Isotrivially fibred isles in the moduli space of surfaces of general type

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Abstract. We complement Catanese's results on isotrivially fibred surfaces by completely describing the components containing an isotrivial surface with monodromy group $\mathbb{Z}/2\mathbb{Z}$. We also give an example for deformation equivalent isotrivial surfaces with different monodromy groups.

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1 Introduction

In [Ca00], Catanese introduced the notion of 'surfaces isogenous to a product', partly because the notion of 'isotrivially fibred surface' is not stable under deformation. For the former he obtains a complete description of the moduli space (of surfaces of general type) by means of fixed loci in products of Teichmüller spaces, while for isotrivially fibred surfaces his description is restricted to the sublocus of surfaces having the same topological type as a given isotrivially fibred surface minus their singular fibres.

In this paper we complement these results in two ways. First we describe entirely the connected components N_2 of the moduli space, which contain these isotrivial isles in the simplest case of the monodromy group $G = \mathbb{Z}/2\mathbb{Z}$. We use the fact (Theorem 3.1) that these surfaces are double coverings of a product of curves if we impose some restrictions on the irregularity. This enables us to enumerate the connected components of N_2, and to calculate its dimension and that of its isotrivially fibred subloci (Corollaries 3.3 and 3.4).

Secondly we give an example, that this method does not extend to bigger monodromy groups, by showing that isotrivially fibred surfaces with different monodromy groups can occur in one connected component of the moduli space (Proposition 4.1). For this purpose we use the description of abelian coverings given in [FP97].

The results of this paper are part of the author's dissertation ([Mo02]), which describes moduli spaces for different types of fibred surfaces, by exploiting high irregularity.

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Notation

- We use the complex numbers $\mathbb{C}$ as our base field throughout.
- By a surface of general type X we think of its canonical model, i.e. a normal surface with K_X ample and at most rational double points.
- $\mathfrak{S}(\cdot)$ is the functor, that associates to a scheme T the isomorphism classes of families of surfaces of general type over T. 'Family' always implies proper and flat. We denote by N its coarse moduli space.
- A fibration of X is a morphism $X \to B$ to a smooth curve B of genus b with connected fibres. Let g denote the genus of the fibres. We call (g, b) the type of the fibration. We will *suppose that* $g \geq 2$ *(fibre condition (FC)) and* $b \geq 2$ *(base condition (BC))*, so that by [Be91] and [Ca00] the property of having a type (g, b)-fibration is deformation invariant. This also implies that minimal models and relatively minimal models coincide, and X admits such a fibration if and only if the minimal model does.
- We denote by $\mathfrak{S}_{g,b}(\cdot)$ the subfunctor of $\mathfrak{S}(\cdot)$ parametrizing (g, b)-fibred surfaces. We only employ this notation when (BC) and (FC) are fulfilled.
- We use different notations to distinguish between a line bundle L on a surface $X \in \mathfrak{S}(\operatorname{Spec}\mathbb{C})$ and a family of line bundles $\mathcal{L}$ on a family of surfaces $X \to T$. Given a locally free sheaf L on X, we denote the associated geometric vector bundle by $\mathbb{V}(L) \to X$.

2 Monodromy and Albanese-image

Given a fibration $h : X \to B$, we let B' be the locus where the fibres are smooth. Recall that h is said to be *isotrivial* (or *of constant moduli*) if the image of the canonical mapping $B' \to M_b$ to the moduli space for smooth curves is just one point.
Due to (FC) there is a group G, the *monodromy group*, such that after a Galois base change $C_1 \to B$ with group G, étale outside the images of degenerate fibres, we have

$$C_1 \times C_2 \cong C_1 \times_B X,$$

where C_2 is a fibre of h. G acts on both C_1 and C_2, and we denote the quotient under the diagonal action by $Q = (C_1 \times C_2)/G$.
For symmetry let $B = B_1$, $b = b_1$, $B_2 = C_2/G$ and $g(B_2) = b_2$. We abbreviate $Y = B_1 \times B_2$ and call the projections $p_i : Y \to B_i$. By [Ser96] we know that $q(X) = b_1 + b_2$, and if $b_2 > 0$, the natural birational map $\varepsilon : X \to Q$ is a morphism, i.e. X has two fibrations $h_1 : X \to B_1$ and $h_2 : X \to B_2$.
Details for this can be found in [Ser96] and [Ca00].

