The Deligne-Mumford compactification of the real multiplication locus and Teichmüller curves in genus three

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Abstract

In the moduli space $\mathcal{M}_g$ of genus g Riemann surfaces, consider the locus $\mathcal{RM}_\mathcal{O}$ of Riemann surfaces whose Jacobians have real multiplication by the order $\mathcal{O}$ in a totally real number field F of degree g. If $g = 3$, we compute the closure of $\mathcal{RM}_\mathcal{O}$ in the Deligne-Mumford compactification of $\mathcal{M}_g$ and the closure of the locus of eigenforms over $\mathcal{RM}_\mathcal{O}$ in the Deligne-Mumford compactification of the moduli space of holomorphic one-forms. For higher genera, we give strong necessary conditions for a stable curve to be in the boundary of $\mathcal{RM}_\mathcal{O}$. Boundary strata of $\mathcal{RM}_\mathcal{O}$ are parameterized by configurations of elements of the field F satisfying a strong geometry of numbers type restriction.

We apply this computation to give evidence for the conjecture that there are only finitely many algebraically primitive Teichmüller curves in

$\mathcal{M}_3$. In particular, we prove that there are only finitely many algebraically primitive Teichmüller curves generated by a one-form having two zeros of order 3 and 1. We also present the results of a computer search for algebraically primitive Teichmüller curves generated by a one-form having a single zero.

1 Introduction

Each Hilbert modular surface has a beautiful minimal smooth compactification due to Hirzebruch. Higher-dimensional Hilbert modular varieties instead admit many toroidal compactifications none of which is clearly the best. In this paper, we consider canonical compactifications of closely related varieties, namely the real multiplication locus $\mathcal{RM}_\mathcal{O}$ in the moduli space $\mathcal{M}_g$ of genus g Riemann surfaces, as well as the locus of eigenforms $\Omega\mathcal{E}_\mathcal{O}$ in the bundle $\Omega\mathcal{M}_g \to \mathcal{M}_g$ of holomorphic one-forms.

If g is 3, we give a complete description of the stable curves in the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$ which are in the boundary of $\mathcal{RM}_\mathcal{O}$ and we describe which stable curves equipped with holomorphic one-forms are in the boundary of the eigenform locus $\Omega\mathcal{E}_\mathcal{O}$. The case $g = 2$ is treated in [Bai07]. If $g > 3$, we give strong restrictions on the stable curves in the boundary of $\mathcal{RM}_\mathcal{O}$. This allows one to reduce many difficult questions about Riemann surfaces with real multiplication to concrete problems in algebraic geometry and number theory by passing to the boundary of $\overline{\mathcal{M}}_g$. In this paper, we apply our boundary classification to obtain finiteness results for Teichmüller curves in $\mathcal{M}_3$ and noninvariance of the eigenform locus under the action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\Omega\mathcal{M}_3$.

Boundary of the eigenform locus. We now state a rough version of our calculation of the boundary of the eigenform locus. See Theorems 5.2, 8.1, and 8.5 for precise statements. Consider a totally real cubic field F, and let $\mathcal{O} \subset F$ be the ring of integers (we handle arbitrary orders $\mathcal{O} \subset F$, but stick to the ring of integers here for simplicity). The Jacobian of a Riemann surface X has real multiplication by $\mathcal{O}$ roughly speaking if the endomorphism ring of $\mathrm{Jac}(X)$ contains a copy of $\mathcal{O}$ (see §2 for details). We denote by $\mathcal{RM}_\mathcal{O} \subset \mathcal{M}_3$ the locus of Riemann surfaces whose Jacobians have real multiplication by $\mathcal{O}$. Real multiplication on $\mathrm{Jac}(X)$ determines an eigenspace decomposition of $\Omega(X)$, the space of holomorphic one-forms on X. The eigenform locus $\Omega\mathcal{E}_\mathcal{O} \subset \Omega\mathcal{M}_3$ is the locus of pairs (X, ω), where $\mathrm{Jac}(X)$ has real multiplication by $\mathcal{O}$, and $\omega \in \Omega(X)$ is an eigenform.

The bundle $\Omega\mathcal{M}_g \to \mathcal{M}_g$ extends to a bundle $\Omega\overline{\mathcal{M}}_g \to \overline{\mathcal{M}}_g$ whose fiber over a stable curve X is the vector space of stable forms on X. A stable form over a stable curve is a form which is holomorphic, except for possibly simple poles at the nodes, such that the two residues at a single node are opposite (see §3 for details). We describe here the closure of $\Omega\mathcal{E}_\mathcal{O}$ in $\Omega\overline{\mathcal{M}}_3$, which also determines the closure of $\mathcal{RM}_\mathcal{O}$ in $\overline{\mathcal{M}}_3$.

Consider the quadratic map $Q\colon F \to F$, defined by

$$Q(x) = N_{\mathbb{Q}}^F(x)/x. \tag{1.1}$$

We say that a finite subset $S \subset F$ satisfies the *no-half-space condition* if the interior of the convex hull of $Q(S)$ in the $\mathbb{R}$-span of $Q(S)$ in $F \otimes_{\mathbb{Q}} \mathbb{R}$ contains 0.

It is well known that every stable curve which is in the closure of the real multiplication locus $\mathcal{RM}_{\mathcal{O}} \subset \mathcal{M}_g$ has geometric genus 0 or g (we give a proof via complex analysis in §5). Our description of the closure of the eigenform locus is as follows.

Theorem 1.1. *A geometric genus zero stable form $(X, \omega) \in \Omega\mathcal{M}_3$ lies in the boundary of the eigenform locus $\Omega\mathcal{E}_{\mathcal{O}}$ if and only if*

- *the set of residues of ω is a multiple of $\iota(S)$, for some subset $S \subset F$, satisfying the no-half-space condition and $\mathbb{Z}$-spanning an ideal $\mathcal{I} \subset \mathcal{O}$, and for some embedding $\iota\colon F \to \mathbb{R}$.*
- *and, furthermore, if $Q(S)$ lies in a $\mathbb{Q}$-subspace of F, then an explicit additional equation (see Theorem 8.5), involving cross-ratios of the nodes of X, is satisfied.*

Remark. The more precise version of this theorem, which we state in §5, gives a necessary condition which holds more generally in any genus. In §8, we show that this condition is sufficient in genus three. In fact, it is sufficient also in genus two, but we ignore this case as the boundary of the eigenform locus was previously calculated in the genus two case in [Bai07]. The higher genus cases are more difficult, as the Torelli map $\mathcal{M}_g \to \mathcal{A}_g$ is no longer dominant.

The boundary of $\mathcal{E}_{\mathcal{O}} := \mathbb{P}\Omega\mathcal{E}_{\mathcal{O}}$ has a stratification into topological types, where two stable forms are of the same topological type if there is a homeomorphism between them which preserves residues up to constant multiple. We may encode a topological type by a directed graph with the edges weighted by elements of an ideal $\mathcal{I} \subset \mathcal{O}$. Vertices represent irreducible components, edges represent nodes, and weights represent residues. The corresponding *boundary stratum* of $\mathcal{E}_{\mathcal{O}}$ is a product of moduli spaces $\mathcal{M}_{0,n}$, or a subvariety thereof. For $g = 3$ the possible topological types arising in the boundary of $\mathcal{RM}_{\mathcal{O}}$ are shown in Figure 1 on p. 39. In Appendix A, we give an algorithm for enumerating all boundary strata of $\mathcal{E}_{\mathcal{O}}$ associated to a given ideal $\mathcal{I}$. In Figure 4, we tabulate the number of two-dimensional boundary strata for many different fields.

An important special case is boundary strata parameterizing irreducible stable curves, otherwise known as *trinodal curves.* Consider a basis $\boldsymbol{r} = (r_1, r_2, r_3)$ of an ideal $\mathcal{I} \subset \mathcal{O}$. We say that $\boldsymbol{r}$ is an *admissible basis* of $\mathcal{I}$ if the r_i satisfy the no-half-space condition. Let $\mathcal{S}_{\boldsymbol{r}}^{\iota} \subset \mathbb{P}\Omega\overline{\mathcal{M}}_3$ be the locus of trinodal forms having residues $(\pm\iota(r_1), \pm\iota(r_2), \pm\iota(r_3))$. Since a trinodal curve may be represented by 6 points in $\mathbb{P}^1$ identified in pairs, we may identify $\mathcal{S}_{\boldsymbol{r}}^{\iota}$ with the moduli space $\mathcal{M}_{0,6}$ of such points. Suppose $\boldsymbol{r}$ is admissible. As three points in $\mathbb{R}^3$ whose convex hull contains 0 must be contained in a subspace, we are in the second case of Theorem 1.1, so $\mathcal{E}_{\mathcal{O}} \cap \mathcal{S}^{\iota}_{\boldsymbol{r}}$ is cut out by a single polynomial equation on $\mathcal{S}^{\iota}_{\boldsymbol{r}} \cong \mathcal{M}_{0,6}$. We see in Theorem 8.5 that this equation is

$$R_1^{a_1} R_2^{a_2} R_3^{a_3} = 1, \tag{1.2}$$

where $R_i \colon \mathcal{M}_{0,6} \to \mathbb{C}^*$ are certain cross-ratios of four points and where the a_i are integers determined explicitly by the r_i.

For example, for the integers in the field of discriminant 49 there is just one admissible basis up to scalar multiple. This basis and the complete boundary of $\mathcal{RM}_{\mathcal{O}}$ for this order are given in Appendix A.

Intersecting flats in $\mathrm{SL}_g(\mathbb{Z})\backslash\mathrm{SL}_g(\mathbb{R})/\mathrm{SO}_g(\mathbb{R})$. In §7, we show that the notion of an admissible basis of a lattice in a totally real cubic number field is equivalent to a second condition on bases of totally real number fields, which we call *rationality* and *positivity*. Namely, a $\mathbb{Q}$-basis $r_1, \dots, r_g$ of F is rational and positive if

$$\frac{r_i}{s_i} / \frac{r_j}{s_j} \in \mathbb{Q}^+ \quad \text{for all } i \neq j,$$

where $s_1, \dots, s_g$ is the dual basis of F with respect to the trace pairing (that is such that $\mathrm{Tr}^F_{\mathbb{Q}}(r_i s_j) = \delta_{ij}$). We highlight the conditions of rationality and positivity here, since rationality is on one hand a familiar condition of commensurable moduli (see end of §10) in the framework of Teichmüller curves. On the other hand for irreducible stable curves, i.e. g-nodal curves, this condition is (together with positivity) a characterization of the residues of eigenforms.

There is a classical correspondence between ideal classes in totally real degree g number fields and compact flats in the locally symmetric space $X_g = \mathrm{SL}_g(\mathbb{Z})\backslash\mathrm{SL}_g(\mathbb{R})/\mathrm{SO}_g(\mathbb{R})$, the moduli space of lattices in $\mathbb{R}^g$. Given a lattice $\mathcal{I}$ in a totally real number field F, let $U(\mathcal{I}) \subset F^*$ be the group of totally positive units preserving $\mathcal{I}$, embedded in the group $D \subset \mathrm{SL}_g(\mathbb{R})$ of positive diagonal matrices via the g real embeddings of F. There is an isometric immersion $p_{\mathcal{I}}$ of the flat torus $T(\mathcal{I}) = U(\mathcal{I})\backslash D$ into X_g arising from the right action of D on $\mathrm{SL}_g(\mathbb{Z})\backslash\mathrm{SL}_g(\mathbb{R})$. Let $\mathrm{Rec} \subset X_g$ be the locus of lattices in $\mathbb{R}^g$ which have an orthogonal basis. Rec is a closed, but not compact, $(g-1)$-dimensional flat. In §7, we show that rational and positive bases of lattices in number fields correspond to intersections of the corresponding compact flat with Rec.

Theorem 1.2. *Given an lattice $\mathcal{I}$ in a totally real number field, there is a natural bijection between the set $p_{\mathcal{I}}^{-1}(\mathrm{Rec})$ and the set of rational and positive bases of $\mathcal{I}$ up to multiplication by units, changing signs, and reordering.*

To sketch the bijection, we remark that given a basis of a lattice there is a unique element in D whose action makes the first basis vector parallel to the first vector of the dual basis. Rationality ensures that this automatically also holds for the other basis vectors.

Specializing back to $g = 3$, Theorems 1.1 and 1.2 together imply that there is a natural bijection between boundary strata of eigenform loci $\mathcal{E}_{\mathcal{O}} \subset \mathbb{P}\Omega\mathcal{M}_3$ and intersection points of compact flats in X_3 with the distinguished flat Rec.

Note that X_3 is 5-dimensional, while each flat in X_3 is at most 2-dimensional, so one would not expect many intersections between these flats. Nevertheless, we show in §9 that the ring of integers in each totally real cubic field has some ideal which has an admissible basis. In fact, the computations described in Appendix A suggest that most lattices in cubic fields have many admissible bases, although there are also examples of lattices which have none. It would be an interesting problem to study the asymptotics of counting these bases.

Algebraically primitive Teichmüller curves. There is an important action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\Omega\mathcal{M}_g$, the study of which has many applications to the dynamics of billiards in polygons and translation flows. A major open problem is the classification of $\mathrm{GL}_2^+(\mathbb{R})$-orbit-closures. In genus two, this was solved by McMullen in [McM07], while not even partial classification results are known for higher genera.

Very rarely, a form (X, ω) has a $\mathrm{GL}_2^+(\mathbb{R})$-stabilizer which is a lattice in $\mathrm{SL}_2(\mathbb{R})$. In that case, the $\mathrm{GL}_2^+(\mathbb{R})$-orbit of (X, ω) projects to an algebraic curve in $\mathcal{M}_g$ which is isometrically immersed with respect to the Teichmüller metric. Such a curve in $\mathcal{M}_g$ is called a *Teichmüller curve*. A Teichmüller curve C is uniformized by a Fuchsian group, called the Veech group of C ([Vee89]). The field F generated by the traces of elements in the Veech group is called the *trace field* of C. The trace field is a totally real field of degree at most g. See §10 for basic definitions about Teichmüller curves and the $\mathrm{GL}_2^+(\mathbb{R})$-action.

Our main motivation for this work was the problem of classifying *algebraically primitive* Teichmüller curves in $\mathcal{M}_g$, that is Teichmüller curves whose trace field has degree g. Every algebraically primitive Teichmüller curve lies in $\mathcal{RM}_\mathcal{O}$ for some order $\mathcal{O}$ in its trace field by [Möl06b], and every Teichmüller curve has a cusp, so Theorem 1.1 allows one to approach the classification of Teichmüller curves by studying the possible stable curves which are limits of their cusps.

In $\Omega\mathcal{M}_2$, each eigenform locus $\Omega\mathcal{E}_\mathcal{O}$ is $\mathrm{GL}_2^+(\mathbb{R})$-invariant and contains one or two Teichmüller curves (see [McM03, McM05]). With the exception of the decagon, these Teichmüller curves lie in the stratum $\Omega\mathcal{M}_2(2)$ (where we write $\Omega\mathcal{M}_g(n_1, \dots, n_k) \subset \Omega\mathcal{M}_g$ for the stratum of forms having zeros of order $n_1, \dots, n_k$). These Teichmüller curves were discovered independently by Calta in [Cal04].

