace $\langle \mathrm{Re}\, \omega, \mathrm{Im}\, \omega \rangle$. Thus the Fuchsian embedding $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ we started with factors through $\mathrm{SL}_2(K)$. Hence $K \cdot \Gamma = K \cdot \mathrm{SL}_2(K) = M_2(K)$. $\square$

2.2. **Modular embeddings.** The expression "modular embedding" is used in the literature for maps from $\mathbb{H}^n$ to the Siegel half-space $\mathbb{H}_n$ equivariantly for some group actions (see [vG87]) and for maps $\mathbb{H} \to \mathbb{H}^n$ with some equivariance conditions (see [CoWo90], [ScWo00]). We compare that latter type with maps induced from Teichmüller curves.

A subgroup Δ of $\mathrm{SL}_2(\mathbb{R})$ acting on $\mathbb{H}^n$ is called *arithmetic* if there is a quaternion algebra Q with n unramified infinite places and an order $\mathcal{O}$ in Q such that Δ is commensurable to the norm unit group $\{M \in \mathcal{O} : M\mathcal{O} \subset \mathcal{O}, \det(M) = 1\}$. A Fuchsian group Γ is said to have a *modular embedding* if there exists an arithmetic group Δ acting on $\mathbb{H}^n$ for an appropriate n, an inclusion $\varphi : \Gamma \hookrightarrow \Delta$ and a holomorphic embedding

$$\phi = (\phi_1, \dots, \phi_n) : \mathbb{H} \to \mathbb{H}^n$$

such that $\phi_1 = \mathrm{id}$ and $\phi(\gamma z) = \varphi(\gamma)\phi(z)$.

Corollary 2.11. *Let Γ be a subgroup of finite index in the affine group $SL(X, \omega)$ of a Teichmüller curve, chosen as in Section 1.4. Then Γ admits a modular embedding $\mathbb{H} \to \mathbb{H}^r$, where $r = [K : \mathbb{Q}]$.*

Proof: With the choices for the symplectic basis adapted to real multiplication made in the setup (1.6), the universal property of the moduli space of abelian varieties with real multiplication gives a holomorphic map $\phi : \mathbb{H} \hookrightarrow \mathbb{H}^r$. Its components are given by

$$\phi_i : \tau \mapsto \left(\int_{b_i} \omega_i(\tau) \right) / \left(\int_{a_i} \omega_i(\tau) \right),$$

where $\omega_i(\tau)$ are eigenforms for real multiplications on the fiber of $A \to C$ pulled back to $\mathbb{H}$. The rest is parallel to the case $r = 2$ (see [McM1] Thm 10.1). ϕ_1 is an isometry ([McM1] Thm. 4.1), hence a Möbius transformation. Replacing the generating flat surface (X, ω) by some $M \cdot (X, \omega)$ we may suppose it is the identity. Consider a decomposition $A \hookrightarrow A_1 \times_C A_2$ up to isogeny. The endomorphisms of A_1/C are an order $\mathfrak{o} \subset K$. The moduli space of abelian varieties with real multiplication by $\mathfrak{o}$ is $\mathbb{H}^r/\mathrm{SL}_2(\mathfrak{o})$, see Section 1.6, and $\mathrm{SL}_2(\mathfrak{o})$ is an arithmetic subgroup of $\mathrm{SL}_2(\mathbb{R})$.

Hence φ descends to the moduli map for A_1/C given by $C \to \mathbb{H}^r/\mathrm{SL}_2(\mathfrak{o})$. This implies that ϕ is equivariant with respect to an inclusion $\varphi : \Gamma \to \Delta := \mathrm{SL}_2(\mathfrak{o})$, which is of course the same as the map ψ in the proof of the previous corollary. □

Note that in the above proof the decomposition $A \hookrightarrow A_1 \times_C A_2$ up to isogeny is by no means unique. Choosing a different decomposition replaces the group Δ by a commensurable arithmetic subgroup in $\mathrm{SL}_2(K)$.

Corollary 2.11 admits a converse, but depends on being able to detect the image of M_g inside A_g (Schottky problem).

Theorem 2.12. *Let Γ be a Fuchsian group with totally real trace field K such that $g = [K : \mathbb{Q}]$ and such that Γ admits a modular embedding $\phi : \mathbb{H} \to \mathbb{H}^g$. If the composition of ϕ with the map to the moduli space of abelian varieties, i.e. the map*

$$\mathbb{H} \to \mathbb{H}^g \to \mathbb{H}_g \to A_g,$$

factors through the moduli space of curves M_g, then $\mathbb{H}/\Gamma$ is a finite unramified cover of a Teichmüller curve in M_g.