Lemma 2.1. *Let $X \to T \in \mathfrak{S}_{g,b}$ be a family of surfaces. Suppose that (at least) one fibre admits an isotrivial fibration with $q \geq b+2$. If $X \to T$ admits a section, the image of the (relative) Albanese-map α is a family of products of smooth curves B_i ($i = 1, 2$) over T of genera b_1 and b_2.*

Proof: Suppose $T = \operatorname{Spec}\mathbb{C}$ and X is isotrivial. The Albanese variety has dimension $b_1 + b_2$, and its universal property applied to $h_1 \times h_2$ induces an isogeny $\mathrm{Alb}(X) \to \mathrm{Jac}(B_1) \times \mathrm{Jac}(B_2)$ and hence a finite morphism $\alpha(X) \to Y$. The image of Y under the inverse isogeny generates the Albanese variety. Since we excluded the cases $b_1 = 1$ and $b_2 = 1$, this is only possible when $\alpha(X) = Y$.
This argument did not make full use of the isotriviality but only of the existence of both fibrations and numerical conditions, i.e. deformation invariant properties. Hence it also applies to the other fibres and in the relative setting by rigidity of a product of curves. □

Remark 2.2. If $q = b_1 + 1$, the Albanese image need not be a product of B_1 and an elliptic curve. This is the reason why we exclude this case, although similar investigations can be made also for coverings of elliptic fibrations.

3 Isotrivial surfaces with monodromy group $G = \mathbb{Z}/2\mathbb{Z}$

We restrict ourselves in this section to surfaces in $\mathfrak{S}^2_{g,b}$, which is the subfunctor of $\mathfrak{S}_{g,b}$ with $q \geq b_1 + 2$ parametrizing surfaces, whose (generic) degree of the Albanese map equals 2 and where the corresponding components of the moduli space contain an isotrivially fibred surface, i.e. such that Lemma 2.1 applies. We shall exclude the case that G acts freely on $C_1 \times C_2$, which is dealt with in [Ca00] (those surfaces will be called isogenous to a product*).*

Under these conditions Lemma 2.1 ensures for each $X \in \mathfrak{S}^2_{g,b}(T)$ the existence of fibrations $h_i : X \to B_i$: The section required there can be created after an étale base change and these fibrations (though not the Albanese morphism) descend to T. The next theorem will show that α is not only generically of degree 2 but in fact finite.

Recall that a double covering $X \to Y$ of surfaces over T is given by data (L, D), where D is an effective, flat divisor on Y and L is a line bundle on Y with $L^{\otimes 2} = \mathcal{O}_Y(D)$.

Theorem 3.1. *The surfaces in $\mathfrak{S}^2_{g,b}(\mathbb{C})$ are double coverings of a product of curves $Y = B_1 \times B_2$ of genus b_i, $b_i \geq 2$, ramified over a curve with at most simple singularities.*
A general surface in $\mathfrak{S}^2_{g,b}(\mathbb{C})$ is smooth, i.e. given $X_0 \in \mathfrak{S}^2_{g,b}(\mathbb{C})$, there is a family $X \to T \in \mathfrak{S}^2_{g,b}(T)$ over a 1-dimensional pointed base $(T, 0)$, whose fibre over 0 is X_0 and where all the other fibres are smooth.
A surface X in $\mathfrak{S}^2_{g,b}(\mathbb{C})$ is isotrivially fibred if and only if the branch divisor of $X \to Y$ is composed of horizontal and vertical curves.