A major obstruction to the existence of algebraically primitive Teichmüller curves in higher genus is that the eigenform loci are no longer $\mathrm{GL}_2^+(\mathbb{R})$-invariant. McMullen showed in [McM03] that $\Omega\mathcal{E}_\mathcal{O}$ is not $\mathrm{GL}_2^+(\mathbb{R})$-invariant for $\mathcal{O}$ the ring of integers in $\mathbb{Q}(\cos(2\pi/7))$. We prove in §11 the following stronger non-invariance statement

Theorem 1.3. *The eigenform locus $\Omega\mathcal{E}_\mathcal{O}$ is not invariant for $\mathcal{O}$ the ring of integers in any totally real cubic field.*

This statement is likely to hold for all orders. See the end of §11 for more details. In contrast to the situation in $\mathcal{M}_2$, we give in this paper strong evidence for the following conjecture.

Conjecture 1.4. *There are only finitely many algebraically primitive Teichmüller curves in $\mathcal{M}_3$.*

In §13, we prove the following instance of this conjecture.

Theorem 1.5. *There are only finitely many algebraically primitive Teichmüller curves generated by a form in the stratum $\Omega\mathcal{M}_3(3,1)$.*

The proof uses the cross-ratio equation (1.2) together with a torsion condition from [Möl06a] which gives strong restrictions on Teichmüller curves generated by forms with more than one zero. This torsion condition was used previously in [McM06b] to show that there is a unique primitive Teichmüller curve in $\Omega\mathcal{M}_2(1,1)$ and in [Möl08] to show finiteness of algebraically primitive Teichmüller curves in the hyperelliptic components $\Omega\mathcal{M}_g(g-1,g-1)^{\mathrm{hyp}}$ of $\Omega\mathcal{M}_g(g-1,g-1)$. Similar ideas should establish finiteness in the strata of $\Omega\mathcal{M}_3$ with more than two zeros. In the remaining cases, the two components of $\Omega\mathcal{M}_3(4)$ and the component $\Omega\mathcal{M}_3(2,2)^{\mathrm{odd}}$ of $\Omega\mathcal{M}_3(2,2)$ more ideas are needed. In the component $\Omega\mathcal{M}_3(2,2)^{\mathrm{odd}}$, which does not entirely consist of hyperelliptic curves, the torsion condition gives no information due to the presence of a sublocus parameterizing hyperelliptic curves.

While we cannot rule out infinitely many algebraically primitive Teichmüller curves in the stratum $\Omega\mathcal{M}_3(4)$, Theorem 1.1 gives an efficient algorithm for searching any given eigenform locus $\Omega\mathcal{E}_{\mathcal{O}}$ for Teichmüller curves in this stratum. Given an order $\mathcal{O}$, first one lists all admissible bases of ideals in $\mathcal{O}$ as described in Appendix A. For each admissible basis, there are a finite number of irreducible stable forms having these residues and a fourfold zero. One then lists these possible stable forms and then checks each to see if the cross-ratio equation (1.2) holds. If it never holds, then there are no possible cusps of Teichmüller curves in $\Omega\mathcal{M}_3(4) \cap \Omega\mathcal{E}_{\mathcal{O}}$, so there are no Teichmüller curves.

Due to numerical difficulties with the odd component, we have only applied this algorithm to the hyperelliptic component $\Omega\mathcal{M}_3(4)^{\mathrm{hyp}}$. The algorithm recovers the one known example in this stratum, Veech's 7-gon curve, contained in $\Omega\mathcal{E}_{\mathcal{O}}$ for $\mathcal{O}$ the ring of integers in the unique cubic field of discriminant 49; it has ruled out algebraically primitive Teichmüller curves in $\Omega\mathcal{M}_3(4)^{\mathrm{hyp}}$ for every other eigenform locus it has considered.

Theorem 1.6. *Other than Veech's 7-gon curve there are no algebraically primitive Teichmüller curves generated by a form in $\Omega\mathcal{E}_{\mathcal{O}} \cap \Omega\mathcal{M}_3(4)^{\mathrm{hyp}}$ for $\mathcal{O}$ the ring of integers in any of the 1778 totally real cubic fields of discriminant less than 40000.*

We discuss the algorithm on which this theorem is based in §14. We also give in this section some further evidence for Conjecture 1.4 in $\Omega\mathcal{M}_3(4)^{\mathrm{hyp}}$, that an infinite sequence of algebraically primitive Teichmüller curves in this stratum would have to satisfy some unlikely arithmetic restrictions on the widths of cylinders in periodic directions. We have not yet made an attempt to obtain a statement like Theorem 1.6 for the stratum $\Omega\mathcal{M}_3(3,1)$. The bounds used in

Theorem 1.5 are effective but rely on height bounds that are so bad, that new ideas are needed to make the algorithm feasible.

For completeness we mention that there is no hope of proving a finiteness theorem for algebraically primitive Teichmüller curves in $\overline{\mathcal{M}}_g$ without bounding g. Already Veech's fundamental paper [Vee89] and also [War98] and [BM10] contain infinitely many algebraically primitive Teichmüller curves for growing genus g.

The eigenform locus is generic. A rough dimension count leads one to expect Conjecture 1.4 to hold for the stratum $\Omega\mathcal{M}_3(4)$, as the expected dimension of $\mathcal{E}_\mathcal{O} \cap \mathbb{P}\Omega\mathcal{M}_3(4)$ is zero, which is too small to contain a Teichmüller curve. On the other hand, if the eigenform locus $\Omega\mathcal{E}_\mathcal{O} \subset \Omega\mathcal{M}_3$ is contained in some stratum besides the generic one $\Omega\mathcal{M}_3(1,1,1,1)$, one would expect this intersection to be positive dimensional. This would be a source of possible Teichmüller curves. In §12, we prove that the eigenform locus is indeed generic, i.e. a dense open set is contained in $\Omega\mathcal{M}_3(1,1,1,1)$.

Theorem 1.7. *For any order $\mathcal{O}$ in a totally real cubic field, each component of the eigenform locus $\Omega\mathcal{E}_\mathcal{O}$ lies generically in $\Omega\mathcal{M}_3(1,1,1,1)$.*

The proof uses Theorem 1.1 to construct a stable curve in the boundary of $\Omega\mathcal{E}_\mathcal{O}$ with the property that each irreducible component is a thrice-punctured sphere. A limiting eigenform on this curve must have a simple zero in each component.

Primitive but not algebraically primitive Teichmüller curves. ¿From a Teichmüller curve in $\mathcal{M}_g$, one can construct many Teichmüller curves in higher genus moduli spaces by a branched covering construction. A Teichmüller curve is *primitive* if it does not arise from one in lower genus via this construction. Every algebraically primitive Teichmüller curve is primitive, but the converse does not hold. In $\mathcal{M}_3$, McMullen exhibited in [McM06a] infinitely many primitive Teichmüller curves with quadratic trace field. These curves lie in the intersection of $\Omega\mathcal{M}_3(4)$ with the locus of *Prym eigenforms*, that is, forms (X,ω) with an involution $i\colon X \to X$ such that the -1 part of $\mathrm{Jac}(X)$ is an Abelian surface with real multiplication having ω as an eigenform. It is not known whether all primitive Teichmüller curves in $\mathcal{M}_3$ with quadratic trace fields arise from this Prym construction.

Our approach to classifying algebraically primitive Teichmüller curves could also be applied to the classification of (say) primitive Teichmüller curves in $\mathcal{M}_3$ with quadratic trace field. Given a positive integer d and an order $\mathcal{O}$ in a real quadratic field F, there is the locus $\mathcal{E}_\mathcal{O}(d) \subset \mathbb{P}\Omega\mathcal{M}_3$ of forms (X,ω) such that there exists a degree d map of X onto an elliptic curve E with the kernel of the induced map $\mathrm{Jac}(X) \to E$ having real multiplication by $\mathcal{O}$ with ω as an eigenform. The locus $\mathcal{E}_\mathcal{O}(d)$ is three-dimensional, and $\mathcal{E}_\mathcal{O}(2)$ coincides with McMullen's Prym eigenform locus. Teichmüller curves in $\mathcal{M}_3$ having quadratic trace field must be generated by a form in some $\mathcal{E}_\mathcal{O}(d)$. There is a classification of the geometric genus zero forms in the boundary of $\mathcal{E}_{\mathcal{O}}(d)$, similar to that of Theorem 1.1, with the map Q replaced by a quadratic map

$$\widetilde{Q}\colon F \oplus \mathbb{Q} \to F \oplus \mathbb{Q}.$$

Each boundary stratum of $\mathcal{E}_{\mathcal{O}}(d)$ parameterizing trinodal curves is again a subvariety of $\mathcal{M}_{0,6}$ cut out by an equation in cross-ratios similar to (1.2).

Since the cross-ratio equation (1.2) is responsible for ruling out algebraically primitive Teichmüller curves in $\Omega\mathcal{M}_3(4)$, one might wonder why its analogue does not also rule out McMullen's Teichmüller curves in $\mathcal{E}_{\mathcal{O}}(2)$. The difference is that the cross-ratio equation cutting out the trinodal boundary strata of $\mathcal{E}_{\mathcal{O}}(2)$ no longer depends on the associated residues $r_i \in F$ as in (1.2). Moreover, each such boundary stratum contains forms having a four-fold zero that one can explicitly exhibit for all $\mathcal{O}$, as opposed to the algebraically primitive case where these forms almost never exist. We hope to provide the details of this discussion in a future paper.

Towards the proof of Theorem 1.1. We conclude by summarizing the proof of Theorem 1.1. For simplicity, we continue to assume that $\mathcal{O}$ is a maximal order. The reader may also wish to ignore the case of nonmaximal orders on a first reading.

The real multiplication locus $\mathcal{RM}_{\mathcal{O}} \subset \mathcal{M}_g$ (or more precisely, its lift to the Teichmüller space) is cut out by certain linear combinations of period matrices. To better understand the equations which cut out the real multiplication locus, in §4 we give a coordinate-free description of period matrices. Given an Abelian group L, we define a cover $\mathcal{M}_g(L) \to \mathcal{M}_g$, the moduli space of Riemann surfaces X equipped with a *Lagrangian marking*, that is, an isomorphism of L onto a Lagrangian subspace of $H_1(X;\mathbb{Z})$. We define a homomorphism

$$\Psi\colon\ \mathbf{S}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z})) \to \mathrm{Hol}^*\mathcal{M}_g(L),$$

where $\mathbf{S}_{\mathbb{Z}}(\cdot)$ denotes the symmetric square, and $\mathrm{Hol}^*\mathcal{M}_g(L)$ is the group of nowhere vanishing holomorphic functions on $\mathcal{M}_g(L)$. Each function $\Psi(a)$ is a product of exponentials of entries of period matrices. There is a Deligne-Mumford compactification $\overline{\mathcal{M}}_g(L)$ of $\mathcal{M}_g(L)$ with a boundary divisor D_γ for each $\gamma \in L$, consisting of stable curves where a curve homologous to γ has been pinched. In Theorem 4.1 we show that each $\Psi(a)$ is meromorphic on $\overline{\mathcal{M}}_g(L)$ with order of vanishing

$$\mathrm{ord}_{D_\gamma}\Psi(a) = \langle a, \gamma\otimes\gamma\rangle$$

along D_γ.

Cusps of the real multiplication locus correspond to ideal classes in $\mathcal{O}$ (or extensions of ideal classes if $\mathcal{O}$ is nonmaximal). Given an ideal $\mathcal{I} \subset \mathcal{O}$, we define in §5 a real multiplication locus $\mathcal{RM}_{\mathcal{O}}(\mathcal{I}) \subset \mathcal{M}_3(\mathcal{I})$, covering $\mathcal{RM}_{\mathcal{O}} \subset \mathcal{M}_3$, of surfaces which have real multiplication in a way which is compatible with the Lagrangian marking by $\mathcal{I}$. The closure of $\mathcal{RM}_{\mathcal{O}}(\mathcal{I})$ in $\overline{\mathcal{M}}_3(\mathcal{I})$ covers the closure of the cusp of $\mathcal{RM}_\mathcal{O}$ corresponding to $\mathcal{I}$, so it suffices to compute the closure in $\overline{\mathcal{M}}_3(\mathcal{I})$. In §5, we construct a rank 3 subgroup Γ of $\mathbf{S}_\mathbb{Z}(\mathrm{Hom}(\mathcal{I},\mathbb{Z})) \cong \mathbf{S}_\mathbb{Z}(\mathcal{I}^\vee)$ (where $\mathcal{I}^\vee \subset F$ is the inverse different of $\mathcal{I}$) such that $\mathcal{RM}_\mathcal{O}(\mathcal{I})$ is cut out by the equations

$$\Psi(a) = 1 \tag{1.3}$$

for all $a \in \Gamma$. The proof of Theorem 6.1 yields an identification of Γ with a lattice in F with the property that for each $a \in \Gamma$ and $t \in \mathcal{I}$, the order of vanishing of $\Psi(a)$ along the divisor $D_t \subset \overline{\mathcal{M}}_g(\mathcal{I})$ is

$$\mathrm{ord}_{D_t} \Psi(a) = \langle a, Q(t) \rangle \tag{1.4}$$

with the pairing the trace pairing on F, and with $Q(t)$ as in (1.1).

Now suppose that $\mathcal{S} \subset \overline{\mathcal{M}}_g(\mathcal{I})$ is a boundary stratum which is the intersection of the divisors D_{t_i} for $t_1, \ldots, t_n \in \mathcal{I}$, and suppose that the t_i do not satisfy the no-half-space condition. This means that we can find a vector $a \in F$ such that $\langle a, Q(t_i)\rangle \geq 0$ for each t_i with strict inequality for at least one. Multiplying a by a sufficiently large integer, we may assume $a \in \Gamma$. From (1.3) we see that $\Psi(a) \equiv 1$ on $\mathcal{RM}_\mathcal{O}(\mathcal{I})$, and from (1.4) we see that $\Psi(a) \equiv 0$ on $\mathcal{S}$. It follows that $\overline{\mathcal{RM}_\mathcal{O}(\mathcal{I})} \cap \mathcal{S} = \emptyset$, from which we conclude the first part of Theorem 1.1.

If the $Q(t_i)$ lie in a subspace of F, then we may choose $a \in \Gamma$ to be orthogonal to each $Q(t_i)$. By (1.4), the function $\Psi(a)$ is nonzero and holomorphic on $\mathcal{S}$. The equation $\Psi(a) = 1$ restricted to $\mathcal{S}$ cuts out a codimension-one subvariety of $\mathcal{S}$, which yields the second part of Theorem 1.1. In the case where $\mathcal{S}$ parameterizes trinodal curves, the equation $\Psi(a) = 1$ is exactly the cross-ratio equation (1.2). This concludes the necessity of the conditions of Theorem 1.1.