Proof: If we lift the factorization through M_g to the universal covers we obtain maps

$$\mathbb{H} \to T_g \to \mathbb{H}_g \to \mathbb{H},$$

where the last map is $\mathbb{H}_g \ni Z \mapsto z_{11}$. By definition of a modular embedding the composition of these maps is the identity, in particular an isometry with respect to the Kobayashi (or equivalently Poincaré) metric. Since a composition of maps is a Kobayashi isometry if and only if each single map is, we conclude that $\mathbb{H} \to T_g$ is a geodesic. Since $\mathbb{H} \to \mathbb{H}_g$ is equivariant with respect to $\Gamma \to \Delta$, the stabilizer of $\mathbb{H}$ is a lattice and $\mathbb{H} \to T_g$ a Teichmüller curve. □

2.3. **A converse.** We now show that Teichmüller curves are characterized by a splitting of the local system as in Prop. 2.4.

Theorem 2.13. *Let $\pi : \mathcal{X} \to C = \mathbb{H}/\Gamma$ be a family of curves of genus g and suppose that the local system of the family $R^1\pi_*\mathbb{R}_{\mathcal{X}}$ has a direct summand $\mathbb{L}_{\mathbb{R}}$ of rank 2 given by the Fuchsian embedding $i : \Gamma \hookrightarrow \mathrm{SL}_2(\mathbb{R})$ (up to $\mathrm{SL}_2(\mathbb{R})$-conjugation). Then $C \to M_g$ is a finite covering of a Teichmüller curve.*

Proof: By semisimplicity of the VHS (Thm. 2.1) we have a decomposition of the local system

$$R^1 f_*\mathbb{C}_{\mathcal{X}} =: \mathbb{V}_{\mathbb{C}} = \bigoplus_{i=1}^{n} (\mathbb{L}_i \otimes W_i)$$

with $\mathbb{L}_i$ irreducible local systems and W_i vector spaces. Since $\mathbb{L}_{\mathbb{R}}$ is defined via the Fuchsian embedding of Γ it is irreducible and we may suppose $(\mathbb{L}_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{L}_1$. We do a priori not know that the W_i are one-dimensional. But for a one-dimensional subspace $U \subset W_i$ of fixed bidegree Thm. 2.1 shows that $\mathbb{L}_1 \otimes U$ is a sub-$\mathbb{C}$-VHS of $\mathbb{V}_{\mathbb{C}}$.
We consider the classifying map for $\mathbb{L}_1 \otimes U$. In more concrete terms, let $\omega_1(\tau)$ be a section over $\mathbb{H}$ of the pullback of the bundle $E^{1,0}(\mathbb{L}_1 \otimes U) \subset E^{1,0}(\mathbb{V}_C) = f_*\omega_{\mathcal{X}/C}$ to $\mathbb{H}$ and complete it to a basis $\omega_i(\tau)$, $i = 1, \ldots, g$, of sections of the pullback of $f_*\omega_{\mathcal{X}/C}$. Choose a symplectic basis of sections $\{a_i, b_i\}$ of the pullback of $R^1 f_*\mathbb{C}_{\mathcal{X}}$ to $\mathbb{H}$ such that $\int_{a_i} \omega_j(\tau) = \delta_{ij}$ and $(\int_{b_i} \omega_j(\tau))_{i,j} \in \mathbb{H}_g$. Then the classifying map is given by the composition

$$\phi : \mathbb{H} \to T_g \to \mathbb{H}_g \to \mathbb{H},$$

where the last map is $\mathbb{H}_g \ni Z \mapsto z_{11}$.

The hypothesis states that ϕ is equivariant with respect to Γ acting on domain and range via its (Fuchsian) embeddings to $\mathrm{SL}_2(\mathbb{R})$. Hence ϕ descends to a holomorphic endomorphism $\overline{\phi}$ of $\mathbb{H}/\Gamma$. Replacing Γ by a subgroup of finite index if necessary, we may suppose $g(\mathbb{H}/\Gamma) \geq 2$. Since $\overline{\phi}$ cannot be constant it has to be an isomorphism. Thus ϕ is an isometry for the Kobayashi metric. As in the previous theorem this implies that $\mathbb{H} \to T_g$ is an isometry. $\square$