Proof: Let $X \in \mathfrak{S}^2_{g,b}(\mathbb{C})$. Stein factorisation gives $X \to X' \to Y$, where X' is a double covering of Y. X' has only rational double points (and hence coincides with X) if and only if D has only simple singularities ([BPV84] Th. II.5.1). Thus we have to show that the open subfunctor

$$\widetilde{\mathfrak{S}^2_{g,b}}(T) = \{X \to T \in \mathfrak{S}^2_{g,b}(T) | X \to \alpha(X) \text{ is finite}\}$$

is also closed. Let $X \to T \in \mathfrak{S}^2_{g,b}(T)$ be a family over the pointed scheme $(T, 0)$, such that the restriction to $T' = T \setminus 0$ is in $\widetilde{\mathfrak{S}^2_{g,b}}(T')$. After an étale base change we may suppose that X admits a simultaneous resolution of singularities. $X|_{T'}$ comes with an involution τ whose quotient is $\alpha(X)|_{T'}$. Since X has an ample canonical divisor, we can apply [FP97] Prop. 4.4 to extend τ to an involution on X. All we need is to show that for the fibre over 0 we have $X_0/\tau \cong \alpha(X_0)$. But this follows noting that α factors via X/τ and that $X/\tau \to T$ is flat (see e.g. [Ma97]).
The second assertion follows from the first and the theorem of Bertini, once we have shown that L is ample. This immediately follows from the exclusion of the case of surfaces isogenous to a product, and the last assertion should be clear by the given description. □

Lemma 3.2. *If $X \to T \in \mathfrak{S}^2_{g,b}(T)$ admits a section, there are line bundles $\mathcal{L}_i$ on B_i, such that $\mathcal{L} = p_1^*\mathcal{L}_1 \otimes p_2^*\mathcal{L}_2$.*

Proof: Due to the section we have

$$\mathrm{Pic}_{B_1 \times B_2}/T \cong \mathrm{Pic}_{B_1/T} \times \mathrm{Pic}_{B_2/T} \times \mathrm{Corr}(B_1, B_2),$$

where $\mathrm{Corr}(B_1, B_2)$ is the functor of divisorial correspondences between B_1 and B_2. If $T = \mathrm{Spec}\,\mathbb{C}$ and X is isotrivially fibred, we know that $L^2 = \mathcal{O}_Y(D) = p_1^*\mathcal{O}_{B_1}(D_1) \otimes p_2^*\mathcal{O}_{B_2}(D_2)$. Since $\mathrm{Corr}(B_1, B_2)$ has no nontrivial nilpotent elements, this splitting is also possible for L, and since the zero section of $\mathrm{Corr}(B_1, B_2)$ is an open and closed immersion, this remains also valid for arbitrary $X \to T \in \mathfrak{S}^2_{g,b}(T)$. □

By an étale base change we can always suppose that a section exists. To analyse dimensions and connected components of the moduli space, we can hence use in the sequel the functor

$$Q(T) = \{(Y/T, \mathcal{L}_1, \mathcal{L}_2, s)\}$$

where

$$Y = B_1 \times B_2 \in \mathfrak{C}_{b_1}(T) \times \mathfrak{C}_{b_2}(T), \mathcal{L}_i \in \mathrm{Pic}_{B_i/T}, s \in \Gamma(Y, p_1^*\mathcal{L}_1 \otimes p_2^*\mathcal{L}_2)$$

instead of $\mathfrak{S}^2_{g_1,b_1}(T)$.
If we let $d_i = \deg L_i$, our description yields:

Corollary 3.3. *Two surfaces in $\mathfrak{S}^2_{g_1,b_1}(\mathbb{C})$ lie in the same connected components, if and only if their invariants b_1, b_2, d_1 and d_2 coincide. We hence denote these components by $N_2(b_1, b_2, d_1, d_2)$.*

Proof: By the above theorem the conditions are clearly necessary.
Let $\mathcal{Y} = \mathcal{B}_1 \times \mathcal{B}_2 \to M^{[n]}$ denote the universal family of products of smooth curves together with level-n-structures (to obtain a fine moduli space). Fix a line bundle $L =$