To obtain sufficiency of these conditions, in §8 we show that one can often define, using the functions $\Psi(a)$, local coordinates from a neighborhood of a boundary stratum $\mathcal{S}$ in $\overline{\mathcal{M}}_g(L)$ into $(\mathbb{C}^*)^m \times \mathbb{C}^n$. In these coordinates, $\mathcal{S}$ is $(\mathbb{C}^*)^m \times \{\mathbf{0}\}$, and the real multiplication locus $\mathcal{RM}_\mathcal{O}(\mathcal{I})$ is a subtorus of $(\mathbb{C}^*)^{m+n}$. The computation of the boundary of the real multiplication locus is thus reduced to the computation of the closure of an algebraic torus in $(\mathbb{C}^*)^{m+n}$. This is taken care of by Theorem 8.14.

Hilbert modular varieties and the locus of real multiplication. We conclude with a discussion of the relation between Hilbert modular varieties and the real multiplication locus. In several textbooks (e.g. [Fre90]) Hilbert modular varieties are defined as the quotients $\mathbb{H}^g/\Gamma$, where $\Gamma = \mathrm{SL}(\mathcal{O} \oplus \mathcal{O}^\vee) \cong \mathrm{SL}_2(\mathcal{O})$ for some order $\mathcal{O} \subset F$, or even more restrictively for $\mathcal{O}$ the ring of integers [Gor02]. There is a natural map from $\mathbb{H}^g/\Gamma$ to the moduli space of Abelian varieties whose image is a component of the locus of Abelian varieties with real multiplication by $\mathcal{O}$. In Appendix B, we provide an example showing that the real multiplication locus need not be connected, so it is in general not the image of $\mathbb{H}^g/\Gamma$. This phenomenon is surely known to experts but is often swept under the rug. If one restricts to quadratic fields (as in [vdG88]) or to maximal orders (as in [Gor02]) this phenomenon disappears.

In this paper, we regard a Hilbert modular variety more generally as a quotient $\mathbb{H}^g/\Gamma'$ for any Γ' commensurable with $\mathrm{SL}_2(\mathcal{O})$. With this more general definition, the locus $\mathcal{RA}_\mathcal{O} \subset \mathcal{A}_g$ of Abelian varieties with real multiplication by $\mathcal{O}$ is parameterized by a union $X_\mathcal{O}$ of Hilbert modular varieties.

The eigenform loci $\mathcal{E}_\mathcal{O} \subset \mathbb{P}\Omega\mathcal{M}_g$ which we compactify are closely related to the Hilbert modular varieties $X_\mathcal{O}$. In genus two, $\mathcal{E}_\mathcal{O}$ is isomorphic to $X_\mathcal{O}$, while in genus three, $\mathcal{E}_\mathcal{O}$ is a (degree-one) branched cover of $X_\mathcal{O}$. In other words, the canonical map $\mathcal{E}_\mathcal{O} \to X_\mathcal{O}$ is one-to-one on the level of points, but the orbifold structures are different. The real multiplication locus $\mathcal{RM}_\mathcal{O} \subset \mathcal{M}_g$ is a quotient of $\mathcal{E}_\mathcal{O}$ by an action of the appropriate Galois group. See §2 for details on Hilbert modular varieties and the various real multiplication loci.

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Notation. Throughout the paper, F will denote a totally real number field, $\mathcal{O}$ an order in F, and $\mathcal{I} \subset F$ a lattice whose coefficient ring contains $\mathcal{O}$.

Given an R-module M, we write $\mathrm{Sym}_R(M)$ for the submodule of $M \otimes_R M$ fixed by the involution $\theta(x \otimes y) = y \otimes x$. We write $\mathbf{S}_R(M)$ for the quotient of $M \otimes_R M$ by the submodule generated by the relations $\theta(z) - z$.

We write $\mathrm{Hom}^+_R(M, N)$ and $\mathrm{Hom}^-_R(M, N)$ for the self-adjoint and anti-self-adjoint maps from M to N when this makes sense (either $N = \mathrm{Hom}_R(M, R)$ or $N = M$ with a bilinear pairing on M).

We write Δ_r for the disk of radius r about the origin in $\mathbb{C}$; we write Δ for the unit disk, and Δ^* for the unit disk with the origin removed.

2 Orders, real multiplication, and Hilbert modular varieties

In this section, we discuss necessary background material on orders in number fields, Abelian varieties with real multiplication, and their various moduli spaces. There are two consequences to keep in mind. First, cusps of Hilbert modular varieties are in bijection with symplectic extensions (Proposition 2.3). We determine in Theorem 2.1 the vector space the extension class lives in. Second, the extension class E will be responsible for a root of unity in the equation alluded to in Theorem 1.1. This root of unity will be introduced in equation (5.5). Unfortunately, with a view towards finiteness results of Teichmüller curves, we know of no *a priori* bound for the order of this root of unity.

Orders. Consider a number field F of degree d. A *lattice in F* (also called *full module*) is a subgroup of the additive group of F isomorphic to a rank d free

Abelian group. An *order* in F is a lattice which is also a subring of F containing the identity element. The ring of integers in F is the unique maximal order.

Given a lattice $\mathcal{I}$ in F, the *coefficient ring* of $\mathcal{I}$ is the order

$$\mathcal{O}_{\mathcal{I}} = \{a \in F : ax \in \mathcal{I} \text{ for all } x \in \mathcal{I}\}.$$

Lattices in finite dimensional vector spaces over F and their coefficient rings are defined similarly.

Ideal classes. Two lattices $\mathcal{I}$ and $\mathcal{I}'$ in F are *similar* if $\mathcal{I} = \alpha\mathcal{I}'$ for some $\alpha \in F$. An *ideal class* is an equivalence class of this relation. Given an order $\mathcal{O}$ the set $\mathrm{Cl}(\mathcal{O})$ of ideal classes of lattices with coefficient ring $\mathcal{O}$ is a finite set (see [BS66]). If $\mathcal{O}$ is the maximal order, $\mathrm{Cl}(\mathcal{O})$ is the ideal class group of $\mathcal{O}$.

Modules over orders. Let $\mathcal{O}$ be an order in a number field F and M a module over $\mathcal{O}$. The *rank* of M is the dimension of $M \otimes \mathbb{Q}$ as a vector space over F. We say M is *proper* if the $\mathcal{O}$-module structure on M does not extend to a larger order in F.

Every finitely generated, torsion-free, rank-one $\mathcal{O}$-module M is isomorphic to a *fractional ideal* of $\mathcal{O}$, that is, a lattice in F whose coefficient ring contains $\mathcal{O}$.

A *symplectic $\mathcal{O}$-module* is a torsion-free $\mathcal{O}$-module M together with a unimodular symplectic form $\langle\ ,\ \rangle \colon M \times M \to \mathbb{Z}$ which is compatible with the $\mathcal{O}$-module structure in the sense that

$$\langle \lambda x, y \rangle = \langle x, \lambda y \rangle$$

for all $\lambda \in \mathcal{O}$ and $x, y \in M$.

We equip F^2 with the symplectic pairing

$$\langle (\alpha_1, \beta_1), (\alpha_2, \beta_2) \rangle = \mathrm{Tr}_{\mathbb{Q}}^{F}(\alpha_1\beta_2 - \alpha_2\beta_1). \tag{2.1}$$

Every rank-two symplectic $\mathcal{O}$-module is isomorphic to a lattice L in F^2 whose coefficient ring contains $\mathcal{O}$, such that the symplectic form on F induces a unimodular symplectic pairing $L \times L \to \mathbb{Z}$.

Inverse different. Given a lattice $\mathcal{I} \subset F$ with coefficient ring $\mathcal{O}$, the *inverse different* of $\mathcal{I}$ is the lattice

$$\mathcal{I}^\vee = \{x \in F : \mathrm{Tr}_{\mathbb{Q}}^{F}(xy) \in \mathbb{Z} \text{ for all } y \in M\}.$$

$\mathcal{I}^\vee$ and $\mathcal{I}$ have the same coefficient rings. The trace pairing induces an $\mathcal{O}$-module isomorphism $\mathcal{I}^\vee \to \mathrm{Hom}(\mathcal{I}, \mathbb{Z})$.

The sum $\mathcal{I} \oplus \mathcal{I}^\vee$ is a symplectic $\mathcal{O}$-module with the canonical symplectic form (2.1).

Symplectic Extensions. We now discuss the classification of certain extensions of lattices in number fields. This will be important in the discussion of cusps of Hilbert modular varieties below.

Let $\mathcal{I}$ be a lattice in a number field F with coefficient ring $\mathcal{O}_{\mathcal{I}}$. An extension of $\mathcal{I}^\vee$ by $\mathcal{I}$ over an order $\mathcal{O} \subset \mathcal{O}_{\mathcal{I}}$ is an exact sequence of $\mathcal{O}$-modules,

$$0 \to \mathcal{I} \to M \to \mathcal{I}^\vee \to 0,$$

with M a proper $\mathcal{O}$-module. Given such an extension, a $\mathbb{Z}$-module splitting $s\colon \mathcal{I}^\vee \to M$ determines a $\mathbb{Z}$-module isomorphism $\mathcal{I} \oplus \mathcal{I}^\vee \to M$. The module M inherits the symplectic form (2.1), which does not depend on the choice of the splitting s since the determinant of an upper triangular matrix does not depend on the off-diagonal entries. We say that this is a *symplectic extension* if the symplectic form is compatible with the $\mathcal{O}$-module structure of M.

Let $E(\mathcal{I})$ be the set of all symplectic extensions of $\mathcal{I}^\vee$ by $\mathcal{I}$ over any order $\mathcal{O} \subset \mathcal{O}_{\mathcal{I}}$ up to isomorphism of exact sequences which are the identity on $\mathcal{I}$ and $\mathcal{I}^\vee$. We give $E(\mathcal{I})$ the usual Abelian group structure: given two symplectic extensions,

$$0 \to \mathcal{I} \xrightarrow{\iota_i} M_i \xrightarrow{\pi_i} \mathcal{I}^\vee \to 0,$$

define $\pi\colon M_1 \oplus M_2 \to \mathcal{I}^\vee$ by $\pi(\alpha, \beta) = \pi_1(\alpha) - \pi_2(\beta)$ and $\iota\colon \mathcal{I} \to M_1 \oplus M_2$ by $\iota = \iota_1 \oplus (-\iota_2)$. The sum of the two extensions is

$$0 \to \mathcal{I} \to \operatorname{Ker}(\pi)/\operatorname{Im}(\iota) \to \mathcal{I}^\vee \to 0.$$

and the identity element is the trivial extension $\mathcal{I} \oplus \mathcal{I}^\vee$.

Let $\operatorname{Hom}^+_{\mathbb{Q}}(F, F)$ be the vector space of endomorphisms of F that are self-adjoint with respect to the trace pairing. Note that $\operatorname{Hom}_F(F, F) \subset \operatorname{Hom}^+_{\mathbb{Q}}(F, F)$. For $x \in F$, let $M_x \in \operatorname{Hom}_F(F, F)$ denote the multiplication-by-x endomorphism.

Given $E \in \operatorname{Hom}^+_{\mathbb{Q}}(F, F)$, let $\mathcal{O}(E)$ be the order

$$\{x \in F : [M_x, E](\mathcal{I}^\vee) \subset \mathcal{I}\},$$

where $[X, Y] = XY - YX$ is the commutator. That $\mathcal{O}(E)$ is a subring of F follows from the formula

$$M_\lambda[M_\mu, E] + [M_\lambda, E]M_\mu = [M_{\lambda\mu}, E].$$

$\mathcal{O}(E)$ is a lattice, as for each $x \in F$, we have $nx \in \mathcal{O}(E)$ for some integer n. Define a symplectic extension $(\mathcal{I} \oplus \mathcal{I}^\vee)_E$ of $\mathcal{I}^\vee$ by $\mathcal{I}$ over $\mathcal{O}(E)$ by giving $\mathcal{I} \oplus \mathcal{I}^\vee$ the $\mathcal{O}(E)$-module structure

$$\lambda \cdot (\alpha, \beta) = (\lambda\alpha + [M_\lambda, E](\beta), \lambda\beta).$$

Theorem 2.1. *The map $E \mapsto (\mathcal{I} \oplus \mathcal{I}^\vee)_E$ induces an isomorphism*

$$\operatorname{Hom}^+_{\mathbb{Q}}(F, F)/(\operatorname{Hom}_F(F, F) + \operatorname{Hom}^+_{\mathbb{Z}}(\mathcal{I}^\vee, \mathcal{I})) \to E(\mathcal{I}).$$

Proof. To see that our map is a well-defined homomorphism is just a matter of working through the definitions, which we leave to the reader.

To show our map is a monomorphism, suppose $(\mathcal{I} \oplus \mathcal{I}^\vee)_E$ is isomorphic to the trivial extension via $\phi\colon (\mathcal{I} \oplus \mathcal{I}^\vee)_E \to \mathcal{I} \oplus \mathcal{I}^\vee$. This isomorphism must be of the form $\phi(\alpha, \beta) = (\alpha + R(\beta), \beta)$ for some self-adjoint $R\colon \mathcal{I}^\vee \to \mathcal{I}$. The condition that this is an $\mathcal{O}(E)$-module isomorphism implies $[M_x, E - R] = 0$ for all $x \in \mathcal{O}(E)$. Since $\mathrm{Hom}_F(F, F)$ is its own centralizer in $\mathrm{Hom}_{\mathbb{Q}}(F, F)$, we must have $E - R \in \mathrm{Hom}_F(F, F)$, so $E \in \mathrm{Hom}_F(F, F) + \mathrm{Hom}^+_{\mathbb{Z}}(\mathcal{I}^\vee, \mathcal{I})$.

Now consider the space $\mathcal{D} = \mathrm{Hom}_{\mathbb{Q}}(F, \mathrm{Hom}^-_{\mathbb{Q}}(F, F))$. We write elements of $\mathcal{D}$ as Q_x with $Q_x \in \mathrm{Hom}^-_{\mathbb{Q}}(F, F)$ for each $x \in F$. Let $\mathcal{C} \subset \mathcal{D}$ be those elements $Q_{\_}$ satisfying

$$M_x Q_y + Q_x M_y = Q_{xy} \tag{2.2}$$

for all $x, y \in F$. We claim that every element of $\mathcal{C}$ is of the form $Q^E_x = [M_x, E]$. To see this, let θ be a generator of F over $\mathbb{Q}$. The map $\mathcal{C} \to \mathrm{Hom}^-_{\mathbb{Q}}(F, F)$ sending $Q_{\_}$ to Q_θ is injective by (2.2), so $\dim \mathcal{C} \leq d(d-1)/2$, where $d = [F : \mathbb{Q}]$. The map $\mathrm{Hom}^+_{\mathbb{Q}}(F, F)/\mathrm{Hom}_F(F, F) \to \mathcal{C}$ sending E to $Q^E_{\_}$ is injective so is an isomorphism because the domain also has dimension $d(d-1)/2$. Thus every element of $\mathcal{C}$ has the desired form.