3. Higgs fields

Consider a weight one $\mathbb{Q}$-VHS $(\mathbb{V}, E^{1,0} \subset E = (\mathbb{V} \otimes_{\mathbb{Q}} \mathcal{O}_C)_{\mathrm{ext}})$ on a curve C with unipotent monodromies. We have in mind that it comes from a family of curves $f : \mathcal{X} \to C$. Recall that we denote the smooth compactification of C by $\overline{C}$ and we let $S = \overline{C} - C$. Let $\mathcal{X}_S := \overline{f}^{-1}(S)$ be the nonsmooth fibers of the semistable family. The connection ∇ composed with the inclusion and projection gives a map

$$\Theta^{1,0} : E^{1,0} \to E \to E \otimes \Omega^1_{\overline{C}}(\log S) \to (E/E^{1,0}) \otimes \Omega^1_{\overline{C}}(\log S)$$

that is $\mathcal{O}_{\overline{C}}$-linear. If we extend $\Theta^{1,0}$ by zero mappings to the whole associated graded sheaf $F := \mathrm{gr}(E) = E^{1,0} \oplus E^{0,1}$ of E we obtain a *Higgs bundle* (F, Θ). By definition this is a vector bundle on $\overline{C}$ with a holomorphic map $\Theta : F \to F \otimes \Omega^1_{\overline{C}}(\log S)$, the additional condition $\Theta \wedge \Theta = 0$ being void if the base is a curve.

Sub-Higgs bundles of a Higgs bundle are subbundles that are stabilized by Θ. Simpson's correspondence ([Si90]) allows us to switch back and forth between sub-Higgs bundles of F and sub-local systems of $\mathbb{V}$.

The reason for working with Higgs bundles rather than sublocal systems is twofold. In Lemma 3.4 it will be useful to argue with degrees and the Higgs field has the simple algebraic description as the edge morphism

$$E^{1,0} = f_* \omega_{\overline{\mathcal{X}}/\overline{C}} \to R^1 f_* \mathcal{O}_{\mathcal{X}} \otimes \Omega^1_{\overline{C}}(\log S) = E^{0,1} \otimes \Omega^1_{\overline{C}}(\log S)$$

of the tautological sequence

$$0 \to f^* \Omega^1_{\overline{C}}(\log S) \to \Omega^1_{\overline{\mathcal{X}}}(\log \mathcal{X}_S) \to \Omega^1_{\overline{\mathcal{X}}/\overline{C}}(\log \mathcal{X}_S) \to 0.$$

Now for a Teichmüller curve $f : \mathcal{X} \to C$ we translate the decomposition in Prop. 2.4 of $\mathbb{V}_{\mathbb{C}} := R^1 f_* \mathbb{C}_{\mathcal{X}}$ in the language of Higgs bundles. As in the general setup let E be the vector bundle associated with $\mathbb{V}_{\mathbb{C}}$ and (F, Θ) the corresponding Higgs bundle. It decomposes into the sum of sub-Higgs bundles

$$(\mathcal{L}_i \oplus \mathcal{L}_i^{\vee}, \tau_i^{1,0} : \mathcal{L}_i \to \mathcal{L}_i^{\vee} \otimes \Omega^1_{\overline{C}}(\log S))$$

stemming from the $\mathbb{L}_i$, $i = 1, \ldots, r$ and the sub-Higgs bundle from $\mathbb{M}$, which plays no role in the sequel.

Lemma 3.1. *The Higgs field of all of the $\mathbb{L}_i$ is generically maximal, i.e., $\tau_i^{1,0}$ is injective and precisely the Higgs field of $\mathbb{L}_1$ is maximal, i.e. $\tau_1^{1,0}$ is an isomorphism.*

Proof: By Prop. 4.10 in [Ko85], one can decompose a Higgs bundle F as a sum of Higgs bundles $F_a \oplus N$ with N the maximal flat subbundle, i.e. $\Theta^{1,0}|_N = 0$. For a weight one pVHS the curvature of the Hodge bundle E with respect to the Hodge metric coming from $H(\cdot, \cdot)$ can be expressed in terms of $\Theta^{1,0}$ (see [Sd73] Lemma 7.18, compare also [ViZu04] Section 1). Hence if N is flat, the curvature vanishes and the bundle N comes from a unitary representation.
We apply this to the Higgs bundles $(\mathcal{L}_i \oplus \mathcal{L}_i^{\vee}, \tau_i^{1,0})$. The corresponding representation is Galois conjugate to the Fuchsian representation of Γ, which contains nontrivial parabolic elements since C is not compact. Thus the representation is not unitary. Hence $\tau^{1,0}$ is nonzero, proving the first statement.