$h_1^*L_1 \otimes h_2^*L_2$ on Y, and let $g : P = \mathrm{Pic}^L_{\mathcal{Y}/M^{[n]}} \to M^{[n]}$ be the component of the relative Picard scheme parametrizing line bundles linearly equivalent to L. Next we take the scheme F representing the functor

$$T \mapsto \{(\mathcal{Y}, \mathcal{L}, s) \,|\, (\mathcal{Y}, \mathcal{L}) \in h_P(T), s \in \Gamma(\mathcal{Y}, \mathcal{L})\},$$

where h_P denotes the functor of points of P. The existence of F is guaranteed by ([EGA] III, Th. 7.7.6) and ([EGA] I, Prop. 9.4.9). We will restrict F to the open locus F', which parametrizes sections s whose zero locus has only simple singularities, and we denote by f the natural morphism $F \to P$. The fibres of f are $H^0(C_1, L_1) \otimes H^0(C_2, L_2)$, where C_i and L_i are the curves and bundles corresponding to the image. Of course, if this vector space is non-zero, the intersection with F' is dense in it.
We claim that $f(F')$ is connected: This is the locus of quadruples (C_i, L_i) $(i = 1, 2)$ such that $H^0(C_1, L_1) \otimes H^0(C_2, L_2)$ is non-zero. g maps $f(F')$ properly onto $M^{[n]}$, which is connected. The fibre of g over $C_1 \times C_2$ is $W^0_{d_1}(C_1) \times W^0_{d_2}(C_2)$, using the notation in [ACGH85]. This space is connected by [ACGH85], Th. V.1.4, and our claim follows by elementary topology.
Suppose F' is the disjoint union of closed subsets A and B. Then $\overline{f(A)}$ and $\overline{f(B)}$ have a common point $p \in f(F')$. Suppose $p \in f(A)$, and hence $f^{-1}(p) \subset A$. Let $B_0 \subset f(B)$ be a subset, over which the fibres of f have constant positive dimension (i.e. $f|_{B_0}$ is a bundle of vector spaces) and such that $p \in \overline{B_0}$. Consider the closure $C = \overline{f^{-1}(B_0)}$ in F, which is still a bundle of vector spaces and whose image under f contains p. $C \cap F'$ is dense in each fibre of $f|_C$, and by construction $B \cap C$ is dense in C. This implies that $f^{-1}(p) \cap B \neq \emptyset$, a contradiction. □

For large d_i we can easily compute the dimension of the moduli space and the locus. Let $h^0_i = \dim H^0(B_i, L_i^{\otimes 2})$ for $i = 1, 2$.

Corollary 3.4. *If $2d_i > 2b_i - 2$ (i.e $L^{\otimes 2} - K_Y$ is ample), the dimension of $N_2(b_1, b_2, d_1, d_2)$ is*

$$\dim N_2 = 4b_1 + 4b_2 - 7 + h^0_1 h^0_2.$$

Isotrivial surfaces form irreducible, closed subvarieties in $N_2(b_1, b_2, d_1, d_2)$ of dimension $4b_1 + 4b_2 + h^0_1$ and of dimension $4b_1 + 4b_2 + h^0_2$.

Proof: $N_2(b_1, b_2, d_1, d_2)$ has a canonical morphism to the moduli space of pairs of curves together with a line bundle of degree d_1 and d_2 respectively. This space is irreducible and has dimension $4b_1 + 4b_2 - 6$. The ampleness of $L^{\otimes 2} - K_Y$ ensures that the fibres are dense in a vector space of dimension $h^0_1 h^0_2 - 1$ and this gives the first assertion.
We obtain the irreducible components of the locus of isotrivial surfaces by taking only sections of the form $s = s^1 \otimes s^2$, i.e. by fixing s^1 and letting s^2 vary or vice versa. The degree hypothesis implies that there are no obstructions to deforming these sections. □

Remark 3.5. These results illustrate in the 'simplest possible' case Catanese's result ([Ca00] Th. 5.4) that if we fix the 'topological type' (see loc. cit. for details) of the isotrivially fibred surface minus the singular fibres, we obtain irreducible components of the isotrivial locus,

4 The monodromy group is not invariant under deformations

This section contains the 'negative' result that in general one cannot hope to analyse components of the moduli space containing isotrivial surfaces by fixing the monodromy group.