Now, every symplectic extension of $\mathcal{I}^\vee$ by $\mathcal{I}$ over an order $\mathcal{O}$ is isomorphic as a symplectic $\mathbb{Z}$-module to $\mathcal{I} \oplus \mathcal{I}^\vee$ with the $\mathcal{O}$-module structure,

$$\lambda \cdot (\alpha, \beta) = (\lambda\alpha + Q_\lambda(\beta), \lambda\beta),$$

with $Q_{\_} \in \mathcal{C}$. Since $Q_{\_} = Q^E_{\_}$ for some E, our map is surjective. ∎

Given an order $\mathcal{O} \subset \mathcal{O}_{\mathcal{I}}$, let $E^{\mathcal{O}}(\mathcal{I}) \subset E(\mathcal{I})$ be the subgroup of extensions over some order $\mathcal{O}'$ such that $\mathcal{O} \subset \mathcal{O}' \subset \mathcal{O}_{\mathcal{I}}$, and let $E_{\mathcal{O}}(\mathcal{I}) \subset E^{\mathcal{O}}(\mathcal{I})$ be the set of extensions over $\mathcal{O}$. From the above description of $E(\mathcal{I})$, we obtain:

Corollary 2.2. *$E(\mathcal{I})$ is a torsion group with $E^{\mathcal{O}}(\mathcal{I})$ a finite subgroup.*

If two lattices $\mathcal{I}$ and $\mathcal{I}'$ are in the same ideal class, then the groups $E(\mathcal{I})$ are canonically isomorphic.

Real multiplication. We now suppose F is a totally real number field of degree g.

Consider a principally polarized g-dimensional Abelian variety A. We let $\mathrm{End}(A)$ be the ring of endomorphisms of A and $\mathrm{End}^0(A)$ the subring of endomorphisms such that the induced endomorphism of $H_1(A; \mathbb{Q})$ is self-adjoint with respect to the symplectic structure defined by the polarization.

Real multiplication by F on A is a monomorphism $\rho\colon F \to \mathrm{End}^0(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. The subring $\mathcal{O} = \rho^{-1}(\mathrm{End}(A))$ is an order in F, and we say that A has real multiplication by $\mathcal{O}$.

There can be many ways for a given Abelian variety to have real multiplication by $\mathcal{O}$. We write $\mathrm{Aut}(\mathcal{O}/\mathbb{Z})$ for the subgroup of the automorphism group $\mathrm{Aut}(F/\mathbb{Q})$ which preserves $\mathcal{O}$. If $\rho\colon \mathcal{O} \to \mathrm{End}^0(A)$ is real multiplication of $\mathcal{O}$ on A, then so is $\rho \circ \sigma$ for any $\sigma \in \mathrm{Aut}(\mathcal{O}/\mathbb{Z})$.

Let $\mathcal{A}_g = \mathbb{H}_g/\mathrm{Sp}_{2g}(\mathbb{Z})$ be the moduli space of g-dimensional principally polarized Abelian varieties (where $\mathbb{H}_g$ is the $g(g+1)/2$-dimensional Siegel upper half space). We denote by $\mathcal{RA}_{\mathcal{O}} \subset \mathcal{A}_g$ the locus of Abelian varieties with real multiplication by $\mathcal{O}$.

Eigenforms. Real multiplication $\rho\colon \mathcal{O} \to \mathrm{End}^0(A)$ induces a monomorphism $\rho\colon \mathcal{O} \to \mathrm{End}\,\Omega(A)$, where $\Omega(A)$ is the vector space of holomorphic one-forms on A. Usually, for $\lambda \in \mathcal{O}$ we just write $\lambda \cdot \omega$, for short, instead of $\rho(\lambda)(\omega)$. If $\iota\colon F \to \mathbb{R}$ is an embedding of F, we say that $\omega \in \Omega(A)$ is an ι-eigenform if

$$\lambda \cdot \omega = \iota(\lambda)\omega$$

for all $\lambda \in \mathcal{O}$. Equivalently, ω is an ι-eigenform if

$$\int_{\lambda\cdot\gamma} \omega = \iota(\lambda) \int_\gamma \omega$$

for all $\lambda \in \mathcal{O}$ and $\gamma \in H_1(A;\mathbb{Z})$. If we do not wish to specify an embedding ι, we just call ω an eigenform.

Given an embedding ι and ι-eigenform (A,ω), there is a unique choice of real multiplication $\rho\colon \mathcal{O} \to \mathrm{End}^0(A)$ which realizes (A,ω) as an ι-eigenform. Thus considering ι-eigenforms allows one to eliminate the ambiguity of the choice of real multiplication.

We denote by $\Omega^\iota(A)$ the one-dimensional space of ι-eigenforms. We obtain the eigenform decomposition,

$$\Omega(A) = \bigoplus_{\iota\colon F\to\mathbb{R}} \Omega^\iota(A), \tag{2.3}$$

where the sum is over all field embeddings ι.

We denote by $\Omega\mathcal{A}_g \to \mathcal{A}_g$ the moduli space of pairs (A,ω) where A is a principally polarized Abelian variety and ω is a nonzero holomorphic one-form on A. We write $\mathcal{EA}_{\mathcal{O}} \subset \mathbb{P}\Omega\mathcal{A}_g$ for the locus of eigenforms for real multiplication by $\mathcal{O}$ and $\mathcal{EA}^\iota_{\mathcal{O}}$ for the locus of ι-eigenforms. Note that for $\mathrm{Gal}(\mathcal{O}/\mathbb{Z})$-conjugate embeddings ι and ι', the eigenform loci $\mathcal{EA}^\iota_{\mathcal{O}}$ and $\mathcal{EA}^{\iota'}_{\mathcal{O}}$ coincide (as an ι-eigenform is simultaneously an ι'-eigenform for a Galois conjugate real multiplication); however, each $(A,\omega) \in \mathcal{EA}^\iota_{\mathcal{O}}$ comes with a canonical choice of real multiplication which depends on ι.

Hilbert modular varieties. Choose an ordering $\iota_1,\dots,\iota_g$ of the g real embeddings of F. We use the notation $x^{(i)} = \iota_i(x)$. The group $\mathrm{SL}_2(F)$ then acts on $\mathbb{H}^g$ by $A \cdot (z_i)_{i=1}^g = (A^{(i)} \cdot z_i)_{i=1}^g$, where $\mathrm{SL}_2(\mathbb{R})$ acts on the upper-half plane $\mathbb{H}$ by Möbius transformations in the usual way.

Given a lattice $M \subset F^2$, we define $\mathrm{SL}(M)$ to be the subgroup of $\mathrm{SL}_2(F)$ which preserves M. The Hilbert modular variety associated to M is

$$X(M) = \mathbb{H}^g/\mathrm{SL}(M).$$

Given an order $\mathcal{O} \subset F$, we define

$$X_{\mathcal{O}} = \coprod_{M} X(M),$$

where the union is over a set of representatives of all isomorphism classes of proper rank two symplectic $\mathcal{O}$-modules. If $\mathcal{O}$ is a maximal order, then every rank two symplectic $\mathcal{O}$-module is isomorphic to $\mathcal{O} \oplus \mathcal{O}^\vee$ (this also holds if $g = 2$; see [McM07]), so in this case $X_{\mathcal{O}}$ is connected. In general, $X_{\mathcal{O}}$ is not connected, as there are nonisomorphic proper symplectic $\mathcal{O}$-modules; see Appendix B.

There are canonical maps $j_\iota \colon X_{\mathcal{O}} \to \mathcal{EA}^\iota_{\mathcal{O}}$ and $j \colon X_{\mathcal{O}} \to \mathcal{RA}_{\mathcal{O}}$ defined as follows. Given a lattice $M \subset F^2$ and $\tau = (\tau_i)_{i=1}^g \in \mathbb{H}^g$, we define $\phi_\tau \colon M \to \mathbb{C}^g$ by

$$\phi_\tau(x, y) = (x^{(i)} + y^{(i)}\tau_i)_{i=1}^g.$$

The Abelian variety $A_\tau = \mathbb{C}^g/\phi_\tau(M)$ has real multiplication by $\mathcal{O}$ defined by $\lambda \cdot (z_i)_{i=1}^g = (\lambda^{(i)} z_i)_{i=1}^g$. The form dz_i is an ι_i-eigenform.

The map $j_\iota \colon X_{\mathcal{O}} \to \mathcal{EA}^\iota_{\mathcal{O}}$ is an isomorphism, so we may regard $X_{\mathcal{O}}$ as the moduli space of principally polarized Abelian varieties A with a choice of real multiplication $\rho \colon \mathcal{O} \to \mathrm{End}^0(A)$.

The Galois group $\mathrm{Gal}(\mathcal{O}/\mathbb{Z})$ acts on $X_{\mathcal{O}}$, and the map j factors through to a generically one-to-one map $j' \colon X_{\mathcal{O}}/\,\mathrm{Gal}(\mathcal{O}/\mathbb{Z}) \to \mathcal{RA}_{\mathcal{O}}$.

Cusps of Hilbert modular varieties. The Baily-Borel-Satake compactification $\widehat{X}(M)$ of $X(M)$ is a projective variety obtained by adding finitely many points to $X(M)$ which we call the *cusps* of $X(M)$. More precisely, we embed $\mathbb{P}^1(F)$ in $(\mathbb{H} \cup \{i\infty\})^g$ by $(x : y) \mapsto (x^{(j)}/y^{(j)})_{j=1}^g$. We define $\mathbb{H}^g_F = \mathbb{H}^g \cup \mathbb{P}^1(F)$ with a certain topology whose precise definition is not needed for this discussion; see [BJ06]. The compactification of $X(M)$ is $\widehat{X}(M) = \mathbb{H}^g_F/\mathrm{SL}(M)$. We define $\widehat{X}_{\mathcal{O}}$ to be the union of the compactifications of its components.

Proposition 2.3. *There is a natural bijection between the set of cusps of $X_{\mathcal{O}}$ and the set of isomorphism classes of symplectic extensions*

$$0 \to \mathcal{I} \to N \to \mathcal{I}^\vee \to 0 \tag{2.4}$$

with N a proper rank-two symplectic $\mathcal{O}$-module and $\mathcal{I}$ a torsion-free rank one $\mathcal{O}$-module. The cusps of $X(M)$ correspond to the isomorphism classes of such extensions where $M \cong N$ as symplectic $\mathcal{O}$-modules.

Sketch of proof. Fix a lattice $M \subset F^2$. We must provide a $\mathrm{SL}(M)$-equivariant bijection between lines $L \subset F^2$ and extensions $0 \to \mathcal{I} \to M \to \mathcal{I}^\vee \to 0$ (up to isomorphism which is the identity on M). We assign to a line L, the extension $0 \to L \cap M \to M \to M/(L \cap M) \to 0$. The line L is recovered from an extension $0 \to \mathcal{I} \to M \to \mathcal{I}^\vee \to 0$ by defining $L = \mathcal{I} \otimes \mathbb{Q}$.

The bijection for cusps of $X_{\mathcal{O}}$ follows immediately. ■

Consider the set of all pairs $(\mathcal{I}, E)$, where $\mathcal{I}$ is a lattice in F whose coefficient ring contains $\mathcal{O}$, and $E \in E_{\mathcal{O}}(\mathcal{I})$. The multiplicative group of F acts on such pairs by $a \cdot (\mathcal{I}, E) = (a\mathcal{I}, E^a)$, where $E^a(x) = aE(ax)$ (using the identification of Theorem 2.1). We define a *cusp packet* for real multiplication by $\mathcal{O}$ to be an equivalence class of a pair $(\mathcal{I}, E)$ under this relation.

We denote by $\mathcal{C}(\mathcal{O})$ the finite set of cusp packets for real multiplication by $\mathcal{O}$. We have seen that there are canonical bijections between $\mathcal{C}(\mathcal{O})$, the set of isomorphism classes of symplectic extensions of the form (2.4), the set of cusps of $X_{\mathcal{O}}$, and the set of cusps of $\mathcal{EA}^{\iota}_{\mathcal{O}}$. Moreover, there is a canonical bijection between the set of cusps of $\mathcal{RA}_{\mathcal{O}}$ and $\mathcal{C}(\mathcal{O})/\operatorname{Aut}(\mathcal{O}/\mathbb{Z})$.

3 Stable Riemann surfaces and their moduli

In this section, we discuss some background material on Riemann surfaces with nodal singularities, holomorphic one-forms, and their various moduli spaces.

Stable Riemann surfaces. A *stable Riemann surface* (or stable curve) is a connected, compact, one-dimensional, complex analytic variety with possibly finitely many nodal singularities – that is, singularities of the form $zw = 0$ – such that each component of the complement of the singularities has negative Euler characteristic (equivalently finite automorphism group). In other terms, a stable Riemann surface can be regarded a disjoint union of finite volume hyperbolic Riemann surfaces with cusps, together with an identification of the cusps into pairs, each pair forming a node. We will refer to a pair of cusps facing a node as *opposite cusps*.

The *arithmetic genus* of a stable Riemann surface is the genus of the non-singular surface obtained by thickening each node to an annulus; the *geometric genus* is the sum of the genera of its irreducible components.

Homology. Given a stable Riemann surface X, let X_0 be the complement of the nodes. For each cusp c of X_0, let $\alpha_c \in H_1(X_0; \mathbb{Z})$ be the class of a positively oriented simple closed curve winding once around c, and let $I \subset H_1(X_0; \mathbb{Z})$ be the subgroup generated by the expressions $\alpha_c + \alpha_d$, where c and d are opposite cusps.

We define $\widehat{H}_1(X; \mathbb{Z}) = H_1(X_0; \mathbb{Z})/I$. Defining $C(X) \subset \widehat{H}_1(X; \mathbb{Z})$ to be the free Abelian subgroup (of rank equal to the number of nodes) generated by the α_c, we have the canonical exact sequence

$$0 \to C(X) \to \widehat{H}_1(X; \mathbb{Z}) \to H_1(\widetilde{X}; \mathbb{Z}) \to 0,$$

where $\widetilde{X} \to X$ is the normalization of X. See e.g. the appendix of [Bai07] for the basic properties of normalization.

Markings. Fix a genus g surface Σ_g, and let X be a genus g stable Riemann surface. A *collapse* is a map $f\colon \Sigma_g \to X$ such that the inverse image of each node is a simple closed curve and f is a homeomorphism on the complement of these curves.

A *marked stable Riemann surface* is a stable Riemann surface X together with a collapse $f\colon \Sigma_g \to X$. Two marked stable Riemann surfaces $f\colon \Sigma_g \to X$ and $g\colon \Sigma_g \to Y$ are equivalent if there is homeomorphism $\phi\colon \Sigma_g \to \Sigma_g$ which is homotopic to the identity and a conformal isomorphism $\psi\colon X \to Y$ such that $g \circ \phi = \psi \circ f$.