Lemma 2.1 in [ViZu04] implies maximality for $\mathbb{L}_1$. If one of the $\mathbb{L}_i$ for $i \neq 1$ was maximal, too, its Higgs field would give an isomorphism $\mathcal{L}_i^{\otimes 2} \cong \Omega^1_{\overline{C}}(\log S)$. Hence $\mathbb{L}_i$ would become isomorphic to $\mathbb{L}_1$ after replacing C by a finite cover. This contradicts Lemma 2.3, which is independent of passing to a finite unramified cover. $\square$

The converse implication of Lemma 2.1 in [ViZu04] gives:

Corollary 3.2. *The groups Γ^σ for $\mathrm{id} \neq \sigma \in \mathrm{Gal}(K/\mathbb{Q})$ are not Fuchsian.*

We let $d_i = \deg(\mathcal{L}_i)$. If we replace C by an unramified covering of degree n the degrees d_i will be multiplied by n. Hence the projectivized r-tuple $(d_1 : \ldots : d_r)$ seems to be an interesting invariant of the Teichmüller curve. Note that Lemma 3.1 says $d_1 > d_i$ for all i.

3.1. **Global sections.** Recall that the local system of a Teichmüller curve has a decomposition $\mathbb{V}_\mathbb{Q} = \mathbb{W}_\mathbb{Q} \oplus \mathbb{M}_\mathbb{Q}$, and over L we have the further decomposition $\mathbb{W}_L = \mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_r$.

Lemma 3.3. *The local system $\mathrm{End}(\mathbb{W}_L \otimes_L \mathbb{C})$ has no global sections of bidegree $(-1,1)$.*

Proof: As above,

$$\mathbb{E}\mathrm{nd}(\mathbb{W}_\mathbb{C}) = \bigoplus_{i=1}^{r} \mathbb{E}\mathrm{nd}((\mathbb{L}_i)_\mathbb{C}) \oplus \bigoplus_{1 \leq i,j \leq r,\, i \neq j} \mathbb{H}om((\mathbb{L}_i)_\mathbb{C}, (\mathbb{L}_j)_\mathbb{C}).$$

Since the $\mathbb{L}_i$ are irreducible and pairwise nonisomorphic, the local system $\mathbb{H}om(\mathbb{L}_i, \mathbb{L}_j)$ has for $i \neq j$ no global sections at all.
We show that $\mathrm{End}(\mathbb{L}_i)$ has only the global sections $\mathbb{C} \cdot \mathrm{id}$, which are in bidegree $(0,0)$. Recall that C is not compact; i.e., the group Γ contains a parabolic element γ. Its action on a fiber of $\mathrm{End}(\mathbb{L}_i)$ decomposes into 2 Jordan blocks, one of size 1 containing the identity and one of size 3. Since global sections are flat, they are contained in a unitary subbundle, on which a unipotent element has to act trivially. Hence they would have to correspond to the one-dimensional eigenspace of the Jordan block. But this contradicts the fact that the flat sections form a direct summand (again Prop. 4.10 in [Ko85]). □

The following Lemma will be used to determine Mumford-Tate groups in Section 4. The facts on Schur functors $\mathbb{S}_\lambda(\cdot)$ for a partition λ of $\{1, \ldots, m\}$ used below can be found in [FuHa91] Chapter 6.

Lemma 3.4. *For all $(m, m') \in \mathbb{N}^2$ the global sections of $\mathbb{W}_\mathbb{Q}^{\otimes m} \otimes (\mathbb{W}_\mathbb{Q}^\vee)^{\otimes m'}$ of bidegree $(0,0)$ are generated by tensor products of global sections of the (trivial) bundles $\det(\mathbb{L}_i)$.*

Proof: We tensor $\mathbb{W}_\mathbb{Q}$ and its summands by $\mathbb{C}$, but we omit this from the notation. Renumber if necessary the indices such that the d_i are nonincreasing. We have

$$\mathbb{W}^{\otimes m} = \bigoplus_{(\lambda_1, \ldots, \lambda_k)} (\bigotimes_{i=1}^{k} \mathbb{S}_{\lambda_i}(\mathbb{L}_i))$$

for partitions λ_i. The Schur functors for partitions with 3 or more rows are zero, because the $\mathbb{L}_i$ are of rank two, while we have for $a + b = \tilde{m}$, $a \geq b$,

$$\mathbb{S}_{\{a,b\}}(\mathbb{L}_i) = \mathrm{Sym}^{\tilde{m}-2b}(\mathbb{L}_i) \otimes \det(\mathbb{L}_i)^a.$$

Since the $\mathbb{L}_i$ have trivial det-bundles and the $\mathbb{L}_i$ are self-dual it suffices to show that $\widetilde{\mathbb{L}} = \otimes_{i \in I} \mathrm{Sym}^{a_i}(\mathbb{L}_i)$ has no global section of bidegree (p,p) for any nonempty subset $I \subset \{1, \ldots, r\}$ and any $a_i > 0$ with $\sum a_i = 2p$ and $1 \in I$. The last property may be assumed since we are interested in sections over $\mathbb{Q}$ and the $\mathbb{L}_i$ are all Galois conjugates.