Proposition 4.1. *If the isotrivial surfaces X_1 and X_2 belong to one connected component of $N_{g,b}$, the orders of the respective monodromy groups G_1 and G_2 are equal, but these groups need not be isomorphic.*

Proof: The first statement is clear because of Lemma 2.1 and the fact that the group order is the (generic) degree of the Albanese map.
To prove the second we can take the simplest possible case $G_1 = \mathbb{Z}/4\mathbb{Z}$ and $G_2 = (\mathbb{Z}/2\mathbb{Z})^2$ and use the description of the isotrivial surfaces as abelian coverings of $Y = B_1 \times B_2$. Fix Y with $b_i \geq 2$, and take line bundles L_i on B_i together with sections $s^i \in \Gamma(B_i, L_i^{\otimes 4})$. Let $L = p_1^* L_1 \otimes p_2^* L_2$ and $s = s^1 \otimes s^2$. The surface

$$X_{G_1} = V(w_1^4 - s) \subset V(L^{\otimes 4}),$$

where w_1 is a local coordinate of L and V denotes the vanishing locus, is an isotrivial G_1-covering of Y. Since $X_{G_1} = (C_1 \times C_2)/G_1$, where C_i is the G_1-covering corresponding to L_i and s^i, this surface has at most A_3 singularities. Due to the conditions on b_i, no rational curve is contained in X_{G_1}, which is hence in $\mathfrak{S}_{g,b}$ (where, as usual, $b = b_1$ and g is the genus of the fibres of $X \to B_1$).
Fix two generators α and β of G_2 and two sections $s_\alpha = s_\alpha^1 \otimes s_\alpha^2 \in \Gamma(Y, L^{\otimes 4})$, $s_\beta = s_\beta^1 \otimes s_\beta^2 \in \Gamma(Y, L^{\otimes 2})$. We can arrange that the zeros of s_α^i and s_β^i ($i = 1, 2$) are disjoint. Hence the surface

$$X_{G_2} = V(w_\alpha^2 - s_\alpha, w_\beta^2 - s_\beta) \subset \mathbb{V}(L^{\otimes 4} \oplus L^{\otimes 2}),$$

where w_α and w_β are local coordinates of $L^{\otimes 2}$ and L, has only A_1-singularities, i.e. is indeed in $\mathfrak{S}_{g,b}$.

We construct a surface X_1 that is easily seen to deform to both X_{G_1} and X_{G_2}. Suppose there are $s_2 \in \Gamma(Y, L^{\otimes 2})$ and $s_4 \in \Gamma(Y, L)$, such that $\mathrm{div}(s_2)$ and $\mathrm{div}(s_4)$ are both smooth and have only normal crossings and such that $\mathrm{div}(s_4 + s_2^2/4)$ is smooth. We claim that, denoting w_i local coordinates of $L^{\otimes i}$,

$$X_1 = V(w_2^2 - s_2 w_2 - s_4, w_1^2 - w_2) \subset \mathbb{V}(L^{\otimes 4} \oplus L^{\otimes 2})$$

is in $\mathfrak{S}_{g,b}$: The quotient $\tilde{X}_1$ of X_1 under $w_1 \mapsto -w_1$ ramifies over $\mathrm{div}(s_4 + s_2^2/4)$ and is hence smooth. Also, $X_1 \to \tilde{X}_1$ ramifies over $w_2 = 0$, which has smooth components and normal crossings by hypotheses.
Obviously X_1 deforms to $V(w_1^4 - s_4) \subset \mathbb{V}(L^{\otimes 4})$, and this is a deformation of X_{G_1}. By change of coordinates $w_\alpha = w_2 - s_2/2$, $w_1 = w_\beta$, we have

$$X_1 \cong V(w_\alpha^2 - (s_4 + s_2^2/4), w_\beta^2 - w_\alpha - s_2/2) \subset \mathbb{V}(L^{\otimes 4} \oplus L^{\otimes 2}),$$

which obviously deforms to X_{G_2}. □

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