Augmented Teichmüller space. The *Teichmüller space* $\mathcal{T}(\Sigma_g)$ is the space of nonsingular marked Riemann surfaces of genus g. It is contained in the *augmented Teichmüller space* $\overline{\mathcal{T}}(\Sigma_g)$, the space of marked stable Riemann surfaces of genus g. We give $\overline{\mathcal{T}}(\Sigma_g)$ the smallest topology such that the hyperbolic length of any simple closed curve is continuous as a function $\overline{\mathcal{T}}(\Sigma_g) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$. Abikoff [Abi77] showed that this topology agrees with other natural topologies on $\overline{\mathcal{T}}$ defined via quasiconformal mappings or quasi-isometries.

Deligne-Mumford compactification. The mapping class group $\operatorname{Mod}(\Sigma_g)$ of orientation preserving homeomorphisms of Σ_g defined up to isotopy acts on $\mathcal{T}(\Sigma_g)$ and $\overline{\mathcal{T}}(\Sigma_g)$ by precomposition of markings. The moduli space of genus g Riemann surfaces is the quotient $\mathcal{M}_g = \mathcal{T}(\Sigma_g)/\operatorname{Mod}(\Sigma_g)$. The Deligne-Mumford compactification of $\mathcal{M}_g$ is $\overline{\mathcal{M}}_g = \overline{\mathcal{T}}(\Sigma_g)/\operatorname{Mod}(\Sigma_g)$, the moduli space of genus g stable curves.

Over $\overline{\mathcal{M}}_g$ is the universal curve $p\colon \overline{\mathcal{C}} \to \overline{\mathcal{M}}_g$, a compact algebraic variety whose fiber over a point representing a stable curve X is a curve isomorphic to X (provided X has no automorphisms).

Stable Abelian differentials. Over $\mathcal{M}_g$ is the vector bundle $\Omega\mathcal{M}_g \to \mathcal{M}_g$ whose fiber over X is the space $\Omega(X)$ of holomorphic one-forms on X. We extend this to the vector bundle $\Omega\overline{\mathcal{M}}_g \to \overline{\mathcal{M}}_g$ whose fiber $\Omega(X)$ over X is the space of stable Abelian differentials on X, defined as follows.

Given a genus g stable Riemann surface X, a *stable Abelian differential* is a holomorphic one-form on X_0, the complement in X of its nodes, such that:

- ω has at worst simple poles at the cusps of X_0.
- If p and q are opposite cusps of X_0, then

$$\operatorname{Res}_p \omega = -\operatorname{Res}_q \omega.$$

The dualizing sheaf ω_X is the sheaf on X of one-forms locally satisfying the two above conditions (see [HM98, p. 82]), so a stable Abelian differential is simply a global section of the dualizing sheaf ω_X. We write $\Omega(X)$ for the space of stable Abelian differentials on X, a g-dimensional vector space by Serre duality.

In the universal curve $p\colon \overline{\mathcal{C}} \to \overline{\mathcal{M}}_g$, let $\overline{\mathcal{C}}_0$ be the complement of the nodes of the fibers. The relative cotangent sheaf of $\overline{\mathcal{C}}_0 \to \overline{\mathcal{M}}_g$ (the sheaf of cotangent vectors to the fibers) is an invertible sheaf which extends in a unique way to an invertible sheaf $\omega_{\overline{\mathcal{C}}/\overline{\mathcal{M}}_g}$ on $\overline{\mathcal{C}}$, the relative dualizing sheaf of this family of curves.

The restriction of $\omega_{\overline{\mathcal{C}}/\overline{\mathcal{M}}_g}$ to a fiber X of this family is simply ω_X. The push-forward $p_*\omega_{\overline{\mathcal{C}}/\overline{\mathcal{M}}_g}$ is the sheaf of sections of the rank g vector bundle $\Omega\overline{\mathcal{M}}_g \to \overline{\mathcal{M}}_g$.

Plumbing coordinates. Following Wolpert [Wol89] we give explicit holomorphic coordinates at the boundary of $\overline{\mathcal{M}}_g$ and a model of the universal curve in these coordinates. See also [Ber74, Ber81] and [Mas76].

Let X be a stable curve with nodes $n_1, \dots, n_k$, and let X_0 be X with the nodes removed, a disjoint union of punctured Riemann surfaces. At each node n_i, let U_i and V_i be small neighborhoods of n_i in each of the two branches of X through n_i, and choose conformal maps $F_i\colon U_i \to \mathbb{C}$ and $G_i\colon V_i \to \mathbb{C}$ whose images contain the unit disk around the origin Δ_1. We write z_i and w_i for the coordinates on U_i and V_i induced by these maps. We define

$$X^* = X \setminus \bigcup_i (\{|z_i| < 1/2\} \cup \{|w_i| < 1/2\}) \quad \text{and}$$
$$M = X^* \times \Delta_1^k.$$

We take a model of a degeneration of a family of curves.

$$\mathbb{V}_i = \{(x_i, y_i, \boldsymbol{t}) \in \Delta_1 \times \Delta_1 \times \Delta_1^k : x_i y_i = t_i\},$$

where $\boldsymbol{t} = (t_i, \dots, t_k)$. The fiber $\mathbb{V}^{\boldsymbol{t}}$ of the projection $(x_i, y_i, \boldsymbol{t}) \mapsto \boldsymbol{t}$ is a nonsingular annulus except when $t_i = 0$, in which case it is two disks meeting at a node.

Let $\mathcal{X} \to \Delta_1^k$ be the family of stable curves obtained by gluing each $\mathbb{V}_i$ to M by the maps

$$\widehat{F}_i(p, \boldsymbol{t}) = (F_i(p), t_i/F_i(p), \boldsymbol{t}) \quad \text{and} \quad \widehat{G}_i(p, \boldsymbol{t}) = (t_i/G_i(p), G_i(p), \boldsymbol{t}),$$

defined on subsets of M. The fiber $X_{\boldsymbol{t}}$ over $\boldsymbol{t}$ is simply the stable Riemann surface obtained by removing the disks $\{|z_i| < |t_i|^{1/2}\}$ and $\{|w_i| < |t_i|^{1/2}\}$ and gluing the boundary circles by the relation $w_i = t_i/z_i$. If $t_i = 0$, the node n_i is unchanged.

Let Q be the space of holomorphic quadratic differentials on X_0 with at worst simple poles at the nodes. Choose $3g-3-k$ Beltrami differentials μ_i on $X_0 \setminus \bigcup(U_i \cup V_i)$ so that no nontrivial linear combination of the μ_i pairs trivially with a quadratic differential in Q (the pairing is defined by integrating the product of the two differentials over X; see [IT92]). Given $\boldsymbol{s} \in \Delta_\epsilon^{3g-3-k}$ for sufficiently small ϵ, the Beltrami differential $\mu_{\boldsymbol{s}} = \sum s_i \mu_i$ satisfies $\|\mu_{\boldsymbol{s}}\|_\infty < 1$ (where $\|\cdot\|_\infty$ denotes the L^∞-norm).

We define a family of stable curves $\mathcal{Y} \to \Delta_\epsilon^{3g-3-k} \times \Delta_1^k$ by endowing $\mathcal{Y} = \mathcal{X} \times \Delta_\epsilon^{3g-3-k}$ with the complex structure on $\mathcal{Y}$ defined by placing on each fiber $X_{\boldsymbol{t}}^{\boldsymbol{s}}$ over $(\boldsymbol{s}, \boldsymbol{t})$ the Beltrami differential $\mu_{\boldsymbol{s}}$.

We obtain a holomorphic (orbifold) coordinate chart $\Delta_s^{3g-3-k} \times \Delta_1^k \to \overline{\mathcal{M}}_g$ sending $(\boldsymbol{s}, \boldsymbol{t})$ to the point representing the stable curve $X_{\boldsymbol{t}}^{\boldsymbol{s}}$. The family $\mathcal{Y}$ is the pullback of the universal curve by this coordinate chart.

Lagrangian markings. Given a genus g stable curve X, a *Lagrangian subgroup* of $\widehat{H}_1(X;\mathbb{Z})$ is a free Abelian subgroup L of rank g such that $\widehat{H}_1(X;\mathbb{Z})/L$ is torsion-free and the restriction of the intersection form on $H_1(\widetilde{X};\mathbb{Z})$ to the image of L under the canonical projection $\widehat{H}_1(X;\mathbb{Z}) \to H_1(\widetilde{X};\mathbb{Z})$ is trivial.

Fix a free Abelian group L of rank g. A *Lagrangian marking* of a genus g stable Riemann surface X by L is a monomorphism $\rho\colon L \to \widehat{H}_1(X;\mathbb{Z})$ whose image is a Lagrangian subgroup. The image $\rho(L)$ necessarily contains the subgroup $C(X)$ of $\widehat{H}_1(X;\mathbb{Z})$ generated by the nodes. Thus we may assign to each node of X its "homology class" in L, an element of L well-defined up to sign.

Let $\overline{\mathcal{M}}_g(L)$ be the space of genus g stable Riemann surfaces with a Lagrangian marking by L and $\mathcal{M}_g(L) \subset \overline{\mathcal{M}}_g(L)$ the subspace of nonsingular surfaces. If we identify L with a Lagrangian subgroup of $H_1(\Sigma_g;\mathbb{Z})$, we have

$$\mathcal{M}_g(L) = \mathcal{T}(\Sigma_g)/\operatorname{Mod}(\Sigma_g, L),$$

where $\operatorname{Mod}(\Sigma_g, L)$ is the subgroup of $\operatorname{Mod}(\Sigma_g)$ fixing L pointwise. Moreover

$$\overline{\mathcal{M}}_g(L) = \overline{\mathcal{T}}(\Sigma_g, L)/\operatorname{Mod}(\Sigma_g, L),$$

where $\overline{\mathcal{T}}(\Sigma_g, L) \subset \overline{\mathcal{T}}(\Sigma_g)$ is the locus of stable Riemann surfaces which can be obtained by collapsing only curves on Σ_g whose homology class belongs to L (including homologically trivial curves).

Given a nonzero $\gamma \in L$, there is the divisor $D_\gamma \subset \overline{\mathcal{M}}_g(L)$ consisting of stable curves where a curve homologous to γ has been pinched. The divisors D_γ and $D_{-\gamma}$ are the same.

The above plumbing coordinates provide in the same way coordinates at the boundary of $\overline{\mathcal{M}}_g(L)$.

Weighted stable curves. Given a free Abelian group L, we define an *L-weighted stable curve* to be a geometric genus 0 stable curve with an element of L associated to each cusp of X, called the weight of that cusp, subject to the following restrictions:

- Opposite cusps of X_0 have opposite weights.
- The sum of the weights of the cusps of an irreducible component of X is zero.
- The weights of X span L.

Note that the first two conditions mean that the weights are subject to the same restrictions as the residues of a stable form.

We say that two L-weighted stable curves X and Y are *isomorphic* (resp. *topologically equivalent*) if there is a weight-preserving conformal isomorphism (resp. homeomorphism) $X \to Y$.

The notion of an L-weighting of a geometric genus 0 stable curve X is in fact equivalent to a Lagrangian marking $\rho\colon L \to \widehat{H}_1(X;\mathbb{Z})$ (necessarily an isomorphism because X is genus 0). If $\alpha_c \in \widehat{H}_1(X;\mathbb{Z})$ is the class of a positively oriented curve around a cusp c with weight w, the marking ρ maps w to α_c.

Weighted boundary strata. An *L-weighted boundary stratum* is a topological equivalence class in the set of all L-weighted stable curves. If X is an L-weighted stable curve having m components C_i, each homeomorphic to $\mathbb{P}^1$ with n_i points removed and with each component having distinct weights, then the corresponding L-weighted boundary stratum is an algebraic variety isomorphic to

$$\prod_{i=1}^{m} \mathcal{M}_{0,n_i},$$

where $\mathcal{M}_{0,n}$ is the moduli space of n labeled points on $\mathbb{P}^1$, with each point being labeled by its weight.

The notion of a L-weighted boundary stratum is in fact equivalent to that of a boundary stratum in $\overline{\mathcal{M}}_g(L)$ parameterizing curves of geometric genus zero. We consider two marked stable curves (X,ρ) and (Y,σ) in $\overline{\mathcal{M}}_g(L)$ to be equivalent if there is a homeomorphism $f\colon X \to Y$ which commutes with the markings, and we define a *Lagrangian boundary stratum* in $\partial\overline{\mathcal{M}}_g(L)$ to be an equivalence class of this relation. A Lagrangian boundary stratum is simply a maximal connected subset of $\partial\overline{\mathcal{M}}_g(L)$ parameterizing homeomorphic stable curves.

In view of the above correspondence between L-weightings and Lagrangian markings by L, every L-weighted boundary stratum $\mathcal{S}$ can be regarded canonically as a geometric genus zero Lagrangian boundary stratum $\mathcal{S} \subset \overline{\mathcal{M}}_g(L)$, and vice-versa.

Given an L-weighted boundary stratum $\mathcal{S}$, we define $\operatorname{Weight}(\mathcal{S}) \subset L$ to be the set of weights of any surface in $\mathcal{S}$.

Embeddings of strata. Suppose now that $\mathcal{I}$ is a lattice in a degree g number field F. Given an $\mathcal{I}$-weighted boundary stratum $\mathcal{S}$ and a real embedding ι of F, we define $p_\iota\colon \mathcal{S} \to \mathbb{P}\Omega\overline{\mathcal{M}}_g$ by associating to a weighted stable curve X the unique stable form on X which has residue $\iota(w)$ at a cusp with weight w. The ι^{th} embedding $\mathcal{S}^\iota$ of $\mathcal{S}$ is its image under p_ι.

Similar strata. Suppose $\mathcal{I}$ and $\mathcal{J}$ are lattices in a number field F. We say that $\mathcal{I}$ and $\mathcal{J}$-weighted stable curves X and Y are *similar* if there is a conformal isomorphism $X \to Y$ which sends each weight x to λx for some fixed $\lambda \in F$.

We say that two weighted boundary strata are *similar* if they parameterize similar weighted stable curves. Note that if the unit group of F is infinite, then

$\mathcal{I}$-weighted boundary stratum is similar to infinitely many distinct $\mathcal{I}$-weighted boundary strata.

Extremal length and the Hodge norm. Given any Riemann surface X, the *Hodge norm* on $H_1(X;\mathbb{R})$ induced by

$$\|\gamma\|_X = \sup_{\omega \in \Omega_1(X)} \left| \int_\gamma \omega \right|,$$

on $H_1(X;\mathbb{Z})$, where $\Omega_1(X)$ denotes the space of forms with unit norm, for the norm

$$\|\omega\| = \left(\int_X |\omega|^2 \right)^{1/2}.$$

Given a curve γ on a Riemann surface X, we write $\operatorname{Ext}(\gamma)$ for the extremal length of the family of curves which are homotopic to γ, that is

$$\operatorname{Ext}(\gamma) = \sup_\rho \frac{L(\rho)^2}{A(\rho)},$$

where the supremum is over all conformal metrics $\rho(z)dz$ with ρ nonnegative and measurable,

$$L(\rho) = \inf_{\delta \simeq \gamma} \int_\delta \rho(z)|dz|,$$

and

$$A(\rho) = \int_X \rho(z)^2 |dz|^2.$$

The relation between curves with small extremal length and homology classes with small Hodge norm is summarized by the following two Propositions.