Suppose the contrary is the case. Denote by $(\widetilde{F}, \theta)$ the Higgs bundle corresponding to $\widetilde{\mathbb{L}}$. The subsystem $\widetilde{\mathbb{U}}$ of $\widetilde{\mathbb{L}}$ generated by all global sections is a polarizable sub-VHS, hence a direct summand by semisimplicity. By the hypothesis we want to contradict that there exists a nontrivial subbundle $\mathbb{U}$ of $\widetilde{\mathbb{U}}$ of type (p,p). Since global sections are flat and the (p,p)-components of a flat section are again flat, the Higgs bundle associated with $\mathbb{U}$ is a direct summand of

$$\widetilde{F}^{p,p} = \bigoplus_{|\alpha| = p} F^\alpha, \quad F^\alpha := \bigotimes_{i \in I} \mathrm{Sym}^{a_i}(\mathcal{L}_i \oplus \mathcal{L}_i^\vee)^{(\alpha_i,\, a_i - \alpha_i)},$$

where $\alpha = \{\alpha_i, i \in I\}$ and $|\alpha| = \sum \alpha_i$.

By Simpson's correspondence ([Si90]) between local systems and Higgs bundles the Higgs subbundle $\mathcal{U}$ is a direct summand and has degree 0. This implies that

$$\mathcal{U} \hookrightarrow \bigoplus_{\deg \alpha = 0} F^{\alpha},$$

where $\deg \alpha = \sum_{i \in I} d_i(2\alpha_i - a_i)$ and $d_i = \deg \mathcal{L}_i$.

Denote by Λ the set of α^{λ} such that $\mathcal{U} \to F^{\alpha^{\lambda}}$ is nonzero. Order Λ lexicographically starting with the α^{λ} where the first component is maximal or if they are equal the second is, etc. Since $1 \in I$ we have $\alpha_1^{\lambda} > 0$ for one α^{λ}; hence by the ordering also $\alpha_1^1 > 0$.

The Higgs field of F^{α} is the sum with appropriate signs

$$\Theta^{p,p} : F^{\alpha} \longrightarrow \Big(\bigoplus_{\substack{|\beta|=1 \\ \alpha-\beta>0}} F^{\alpha-\beta} \Big) \otimes \Omega^1_{\overline{C}}(\log S)$$

of the Higgs fields $\tau_i^{\alpha_i, a_i - \alpha_i}$ of the $\mathrm{Sym}^{a_i}(\mathbb{L}_i)$. Since $d_1 > d_2$, since the d_i are non-increasing and since the $\alpha^{\lambda} \in \Lambda$ satisfy $\deg \alpha^{\lambda} = 0$ the subbundle $F^{\alpha-\beta} \otimes \Omega^1_{\overline{C}}(\log S)$ with $\alpha := \alpha^1$ and $\beta = (1, 0, \ldots, 0)$ only receives a Higgs field from F^{α^1} but from no other $F^{\alpha^{\lambda}}$ for $\lambda \in \Lambda$.

The Higgs field $\tau_1^{1,0}$ is nontrivial by Lemma 3.1, and hence the same is true for all the $\tau_i^{\alpha_i, a_i - \alpha_i}$. Since there are no cancellations by the above argument, the composition

$$\mathcal{U} \longrightarrow \bigoplus\nolimits_{\deg \alpha = 0} F^{\alpha} \xrightarrow{\Theta^{p,p}} \Big(\bigoplus\nolimits_{\substack{|\beta|=1 \\ \alpha-\beta>0}} F^{\alpha-\beta}\Big) \otimes \Omega^1_{\overline{C}}(\log S) \xrightarrow{pr} F^{\alpha^1 - (1,0,\ldots,0)} \otimes \Omega^1_{\overline{C}}(\log S)$$

is nontrivial. This contradicts the flatness of $\mathbb{U}$. □

4. Shimura varieties

Suppose we examine a Teichmüller curve where the local system $\mathbb{M}_{\mathbb{Q}}$ of Prop. 2.4 vanishes, i.e. such that $r = g$. This implies that $\mathbb{V}$ does not split over $\mathbb{Q}$. We will refer to this condition by saying that the family of Jacobians A/C (or equivalently: its generic fiber) is simple. For dimension reasons the endomorphism ring of the generic fiber of A cannot contain a bigger field than K. Nevertheless it might be possible that a Teichmüller curve C maps to a Shimura subvariety properly contained in the locus of real multiplication. This Shimura subvariety could parameterize abelian varieties whose endomorphism ring contains a quaternion algebra, but there are also Shimura subvarieties that are not characterized by endomophisms of abelian varieties; see [Mu69].