Proposition 3.1. *For any curve γ on a Riemann surface X, we have*

$$\|\gamma\|_X^2 \le \operatorname{Ext}(\gamma).$$

Proof. Choose a form ω such that $\|\omega\| = 1$ and $|\int_\gamma \omega| = \|\gamma\|_X$. Regarding $|\omega|$ as a conformal metric on X, we obtain

$$\|\gamma\|_X = \left| \int_\gamma \omega \right| \le \int_\gamma |\omega|,$$

thus, using the compactness of X,

$$\|\gamma\|_X^2 \le L(|\omega|)^2 \le \operatorname{Ext}(\gamma). \qquad \blacksquare$$

Proposition 3.2. *Given any Riemann surface X with $g(X) > 1$, there is a constant C, depending only on the genus of X, such that any cycle $\gamma \in H_1(X;\mathbb{Z})$ is homologous to a sum of simple closed curves $\gamma_1, \dots, \gamma_n$ such that for each i,*

$$\operatorname{Ext}(\gamma_i) \le C\|\gamma\|_X^2 \tag{3.1}$$

Proof. Let ω be a holomorphic one-form on X such that $\operatorname{Im}\omega$ is Poincaré dual to γ. Since $\operatorname{Im}\omega$ has integral periods, the map $f\colon X \to \mathbb{R}/\mathbb{Z}$ defined by $f(q) = \int_p^q \operatorname{Im}\omega$ (with p a chosen basepoint) is well-defined. The horizontal foliation of ω (that is, the kernel foliation of $\operatorname{Im}\omega$) is periodic, and each fiber $\gamma_r = f^{-1}(r)$ is a union of closed, horizontal leaves of ω. Giving the leaves of γ_r the orientation defined by $\operatorname{Re}\omega$, we can regard γ_r as a cycle in $H_1(X;\mathbb{Z})$ which is homologous to γ. By Poincaré duality,

$$\operatorname{length}(\gamma_r) = \int_{\gamma_r} \operatorname{Re}\omega = \int_X \operatorname{Re}\omega \wedge \operatorname{Im}\omega = \frac{1}{2}\|\omega\|^2,$$

so each component of γ_r has length at most $\|\omega\|^2/2$.

Since ω has at most $2g-2$ distinct zeros, there is an open interval $I \subset \mathbb{R}/\mathbb{Z}$ of length at least $1/(2g-2)$ which is disjoint from the images of the zeros of ω. Choose some $r \in I$. The inverse image $f^{-1}(I)$ consists of flat cylinders $C_1, \ldots, C_n$, each of height at least $1/(2g-2)$, and with each C_i containing a component γ_r^i of γ_r. We obtain the bound,

$$\operatorname{Mod}(C_i) \geq \frac{2}{(2g-2)\|\omega\|^2}, \tag{3.2}$$

for the modulus of C_i. From monotonicity of extremal length, (see [Ahl66, Theorem I.2]) we have $\operatorname{Ext}(\gamma_r^i) \leq 1/\operatorname{Mod}(C_i)$, which with (3.2) implies (3.1) (setting $\gamma_i = \gamma_r^i$). ∎

Remark. A similar argument is used by Accola in [Acc60], where he shows that $\|\gamma\|_X^2$ is equal to the extremal length of the homology class γ.

4 Period Matrices

In this section, we study period matrices as functions on $\overline{\mathcal{M}}_g$. We develop a coordinate-free version of the classical period matrices. We see that exponentials of entries of period matrices are canonical meromorphic functions on $\overline{\mathcal{M}}_g(L)$, and we calculate the orders of vanishing of these functions along boundary divisors of $\overline{\mathcal{M}}_g(L)$.

Fix a genus g surface Σ_g and a splitting of $H_1(\Sigma_g;\mathbb{Z})$ into a sum of Lagrangian subgroups,

$$H_1(\Sigma_g;\mathbb{Z}) = L \oplus M.$$

Given a surface $X \in \mathcal{T}(\Sigma_g)$, integration of forms yields isomorphisms

$$P_L^X\colon \Omega(X) \to \operatorname{Hom}_{\mathbb{Z}}(L,\mathbb{C}) \quad \text{and} \quad P_M^X\colon \Omega(X) \to \operatorname{Hom}_{\mathbb{Z}}(M,\mathbb{C}).$$

We obtain a holomorphic map

$$\mathcal{T}(\Sigma_g) \to \operatorname{Hom}_{\mathbb{C}}(\operatorname{Hom}_{\mathbb{Z}}(L,\mathbb{C}), \operatorname{Hom}_{\mathbb{Z}}(M,\mathbb{C})) \xrightarrow{\cong} L \otimes_{\mathbb{Z}} L \otimes_{\mathbb{Z}} \mathbb{C}, \tag{4.1}$$

where the second map uses the isomorphism $L \to M^*$ provided by the intersection form. The Riemann bilinear relations imply that the image of the map (4.1) lies in $\mathrm{Sym}_{\mathbb{Z}}(L)$, so we obtain a holomorphic map,

$$\Phi\colon \mathcal{T}(\Sigma_g) \to \mathrm{Sym}_{\mathbb{Z}}(L) \otimes \mathbb{C},$$

and the dual homomorphism,

$$\Phi^*\colon\ \mathbf{S}_{\mathbb{Z}}(\mathrm{Hom}(L, \mathbb{Z})) \otimes \mathbb{C} \to \mathrm{Hol}\,\mathcal{T}(\Sigma_g),$$

where $\mathrm{Hol}\,\mathcal{T}(\Sigma_g)$ denotes the additive group of holomorphic functions on $\mathcal{T}(\Sigma_g)$.

The map Φ^* is just a coordinate-free version of the classical period matrix. If we choose a basis (α_i) of L and dual bases (β_i) of M and (ω_i) of $\Omega(X)$, the period matrix is (τ_{ij}) where $\tau_{ij} = \omega_i(\beta_j)$. The map Φ^* is simply

$$\Phi^*(\alpha_i^* \otimes \alpha_j^*) = \tau_{ij},$$

where (α_i^*) is the dual basis of $\mathrm{Hom}(L, \mathbb{Z})$.

The map Φ^* depends on the choice of the complementary Lagrangian subgroup M. Every complementary Lagrangian is of the form

$$M_T = \{m + T(m) : m \in M\},$$

for some self-adjoint $T\colon M \to L$. Suppose we choose a different complementary Lagrangian M_T, and Φ_T^* is the corresponding homomorphism. The new homomorphism Φ_T^* is related to the old one by

$$\Phi_T^*(a) = \Phi^*(a) + \langle a, T\rangle,$$

where we are regarding T as an element of $\mathrm{Sym}_{\mathbb{Z}}(L)$ using the canonical isomorphisms $\mathrm{Hom}_{\mathbb{Z}}(M, L) \cong \mathrm{Hom}_{\mathbb{Z}}(L^*, L) \cong L \otimes L$ which identifies self-adjoint homomorphisms with symmetric tensors. It follows that the functions $\Psi(a) = e^{2\pi i \Phi^*(a)}$ do not depend on the choice of M and so descend to nonzero holomorphic functions on $\mathcal{M}_g(L)$. We obtain a canonical homomorphism

$$\Psi\colon\ \mathbf{S}_{\mathbb{Z}}(\mathrm{Hom}(L, \mathbb{Z})) \to \mathrm{Hol}^*\,\mathcal{M}_g(L).$$

We denote by

$$\langle \cdot, \cdot \rangle : \mathbf{S}_{\mathbb{Z}}(\mathrm{Hom}(L, \mathbb{Z})) \times \mathrm{Sym}^2(L) \to \mathbb{Z} \tag{4.2}$$

the extension of the natural contraction to the second symmetric product.

Theorem 4.1. *For each $a \in \mathbf{S}_{\mathbb{Z}}(\mathrm{Hom}(L, \mathbb{Z}))$, the function $\Psi(a)$ is meromorphic on $\overline{\mathcal{M}}_g(L)$. For each nonzero $\gamma \in L$, the order of vanishing of $\Psi(a)$ along D_γ is*

$$\mathrm{ord}_{D_\gamma} \Psi(a) = \langle \gamma \otimes \gamma, a\rangle.$$

The function $\Psi(a)$ is holomorphic and nowhere vanishing along any Lagrangian boundary stratum obtained by pinching a curve homologous to zero.

If $\mathcal{S} \subset \partial\overline{\mathcal{M}}_g(L)$ is a Lagrangian boundary stratum with

$$\langle \gamma \otimes \gamma, a \rangle \geq 0 \tag{4.3}$$

for all $\gamma \in \mathrm{Weight}(\mathcal{S})$, then $\Psi(a)$ is holomorphic on $\mathcal{S}$. If the pairing (4.3) is zero for all $\gamma \in \mathrm{Weight}(\mathcal{S})$, then $\Psi(a)$ is nowhere vanishing on $\mathcal{S}$. Otherwise $\Psi(a)$ vanishes identically on $\mathcal{S}$.

Proof. We use in this proof the plumbing coordinates and related notation introduced in §3. Let X be a stable curve with nodes $n_1, \ldots, n_k$ obtained by pinching curves $\gamma_1, \ldots, \gamma_k$ with homology classes $[\gamma_1], \ldots, [\gamma_k] \in L$. Let

$$\mathcal{Y} \to B := \Delta_\epsilon^{3g-3-k} \times \Delta_1^k$$

be the family of stable curves constructed above, with X the fiber over $(\mathbf{0}, \mathbf{0})$. The nodes of this family are contained in the open sets

$$\mathbb{W}_i := \mathbb{V}_i \times \Delta_\epsilon^{3g-3-k} = \{(x_i, y_i, \boldsymbol{s}, \boldsymbol{t}) \in \Delta_1 \times \Delta_1 \times \Delta_\epsilon^{3g-3-k} \times \Delta_1^k : x_i y_i = t_i\},$$

for $i = 1, \ldots, k$. Define sections $p_i, q_i \colon B \to \mathcal{Y}$ with image in $\partial\mathbb{W}_i$ by

$$p_i(\boldsymbol{s}, \boldsymbol{t}) = (1, t_i, \boldsymbol{s}, \boldsymbol{t}) \quad \text{and} \quad q_i(\boldsymbol{s}, \boldsymbol{t}) = (t_i, 1, \boldsymbol{s}, \boldsymbol{t}).$$

Choose $\alpha_1 \otimes \alpha_2 \in \mathbf{S}_{\mathbb{Z}}(\mathrm{Hom}(L, \mathbb{Z}))$ and let η be the holomorphic section of the relative dualizing sheaf $\omega_{\mathcal{Y}/\mathcal{B}}$ such that each period homomorphism $L \to \mathbb{C}$ defined by each restriction $\eta_{\boldsymbol{t}}^{\boldsymbol{s}}$ to the fiber $X_{\boldsymbol{t}}^{\boldsymbol{s}}$ agrees with $\alpha_1 \colon L \to \mathbb{Z}$.

On $\mathbb{W}_i$ we may express η as

$$\eta = \frac{\alpha_1([\gamma_i])}{2\pi i} \frac{dx_i}{x_i} + f_i\, dx_i + g_i\, dy_i \tag{4.4}$$

with f_i and g_i holomorphic functions of $x_i, y_i, \boldsymbol{s}$, and $\boldsymbol{t}$.

Let $\delta_{\boldsymbol{t},i}^{\boldsymbol{s}} \colon [-1, 1] \to \mathbb{W}_i$ be a path in the fiber of $\mathbb{W}_i$ over $(\boldsymbol{s}, \boldsymbol{t})$ joining $p_i(\boldsymbol{s}, \boldsymbol{t})$ to $q_i(\boldsymbol{s}, \boldsymbol{t})$. We may explicitly parameterize one such as

$$\delta_{\boldsymbol{t},i}^{\boldsymbol{s}}(r) = \begin{cases} (\sqrt{t_i} - r(1 - \sqrt{t_i}), t_i/(\sqrt{t_i} - r(1 - \sqrt{t_i})), \boldsymbol{s}, \boldsymbol{t}) & \text{if } r \leq 0 \\ (t_i/(r(1 - \sqrt{t_i}) + \sqrt{t_i}), r(1 - \sqrt{t_i}) + \sqrt{t_i}, \boldsymbol{s}, \boldsymbol{t}) & \text{if } r \geq 0. \end{cases}$$

We may choose a continuous family of 1-chains $\delta_{\boldsymbol{t},0}^{\boldsymbol{s}}$ in $X_{\boldsymbol{t}}^{\boldsymbol{s}}$ with endpoints in $\{p_i(\boldsymbol{s}, \boldsymbol{t}), q_i(\boldsymbol{s}, \boldsymbol{t})\}_{i=1}^k$ such that

$$\delta_{\boldsymbol{t}}^{\boldsymbol{s}} = \delta_{\boldsymbol{t},0}^{\boldsymbol{s}} + \sum_{i=1}^k \alpha_2([\gamma_i]) \delta_{\boldsymbol{t},i}^{\boldsymbol{s}}$$

is a 1-cycle whose intersection with classes in L agrees with the homomorphism $\alpha_2 \colon L \to \mathbb{Z}$.

We have

$$\Psi(\alpha_1 \otimes \alpha_2)(\boldsymbol{s}, \boldsymbol{t}) = \exp\left(\int_{\delta_{\boldsymbol{t}}^{\boldsymbol{s}}} \eta_{\boldsymbol{t}}^{\boldsymbol{s}} \right), \tag{4.5}$$

where we use the notation $\exp(z) = e^{2\pi i z}$. The integral $\int_{\delta^{\boldsymbol{s}}_{\boldsymbol{t},0}} \eta^{\boldsymbol{s}}_{\boldsymbol{t}}$ is an integral of a holomorphically varying form over a 1-cycle with holomorphically varying endpoints, and so its contribution to (4.5) is holomorphic and nonzero. Thus it does not contribute to the order of vanishing of $\Psi(\alpha_1 \otimes \alpha_2)$.

The integral

$$\int_{\delta^{\boldsymbol{s}}_{\boldsymbol{t},i}} f_i\, dx_i + g_i\, dx_i$$

is a finite holomorphic function of $\boldsymbol{s}$ and $\boldsymbol{t}$ and so does not contribute to the order of vanishing of $\Psi(\alpha_1 \otimes \alpha_2)$. The factor of $\Psi(\alpha_1 \otimes \alpha_2)$ coming from the first term of (4.4) is

$$\exp\left(\alpha_1([\gamma_i])\alpha_2([\gamma_i]) \int_{\delta^{\boldsymbol{s}}_{\boldsymbol{t},i}} \frac{dx_i}{x_i}\right) = t_i^{\alpha_1([\gamma_i])\alpha_2([\gamma_i])}.$$

In our $(\boldsymbol{s}, \boldsymbol{t})$-coordinates for $\overline{\mathcal{M}}_g(L)$, the divisor D_{γ_i} is the locus $\{t_i = 0\}$. We have seen that in these coordinates,

$$\Psi(\alpha_1 \otimes \alpha_2)(\boldsymbol{s}, \boldsymbol{t}) = k(\boldsymbol{s}, \boldsymbol{t}) \prod_i t_i^{\alpha_1([\gamma_i])\alpha_2([\gamma_i])}, \tag{4.6}$$

with k a nonzero holomorphic function. Thus $\Psi(\alpha_1 \otimes \alpha_2)$ is meromorphic with the desired orders of vanishing.

Now suppose $\mathcal{S}$ is a Lagrangian boundary stratum and $a \in \mathrm{Hom}(L, \mathbb{Z})$ with $\langle \gamma \otimes \gamma, a\rangle \geq 0$ for each weight γ. We see from (4.6) that $\Psi(a)$ is holomorphic on $\mathcal{S}$, since each t_i has nonnegative exponent. If $\langle \gamma \otimes \gamma, a\rangle > 0$ for some weight γ, then some t_i has positive exponent, so $\Psi(a)$ vanishes on $\mathcal{S}$. ■

We will also need the following strengthening of this theorem.

Corollary 4.2. *Let $\mathcal{S} \subset \partial\overline{\mathcal{M}}_g(L)$ be a Lagrangian boundary stratum obtained by pinching n curves on Σ_g whose homology classes are $\gamma_1, \ldots, \gamma_n \in L$. Take local coordinates $t_1, \ldots, t_n$ around some $x \in \mathcal{S}$ in which the divisor D_{γ_i} of curves obtained by pinching γ_i is cut out by the equation $t_i = 0$. Then for any $a \in \mathbf{S}_{\mathbb{Z}}(\mathrm{Hom}(L, \mathbb{Z}))$, the function*

$$\prod_{i=1}^{m} t_i^{-\langle \gamma \otimes \gamma, a\rangle} \Psi(a)$$

is holomorphic and nonzero on a neighborhood of x.

Proof. This follows immediately from (4.6). ■

5 Boundary of the eigenform locus: Necessity

In this section we begin the study of the boundary of the locus of Riemann surfaces whose Jacobians have real multiplication. We give an explicit necessary condition for a stable curve to lie in the boundary of the real multiplication locus. In §8, we will see that this condition is also sufficient in genus three.

In all that follows, F will denote a totally real number field of degree $g > 1$, $\mathcal{O}$ will denote an order in F, and $\mathcal{I}$ will denote a lattice in F whose coefficient ring contains $\mathcal{O}$.

The real multiplication locus. The Jacobian of a stable curve X is

$$\mathrm{Jac}(X) = \Omega(X)^*/\widehat{H}_1(X;\mathbb{Z}) = \Omega(X)^*/H_1(X_0;\mathbb{Z}),$$

where $X_0 \subset X$ is the complement of the nodes. The Jacobian is a compact Abelian variety if each node of X is separating, or equivalently if the geometric genus of X is g. Otherwise it is a noncompact semi-Abelian variety. We denote by $\widetilde{\mathcal{M}}_g \subset \overline{\mathcal{M}}_g$ the locus of stable curves with compact Jacobians. The Torelli map $t\colon \widetilde{\mathcal{M}}_g \to \mathcal{A}_g$ maps each Riemann surface to its Jacobian.

Let $\mathcal{RM}_{\mathcal{O}} \subset \widetilde{\mathcal{M}}_g$ be the locus of Riemann surfaces whose Jacobians have real multiplication by $\mathcal{O}$. In other words, $\mathcal{RM}_{\mathcal{O}} = t^{-1}(\mathcal{RA}_{\mathcal{O}})$. This is a slight abuse of notation, since we defined $\mathcal{RM}_{\mathcal{O}}$ in §1 to be subvariety of $\mathcal{M}_g$, but the distinction will never be important. If g is 2 or 3, then t is a bijection, so $\mathcal{RM}_{\mathcal{O}}$ is a g-dimensional subvariety of $\mathcal{M}_g$. In general, it is not known what the dimension of $\mathcal{RM}_{\mathcal{O}}$ is, or even whether $\mathcal{RM}_{\mathcal{O}}$ is nonempty.

We define $\mathcal{E}_{\mathcal{O}} \subset \mathbb{P}\Omega\widetilde{\mathcal{M}}_g$ to be the locus of eigenforms for real multiplication by $\mathcal{O}$ and $\mathcal{E}^\iota_{\mathcal{O}}$ to be the locus of ι-eigenforms. The Torelli map exhibits $\mathcal{E}^\iota_{\mathcal{O}}$ as a one-to-one branched cover of $\mathcal{EA}^\iota_{\mathcal{O}} \cong X_{\mathcal{O}}$.

Admissible strata. The tensor product $F \otimes_{\mathbb{Q}} F$ has the structure of an F-bimodule. We define

$$\Lambda^1 = \{\lambda \in F \otimes_{\mathbb{Q}} F : x \cdot \lambda = \lambda \cdot x \text{ for all } x \in F\}.$$

The proof of the following proposition contains an alternative interpretation of Λ^1 as the vector space $\mathrm{Hom}_F(F, F)$.

Proposition 5.1. *$\Lambda^1 \subset \mathrm{Sym}_{\mathbb{Q}}(F)$.*

Proof. Identify F with $\mathrm{Hom}_{\mathbb{Q}}(F, \mathbb{Q})$ via the trace pairing. This naturally induces an isomorphism $F \otimes_{\mathbb{Q}} F \to \mathrm{Hom}_{\mathbb{Q}}(F, F)$. Under this isomorphism, $\mathrm{Sym}_{\mathbb{Q}}(F)$ corresponds to the self-adjoint endomorphisms $\mathrm{Hom}^+_{\mathbb{Q}}(F, F)$, and Λ^1 corresponds to $\mathrm{Hom}_F(F, F)$. Since left multiplication by $x \in F$ is self-adjoint, $\mathrm{Hom}_F(F, F) \subset \mathrm{Hom}^+_{\mathbb{Q}}(F, F)$. ∎

Identifying F with its dual as above, the dual of $\mathrm{Sym}_{\mathbb{Q}}(F)$ is $\mathbf{S}_{\mathbb{Q}}(F)$. We let $\mathrm{Ann}(\Lambda^1) \subset \mathbf{S}_{\mathbb{Q}}(F)$ denote the annihilator of Λ^1.

Given an $\mathcal{I}$-weighted boundary stratum $\mathcal{S}$, we define the following subsets of the vector space $\mathbf{S}_{\mathbb{Q}}(F)$, where the bilinear form is the one defined in (4.2).

$$C(\mathcal{S}) = \{x \in \mathbf{S}_{\mathbb{Q}}(F) : \langle x, \alpha \otimes \alpha \rangle \geq 0 \text{ for all } \alpha \in \mathrm{Weight}(\mathcal{S})\}$$
$$N(\mathcal{S}) = \{x \in \mathbf{S}_{\mathbb{Q}}(F) : \langle x, \alpha \otimes \alpha \rangle = 0 \text{ for all } \alpha \in \mathrm{Weight}(\mathcal{S})\}.$$

Note that $C(\mathcal{S})$ is a convex cone, and $N(\mathcal{S})$ is a subspace contained in $C(\mathcal{S})$.

We say that an $\mathcal{I}$-weighted boundary stratum $\mathcal{S}$ is *admissible* if

$$C(\mathcal{S}) \cap \mathrm{Ann}(\Lambda^1) \subset N(\mathcal{S}). \tag{5.1}$$

Note that the admissibility of a stratum only depends on the set of residues and not on the topology of the stable curves it parameterizes.

We will see in Corollary 8.2 that if $\mathcal{I}$ is a lattice in a cubic field, then there are only finitely many admissible $\mathcal{I}$-weighted boundary strata up to similarity.

In §1, we defined using the no-half-space condition the notion of an admissible basis of a lattice in a cubic field. We will see in §6 that the no-half-space condition is equivalent to (5.1), so an admissible basis is exactly the set of weights of an admissible stratum parameterizing irreducible stable curves.

Algebraic tori. Fix an $\mathcal{I}$-weighted boundary stratum $\mathcal{S}$. There is a surjective map of algebraic tori:

$$p\colon \mathrm{Hom}(N(\mathcal{S}) \cap \mathbf{S}_{\mathbb{Z}}(\mathcal{I}^\vee), \mathbb{G}_m) \to \mathrm{Hom}(N(\mathcal{S}) \cap \mathrm{Ann}(\Lambda^1) \cap \mathbf{S}_{\mathbb{Z}}(\mathcal{I}^\vee), \mathbb{G}_m). \tag{5.2}$$

The reader unfamiliar with algebraic groups should think of $\mathbb{G}_m$ as the multiplicative group $\mathbb{C}^*$ of nonzero complex numbers.

By the discussion at the end of §3, we may regard $\mathcal{S}$ as a boundary stratum of $\overline{\mathcal{M}}_g(\mathcal{I})$. By Corollary 4.2, for each nonzero $a \in N(\mathcal{S}) \cap \mathbf{S}_{\mathbb{Z}}(\mathcal{I}^\vee)$ the restriction of $\Psi(a)$ to $\mathcal{S}$ is a nonzero holomorphic function on $\mathcal{S}$. We obtain a canonical morphism,

$$\mathrm{CR}\colon \mathcal{S} \to \mathrm{Hom}(N(\mathcal{S}) \cap \mathbf{S}_{\mathbb{Z}}(\mathcal{I}^\vee), \mathbb{G}_m), \tag{5.3}$$

which we call the cross ratio map. (The reason for this terminology will be apparent in Corollary 8.4.) Recall that $E(\mathcal{I})$ is the torsion Abelian group of symplectic extensions of $\mathcal{I}^\vee$ by $\mathcal{I}$. Identifying $\mathrm{Hom}^+_{\mathbb{Q}}(F, F)$ with $\mathrm{Sym}_{\mathbb{Q}}(F)$ via the trace pairing, the isomorphism of Theorem 2.1 becomes an isomorphism,

$$\mathrm{Sym}_{\mathbb{Q}}(F)/(\Lambda^1 + \mathrm{Sym}_{\mathbb{Z}}(\mathcal{I})) \to E(\mathcal{I}). \tag{5.4}$$

Given $E \in \mathrm{Sym}_{\mathbb{Q}}(F)$ and $a \in \mathrm{Ann}(\Lambda^1) \cap \mathbf{S}_{\mathbb{Z}}(\mathcal{I}^\vee)$, we define

$$q(E)(a) = e^{-2\pi i \langle E, a \rangle}. \tag{5.5}$$

Since $q(E)(a) = 1$ if $a \in N(\mathcal{S}) \cap \mathrm{Ann}(\Lambda^1) \cap \mathbf{S}_{\mathbb{Z}}(\mathcal{I}^\vee)$ and if E lies in Λ^1 or $\mathrm{Sym}_{\mathbb{Z}}(\mathcal{I})$, (5.5) defines a homomorphism,

$$q\colon E(\mathcal{I}) \to \mathrm{Hom}(N(\mathcal{S}) \cap \mathrm{Ann}(\Lambda^1) \cap \mathbf{S}_{\mathbb{Z}}(\mathcal{I}^\vee), \mathbb{G}_m).$$

Given a symplectic extension $E \in E(\mathcal{I})$, we define

$$G(E) = p^{-1}(q(E)),$$

a translate of a subtorus of $\mathrm{Hom}(N(\mathcal{S}) \cap \mathbf{S}_{\mathbb{Z}}(\mathcal{I}^\vee), \mathbb{G}_m)$. We then obtain for each extension E a subvariety of $\mathcal{S}$:

$$\mathcal{S}(E) = \mathrm{CR}^{-1}(G(E)) \tag{5.6}$$

We define $\mathcal{S}^\iota(E) \subset \mathbb{P}\Omega\overline{\mathcal{M}}_g$ to be the image of $\mathcal{S}(E)$ under p_ι.

If $\mathcal{S}$ is an $\mathcal{I}$-weighted stratum and $\mathcal{S}'$ is a similar $a\mathcal{I}$-weighted stratum, then the subvarieties $\mathcal{S}(E)$ and $\mathcal{S}'(E^a)$ are identified under the canonical isomorphism $\mathcal{S} \to \mathcal{S}'$. Thus the variety $\mathcal{S}(E)$ can be regarded as depending only on the similarity class of $\mathcal{S}$ and the cusp packet $(\mathcal{I}, E)$.

Boundary of $\mathcal{RM}_\mathcal{O}$. We can now state our necessary condition for a stable curve to be in the boundary of $\mathcal{RM}_\mathcal{O}$.

Theorem 5.2. *Consider an order $\mathcal{O}$ in a degree g totally real number field F, a real embedding ι of F, and a cusp packet $(\mathcal{I}, E) \in \mathcal{C}(\mathcal{O})$. The closure in $\mathbb{P}\Omega\overline{\mathcal{M}}_g$ of the cusp of $\mathcal{E}^\iota_\mathcal{O}$ associated to $(\mathcal{I}, E)$ is contained in the union over all admissible $\mathcal{I}$-weighted boundary strata $\mathcal{S}$ of the varieties $\mathcal{S}^\iota(E)$.*

The closure of the corresponding cusp of $\mathcal{RM}_\mathcal{O}$ in $\overline{\mathcal{M}}_g$ is contained in the union over all $\mathcal{I}$-weighted boundary strata $\mathcal{S}$ of the images of the $\mathcal{S}(E)$ under the forgetful map to $\overline{\mathcal{M}}_g$.

The proof of Theorem 5.2 appears at the end of this section.

Auxiliary real multiplication loci. Given a cusp packet $(\mathcal{I}, E) \in \mathcal{C}(\mathcal{O})$, let

$$\mathcal{RM}_\mathcal{O}(\mathcal{I}, E) \subset \mathcal{M}_g(\mathcal{I})$$

be the locus of Riemann surfaces with Lagrangian marking (X, ρ) such that $\mathrm{Jac}(X)$ has real multiplication by $\mathcal{O}$, the marking $\rho\colon \mathcal{I} \to H_1(X; \mathbb{Z})$ is an $\mathcal{O}$-module homomorphism, and the extension of $\mathcal{O}$-modules

$$0 \to \rho(\mathcal{I}) \to H_1(X; \mathbb{Z}) \to H_1(X; \mathbb{Z})/\rho(\mathcal{I}) \to 0$$

is isomorphic to the extension determined by $(\mathcal{I}, E)$.

We also have bundles of eigenforms over $\mathcal{RM}_\mathcal{O}(\mathcal{I}, E)$. On $\overline{\mathcal{M}}_g(\mathcal{I})$, there is the trivial bundle $\Omega^\iota\overline{\mathcal{M}}_g(\mathcal{I})$ of forms ω such that for some constant c and for each $\lambda \in \mathcal{I}$, we have $\int_{\rho(\lambda)} \omega = c\iota(\lambda)$, where ρ is the Lagrangian marking. The restriction $\Omega^\iota\overline{\mathcal{RM}}_\mathcal{O}(\mathcal{I}, E)$ of $\Omega^\iota\overline{\mathcal{M}}_g(\mathcal{I})$ to $\overline{\mathcal{RM}}_\mathcal{O}(\mathcal{I}, E)$ is the trivial line bundle of ι-eigenforms. We denote its projectivization by $\overline{\mathcal{E}}^\iota_\mathcal{O}(\mathcal{I}, E) \subset \mathbb{P}\Omega\overline{\mathcal{M}}_g(\mathcal{I})$.