The Mumford-Tate group. Let B be an abelian variety with polarization Θ and let $Q(\Theta)$ be the associated alternating form on $H^1(B, \mathbb{Q})$. The *special Mumford-Tate group of* (B, Q) is the largest algebraic $\mathbb{Q}$-subgroup of $\mathrm{Sp}(H^1(B, \mathbb{Q}), Q)$ that leaves the Hodge cycles of $B \times \cdots \times B$ invariant, i.e. all elements in $H^2p(B \times \cdots \times B, \mathbb{Q})^{p,p}$. We denote it by $\mathrm{Hg}(B)$. The *special Mumford-Tate group of a polarized variation of Hodge structures* $\mathbb{W}_{\mathbb{Q}}$, denoted by $\mathrm{Hg} := \mathrm{Hg}(\mathbb{W}_{\mathbb{Q}})$, is the largest $\mathbb{Q}$ subgroup of $\mathrm{Sp}(F, Q)$ fixing the Hodge cycles in $\mathbb{W}^{\otimes m} \otimes (\mathbb{W}^{\vee})^{\otimes m'}$ of bidegree $(0, 0)$ that remain Hodge under parallel transform. Here F is a fiber of $\mathbb{W}_{\mathbb{Q}}$ and Q the polarization. Mumford defines a *Shimura variety* (see [Mu66], [Mu69]) as the coarse moduli space $M(\mathrm{Hg})$ for the functor $\mathcal{M}(\mathrm{Hg})$ representing isomorphism classes of polarized abelian varieties with special Mumford-Tate group equal to a subgroup of Hg. Since the graphs of endomorphisms of abelian varieties are Hodge classes, Shimura varieties of PEL-type (see [Sh66]) fall within this category.
The two definitions of Mumford-Tate groups are related by the following general Lemma (see [Sc96] Lemma 2.3, see also [Mo98] Section 1):

Lemma 4.1. *If $\mathbb{W}_{\mathbb{Q}} = R^1\pi_*\mathbb{Q}$ for a family $g : B \to C$ of abelian varieties, then for a dense subset in C the special Mumford-Tate groups of the fiber coincide with* $\mathrm{Hg}(\mathbb{W}_{\mathbb{Q}})$.

We specialize to a Teichmüller curve and to the family of Jacobians $\pi : A \to C$:

Theorem 4.2. *Suppose that the family of abelian varieties A/C over a Teichmüller curve is simple or equivalently $[K : \mathbb{Q}] = g$. Then the Shimura variety parameterizing abelian varieties with real multiplication by K is the smallest Shimura subvariety of A_g that the Teichmüller curve C maps to.*

Proof: The locus of real multiplication is the moduli space of the Mumford-Tate group Hg_K fixing the Hodge cycles

$$\oplus_\sigma \, \sigma(a) \cdot \mathrm{id}_{\mathbb{L}^\sigma} \quad \text{for} \quad a \in K.$$

By Lemma 3.4 the local system $\mathbb{W}^{\otimes m} \otimes (\mathbb{W}^\vee)^{\otimes m'}$ contains no Hodge cycles other than tensor powers and products of the above; hence $\mathrm{Hg}_K = \mathrm{Hg}(R^1 f_* \mathbb{Q})$. □

5. Rigidity, Galois action

Teichmüller curves with $r = 1$, i.e. with trace field $\mathbb{Q}$, are obtained as unramified coverings of the once-punctured torus by [GuJu00] Thm. 5.5. They are called *square-tiled coverings* or *origamis* because of this topological description. Since they are Hurwitz spaces, they are known ([Lo03], [Mö03]) to be defined over $\overline{\mathbb{Q}}$. We show that this holds for all Teichmüller curves.

Let $\mathbb{W}$ be the local subsystem of $R^1 f_* \mathbb{Q}_{\mathcal{X}}$ from Prop. 2.4, which carries a polarized $\mathbb{Q}$-VHS. The choice of a $\mathbb{Z}$-structure on $\mathbb{W}$ defines a family of abelian subvarieties $A_1/C \to A/C$. We could take the families A_1 and A_2 of Thm. 2.7 and dualize the isogeny to obtain

$$A_1 \hookrightarrow A_1 \times_C A_2 \to A.$$

In any case we do not claim any uniqueness of A_1. We provide A_1 with the pullback of the polarization of A. It will be of type δ, not necessarily principal.

Theorem 5.1. *For a Teichmüller curve C and A_1/C chosen as above the canonical map $C \to A_{r,\delta}$ to the moduli space of δ-polarized abelian varieties of dimension r admits no deformations. In particular the Teichmüller curve C is defined over $\overline{\mathbb{Q}}$.*

Proof: By a theorem of Faltings ([Fa83] Thm. 2) the tangent space to the space of deformations of $C \to A_{r,\delta}$ is a subspace of the global sections of $\mathrm{End}(\mathbb{W}_\mathbb{C})$ of bidegree $(-1, 1)$. The nonexistence of such sections was shown in Lemma 3.3. If the Teichmüller curve were not defined over $\mathbb{Q}$ the transcendental parameters in the defining equations of C would provide a nontrivial deformation of the family of Jacobians $A \to C$. □

Remark 5.2. The proof of Thm. 4.2 gives immediately that the image of $C \to A_{r,\delta}$ lies in the locus of real multiplication but in no smaller Shimura variety. It would be interesting to know whether this is also the case for the map $C \to A_g$ if $[K : \mathbb{Q}] < g$ and whether this map admits deformation.

5.1. An algebraic description of Teichmüller curves.

The hypothesis of unipotent monodromies in the following theorem can always be achieved by passing to a finite covering unramified outside $S = \overline{C} - C$. It is there since "maximal Higgs" does not make sense otherwise.

Theorem 5.3. *Suppose that the Higgs bundle of a family of curves $f : \mathcal{X} \to C = \mathbb{H}/\Gamma$ with unipotent monodromies has a rank two Higgs subbundle with maximal Higgs field. Then $C \to M_g$ is a finite covering of a Teichmüller curve.*

Proof: Let $(\mathcal{L} \oplus \mathcal{L}^{-1}, \tau^{1,0})$ be a maximal Higgs rank-2 subbundle. By Simpson's correspondence (see [Si90] or the summary in [ViZu04]) the local system $\mathbb{V}_\mathbb{C}$ has a rank two direct summand $\mathbb{L}_\mathbb{C}$ that carries a VHS, whose Higgs bundle is $(\mathcal{L} \oplus \mathcal{L}^{-1}, \tau^{1,0})$.

We claim that $\mathbb{L}_{\mathbb{C}}$ is defined over $\mathbb{R}$: By the properties of a VHS the complex conjugate of $\mathbb{L}_{\mathbb{C}}$ has a Higgs bundle $(\mathcal{L} \oplus \mathcal{L}^{-1}, \widetilde{\tau^{1,0}})$. Since the property "maximal Higgs" is equivalent to $2 \deg \mathcal{L} = \deg \Omega^1_{\overline{C}}(\log S)$ we deduce that $\widetilde{\tau^{1,0}}$ is also an isomorphism. Moreover we must have $\mathbb{L}_{\mathbb{C}} \cong \overline{\mathbb{L}_{\mathbb{C}}}$ up to conjugation, since otherwise the argument of [McM1] Thm. 4.2 explained in Remark 2.5 leads to a contradiction. In particular the traces of $\mathbb{L}_{\mathbb{C}}$ are real.

Now consider $\mathbb{L}_{\mathbb{C}}$ as given by a representation

$$\rho : \pi_1(C, c) \to \mathrm{SL}_2(\mathbb{C}).$$

We cannot yet use that the image of ρ equals Γ. But we know that ρ is semisimple and irreducible even if we pass to a subgroup of finite index in $\pi_1(C, c)$. From this one deduces that the image of ρ contains two non-commuting hyperbolic elements. If the eigenvalues of one such element are real, we are done by [Ta69] (compare Lemma 2.2).

So suppose all these eigenvalues are nonreal. If ρ and $\overline{\rho}$ are given in the eigenbasis of a hyperbolic element γ_1, a matrix M that conjugates ρ into $\overline{\rho}$ has to be off diagonal. If γ_2 is hyperbolic and does not commute with γ_1 they do not share an eigenvector. Since M has to be off diagonal, too, if we consider ρ and $\overline{\rho}$ in the eigenbasis for γ_2, we obtain a contradiction.

Using the claim and by Lemma 2.1 in [ViZu04] the property "maximal Higgs" implies that the action of $\pi_1(C)$ on a fiber of $\mathbb{L}$ is just the action of $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$. We can now use Thm. 2.13 to complete the proof. □

This characterization may not be very useful for constructing Teichmüller curves but it has the advantage of being completely algebraic. In [Mö03] we noted that the absolute Galois group $G_{\mathbb{Q}}$ acts on the set of all origamis (in fact faithfully in an appropriate sense).

Corollary 5.4. *The absolute Galois group acts on the set of Teichmüller curves with $r = g$.*

Proof: In case $r = g$ the map $C \to M_g$ is defined over a number field, since $C \to A_g$ is defined over a number field by Thm. 5.1. Hence it makes sense to let $G_{\mathbb{Q}}$ act on the family $f : \mathcal{X} \to C$. The construction of the Higgs bundle (see Section 3) is algebraic. So the Higgs bundle of the Galois conjugate curve will have as many Higgs subbundles of a given rank as the original one; and the $G_{\mathbb{Q}}$-action on families of curves defined over number fields preserves the property of the Higgs field to be an isomorphism on a subbundle. □

Remark 5.5. For the cases $1 < r < g$ we cannot show with these methods that the map $C \to M_g$ is defined over a number field, although we expect this to be true in general. If it is true, then the above proof applies to show that $G_{\mathbb{Q}}$ acts on the set of all Teichmüller curves.

References

[CoWo90] Cohen, P., Wolfart, J., *Modular embeddings for some non-arithmetic Fuchsian groups*, Acta Arithm. 56 (1990), 93–110

[De70] Deligne, P., *Equations Différentielles à Points Singuliers Réguliers*, Springer LNM 163 (1970)

[De71] Deligne, P., *Théorie de Hodge II*, Publ. Math. I.H.E.S. 40 (1971), 5–57

[De87] Deligne, P., *Un théorème de finitude pour la monodromie*, Discrete groups in Geometry and Analysis, Birkhäuser, Progress in Math. 67 (1987), 1–19

[Fa83] Faltings, G., *Arakelov's Theorem for Abelian varieties*, Invent. Math. 73 (1983), 337–347

[FuHa91] Fulton, W., Harris, J., *Representation theory*, GTM 129, Springer (1991)

[GuJu00] Gutkin, E., Judge, C., *Affine mappings of translation surfaces*, Duke Math. J. 103 No. 2 (2000), 191–212

[Ko85] Kollár, J., *Subadditivity of the Kodaira dimension: Fibers of general type*, Adv. Studies in Pure Math. 10 (1987), 361–398

[Lo03] Lochak, P., *On arithmetic curves in the moduli space of curves*, Journal of the Math. Inst. of Jussieu 4, No. 3 (2005), 443–508

[LaBi92] Lange, H., Birkenhake, C., *Complex abelian varieties*, Springer Grundlehren 302 (1992)

[McM1] McMullen, C., *Billiards and Teichmüller curves on Hilbert modular surfaces*, Journal of the AMS 16 (2003), 857–885

[McM2] McMullen, C., *Teichmüller geodesics of infinite complexity*, Acta Math. 191, No. 2 (2003), 191–223

[McM3] McMullen, C., *Dynamics of* $\mathrm{SL}_2(\mathbb{R})$ *over moduli space in genus two*, preprint (2003)

[Mo98] Moonen, B., *Linearity properties of Shimura varieties I*, J. Alg. Geom. 7 No. 3 (1998), 539–567

[Mö03] Möller, M., *Teichmüller curves, Galois actions,* $\widehat{GT}$*-relations*, Math. Nachr. 278, No. 9 (2005), 1061–1077

[Mu66] Mumford, D., *Families of Abelian varieties*, Proc. Symp. Pure Math. 9 (1966), 347–351

[Mu69] Mumford, D., *A note of Shimura's paper: Discontinuous groups and Abelian varieties*, Math. Ann. 181 (1969), 345–351

[Sc96] Schoen, C., *Varieties dominated by product varieties*, Int. J. Math. 7 (1996), 541–571

[Sd73] Schmid, W., *Variation of Hodge Structure: The singularities of the period mapping*, Inv. Math. 22 (1973), 211–319

[ScWo00] Schmutz-Schaller, P., Wolfart, J., *Semi-arithmetic Fuchsian groups and modular embeddings*, J. London Math. Soc. 61 (2000), 13–24

[Si90] Simpson, C., *Harmonic bundles on noncompact curves*, Journal of the AMS 3 (1990), 713–769

[Sh66] Shimura, G., *Moduli of Abelian Varieties and Number Theory*, Proc. Symp. Pure Math. 9 (1966), 313–332

[SmWe04] Smillie, J., Weiss, B., *Minimal sets for flows on moduli space*, Israel J. of Math. 142 (2004), 249–260

[Ta69] Takeuchi, K., *On some discrete subgroups of* $\mathrm{SL}_2(\mathbb{R})$, J. Fac. Sci. Univ. Tokyo 16 (1969), 97–100

[Ve89] Veech, W., *Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards*, Invent. Math. 97 (1989), 533–583

[vG87] van der Geer, G., *Hilbert modular surfaces*, Erg. der Math., Bd. 16, Springer-Verlag (1987)

[ViZu04] Viehweg, E., Zuo, K., *A characterization of Shimura curves in the moduli stack of abelian varieties*, Journal of Diff. Geometry 66 (2004), 233–287

[ViZu05] Viehweg, E., Zuo, K., *Numerical bounds for semi-stable families of curves or of higher-dimensional manifolds*, preprint (2005)

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