Given a cusp packet $(\mathcal{I}, E)$ and a symplectic isomorphism $\eta :: \mathcal{I} \oplus \mathcal{I}^\vee \to H_1(\Sigma_g; \mathbb{Z})$, we define

$$\mathcal{RT}_\mathcal{O}(\mathcal{I}, E, \eta) \subset \mathcal{T}(\Sigma_g)$$

to be the locus of marked Riemann surfaces (X, f) such that $\mathrm{Jac}(X)$ has real multiplication by $\mathcal{O}$ and the symplectic $\mathbb{Z}$-module isomorphism

$$f_* \circ \eta\colon (\mathcal{I} \oplus \mathcal{I}^\vee)_E \to H_1(X; \mathbb{Z})$$

is also an isomorphism of symplectic $\mathcal{O}$-modules.

The homomorphism η determines a Lagrangian splitting of $H_1(\Sigma_g; \mathbb{Z})$, and we obtain as in §4 a holomorphic map $\Phi\colon \mathcal{T}(\Sigma_g) \to \mathrm{Sym}_{\mathbb{Z}}(\mathcal{I}) \otimes \mathbb{C}$.

Proposition 5.3. *We have*

$$\mathcal{RT}_{\mathcal{O}}(\mathcal{I}, E, \eta) = \Phi^{-1}(\Lambda^1 \otimes_{\mathbb{Q}} \mathbb{C} - E)$$

Proof. In this proof, we will identify $\mathrm{Sym}_{\mathbb{Z}}(\mathcal{I})$ with $\mathrm{Hom}^+(\mathcal{I}^\vee, \mathcal{I})$. Under this identification, we have

$$\begin{gathered}\mathrm{Sym}_{\mathbb{Z}}(\mathcal{I}) \otimes \mathbb{C} = \mathrm{Hom}^+_{\mathbb{C}}(\mathcal{I}^\vee \otimes \mathbb{C}, \mathcal{I} \otimes \mathbb{C}),\\ \Lambda^1 \otimes \mathbb{C} = \mathrm{Hom}^+_F(\mathcal{I}^\vee \otimes \mathbb{C}, \mathcal{I} \otimes \mathbb{C})\\ \phi := \Phi(X, f) \in \mathrm{Hom}^+_{\mathbb{C}}(\mathcal{I}^\vee \otimes \mathbb{C}, \mathcal{I} \otimes \mathbb{C}), \quad \text{and}\\ E \in \mathrm{Hom}^+_{\mathbb{Q}}(\mathcal{I}^\vee \otimes \mathbb{Q}, \mathcal{I} \otimes \mathbb{Q}).\end{gathered}$$

We have two splittings of $H_1(X; \mathbb{C})$: the one induced by ρ,

$$H_1(X; \mathbb{C}) = (\mathcal{I} \otimes \mathbb{C}) \oplus (\mathcal{I}^\vee \otimes \mathbb{C}),$$

and the Hodge decomposition,

$$H_1(X; \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(\Omega(X), \mathbb{C}) \oplus \mathrm{Hom}_{\mathbb{C}}(\overline{\Omega(X)}, \mathbb{C}).$$

The Hodge decomposition is determined by the map $\phi\colon \mathcal{I}^\vee \otimes \mathbb{C} \to \mathcal{I} \otimes \mathbb{C}$:

$$\mathrm{Hom}_{\mathbb{C}}(\Omega(X), \mathbb{C}) = \mathrm{Graph}(\phi). \tag{5.7}$$

The $\mathcal{O}$-module structure of $H_1(X; \mathbb{C})$ inherited from that of $(\mathcal{I} \oplus \mathcal{I}^\vee)_E$ induces real multiplication on $\mathrm{Jac}(X)$ if and only if it preserves the Hodge decomposition. By (5.7), the Hodge decomposition is preserved if and only if

$$\phi(\lambda \cdot \alpha) = \lambda \cdot \phi(\alpha) + [M_\lambda, E](\alpha)$$

for all $\alpha \in \mathcal{I}^\vee$ and $\lambda \in \mathcal{O}$, which holds if and only if

$$(\phi + E)(\lambda \cdot \alpha) = \lambda \cdot (\phi + E)(\alpha),$$

that is, if and only if $\phi + E \in \Lambda^1$. ∎

Corollary 5.4. *Given any $a \in \mathrm{Ann}(\Lambda^1) \subset \mathbf{S}_{\mathbb{Z}}(\mathcal{I}^\vee)$, we have*

$$\Psi(a) \equiv q(E)(a)$$

on $\mathcal{RM}_{\mathcal{O}}(\mathcal{I}, E)$.

Proof. This follows directly from Proposition 5.3 and the definition of q. ∎

Invariant vanishing cycles. Consider a family $\mathcal{X} \to \Delta$ of stable curves which is smooth over Δ^* in the sense that the fiber X_p over nonzero p is smooth. Any such family defines a holomorphic map $\Delta \to \overline{\mathcal{M}}_g$ sending p to X_p, and conversely any holomorphic disk $\Delta \to \overline{\mathcal{M}}_g$ sending Δ^* to $\mathcal{M}_g$, after possibly taking a base extension (a cover of Δ ramified only over 0), arises from such a family.

In any smooth fiber X_p, there is a collection of isotopy classes of simple closed curves, which we call the *vanishing curves* which are pinched as $p \to 0$. The vanishing curves are consistent in the sense that given any path in Δ^* joining p to q, the lifted homeomorphism $f\colon X_p \to X_q$ (defined up to isotopy) preserves the vanishing curves. The *vanishing cycles* in $H_1(X_p; \mathbb{Z})$ are those cycles generated by the vanishing curves. Trivializing the family over a path starting and ending at p yields a homeomorphism of X_p which is simply a product of Dehn twists around the vanishing curves. Thus the monodromy action of $\pi_1(\Delta^*, p)$ on $H_1(X_p; \mathbb{Z})$ is unipotent and fixes pointwise the subgroup $V_p \subset H_1(X_p; \mathbb{Z})$ of vanishing cycles.

Real multiplication by $\mathcal{O}$ on the family $\mathcal{X} \to \Delta$ is a monomorphism $\rho\colon \mathcal{O} \hookrightarrow \operatorname{End}^0 \operatorname{Jac}_{\mathcal{X}/\Delta}$, where $\operatorname{Jac}_{\mathcal{X}/\Delta} \to \Delta$ is the relative Jacobian of the family $\mathcal{X} \to \Delta$. This is equivalent to a choice of real multiplication $\rho\colon \mathcal{O} \to \operatorname{Jac}(X_p)$ for each smooth fiber X_p with the requirement that each isomorphism $H_1(X_p; \mathbb{Z}) \to H_1(X_q; \mathbb{Z})$ arising from the Gauss-Manin connection (the canonical flat connection on the bundle of first homology; see [Voi07]) commutes with the action of $\mathcal{O}$.

Proposition 5.5. *Consider a family of genus g stable curves $\mathcal{X} \to \Delta$, smooth over Δ^*, with real multiplication by $\mathcal{O}$. For each nonzero p, the subgroup $V_p \subset H_1(X_p; \mathbb{Z})$ of vanishing cycles is preserved by the action of $\mathcal{O}$ on $H_1(X_p; \mathbb{Z})$.*

Proof. Since the action of $\mathcal{O}$ on first homology commutes with the Gauss-Manin connection, it is enough to show that V_p is invariant for a single p.

Let $\lambda \in \mathcal{O}$ be a primitive element for F. For any $\gamma \in H_1(X_p; \mathbb{Z})$, we have the bound,

$$\|\lambda \cdot \gamma\|_{X_p} \leq \|\lambda\|_\infty \|\gamma\|_{X_p},$$

where $\|\lambda\|_\infty = \sup_\iota |\iota(\lambda)|$, with the supremum over all field embeddings $\iota\colon F \to \mathbb{R}$, and $\| \cdot \|_{X_p}$ is the Hodge norm introduced in §3.

There is a constant D (independent of p) such that $\operatorname{Ext}(\gamma) \geq D$ for any curve γ on X_p which is not a vanishing curve. For any $\epsilon > 0$, we may choose p sufficiently small that $\operatorname{Ext}(\gamma_i) < \epsilon$ for any vanishing curve γ_i. By Proposition 3.1, we have

$$\|\lambda \cdot \gamma_i\|_{X_p} \leq \|\lambda\|_\infty \|\gamma_i\| < \|\lambda\|_\infty \epsilon^{1/2}.$$

By Proposition 3.2, $\lambda \cdot \gamma_i$ is homologous to a sum of simple closed curves δ_j with

$$\operatorname{Ext}(\delta_j) < C\|\lambda\|_\infty^2 \epsilon.$$

Thus $\operatorname{Ext}(\delta_j) < D$ if ϵ is chosen sufficiently small. The δ_j must then be vanishing curves. Thus the action of λ preserves V_p, and since λ is a primitive element, V_p is preserved by $\mathcal{O}$. ∎

Corollary 5.6. *Each stable curve in $\overline{\mathcal{RM}}_{\mathcal{O}} \subset \overline{\mathcal{M}}_g$ has geometric genus either 0 or g.*

Proof. Suppose X is a stable curve in $\overline{\mathcal{RM}}_{\mathcal{O}}$. Choose a family of stable curves $\mathcal{X} \to \Delta$, smooth over Δ^*, with real multiplication by $\mathcal{O}$, with X the fiber over 0. The geometric genus of X is $g - \operatorname{rank} V_p$ for any nonzero p. By Proposition 5.5, $V_p \otimes \mathbb{Q}$ is a vector space over F, so $\dim_{\mathbb{Q}} V_p \otimes \mathbb{Q}$ must be a multiple of $[F : \mathbb{Q}] = g$. ■

Proof of Theorem 5.2. Consider (X_0, ω_0) in the closure of the cusp of $\mathcal{E}^{\iota}_{\mathcal{O}}$ determined by the cusp packet $(\mathcal{I}, E)$. We first claim that (X_0, ω_0) must lie in the image of $\overline{\mathcal{E}}^{\iota}_{\mathcal{O}}(\mathcal{I}, E) \subset \mathbb{P}\Omega\overline{\mathcal{M}}_g(\mathcal{I}, E)$. Since $\overline{\mathcal{E}}^{\iota}_{\mathcal{O}}$ is a variety, we may choose a holomorphic disk $f \colon \Delta \to \overline{\mathcal{E}}^{\iota}_{\mathcal{O}}$ sending 0 to (X_0, ω_0) and Δ^* to the cusp of $\mathcal{E}^{\iota}_{\mathcal{O}}$ determined by $(\mathcal{I}, E)$. Possibly taking a base extension, we may assume f arises from a family of stable curves $\mathcal{X} \to \Delta$ with real multiplication by $\mathcal{O}$. For each $p \in \Delta^*$, the vanishing cycles V_p for the fiber X_p over p are $\mathcal{O}$-invariant by Proposition 5.5, so we obtain an extension of $\mathcal{O}$-modules

$$0 \to V_p \to H_1(X; \mathbb{Z}) \to H_1(X; \mathbb{Z})/V_p \to 0,$$

which must be isomorphic to the extension determined by $(\mathcal{I}, E)$. Since the monodromy action of $\pi_1(\Delta^*, p)$ on V_p is trivial, we may identify each V_q with $\mathcal{I}$ and obtain a consistent Lagrangian marking of $\widehat{H}_1(X_q; \mathbb{Z})$ by $\mathcal{I}$ for each q, which defines a lift $g \colon \Delta \to \overline{\mathcal{E}}^{\iota}_{\mathcal{O}}(\mathcal{I}, E) \subset \mathbb{P}\Omega\overline{\mathcal{M}}_g(\mathcal{I})$. It follows that (X_0, ω_0) lies in the image of some $(Y, \eta) \in \overline{\mathcal{E}}^{\iota}_{\mathcal{O}}(\mathcal{I}, E)$ as claimed.

The form (Y, η) must lie in some boundary stratum $\mathcal{S}^{\iota} \subset \mathbb{P}\Omega\overline{\mathcal{M}}_g(\mathcal{I})$ lying over a boundary stratum $\mathcal{S} \subset \overline{\mathcal{M}}_g(\mathcal{I})$. We must then show that if the intersection $\mathcal{S} \cap \overline{\mathcal{RM}}_{\mathcal{O}}(\mathcal{I}, E)$ is nontrivial, then $\mathcal{S}$ is admissible, and moreover that $\mathcal{S} \cap \overline{\mathcal{RM}}_{\mathcal{O}}(\mathcal{I}, E) \subset \mathcal{S}(E)$.

Suppose that the stratum $\mathcal{S}$ is not admissible, so the cone condition (5.1) does not hold. Then there is some a in $C(\mathcal{S}) \cap \operatorname{Ann}(\Lambda^1) \cap \mathbf{S}_{\mathbb{Z}}(\mathcal{I}^{\vee})$ but not in $N(\mathcal{S})$. By Theorem 4.1, the function $\Psi(a)$ is holomorphic and identically zero on $\mathcal{S}$. By Corollary 5.4, $\Psi(a) \equiv q(E)(a)$, a nonzero constant on $\overline{\mathcal{RM}}_{\mathcal{O}}(\mathcal{I}, E)$. In particular, $\Psi(a)$ is nonzero along $\mathcal{S} \cap \overline{\mathcal{RM}}_{\mathcal{O}}(\mathcal{I}, E) \neq \emptyset$, a contradiction. Thus $\mathcal{S}$ is admissible.

Since $\Psi(a) \equiv q(E)(a)$ on $\overline{\mathcal{RM}}_{\mathcal{O}}(\mathcal{I}, E)$ for all $a \in N(\mathcal{S}) \cap \operatorname{Ann}(\Lambda^1) \cap \mathbf{S}_{\mathbb{Z}}(\mathcal{I}^{\vee})$, it follows immediately that $\overline{\mathcal{RM}}_{\mathcal{O}}(\mathcal{I}, E) \cap \mathcal{S} \subset \mathcal{S}(E)$. ■

6 A geometric reformulation of admissibility

The aim of this section is to give a more explicit reformulation of when an $\mathcal{I}$-weighted boundary stratum is admissible.

The no-half-space condition. Consider a finite dimensional vector space V over $\mathbb{Q}$. We say that a set $S = \{v_1, \ldots, v_n\} \subset V$ satisfies the *no-half-space* *condition* if it is not contained in a closed half-space of its $\mathbb{Q}$-span. Equivalently, S satisfies the no-half space condition if and only if zero is in the interior of the convex hull of S.

The reformulation. Consider a totally real number field F with Galois closure K. Let $G = \mathrm{Gal}(K/\mathbb{Q})$ and $H = \mathrm{Gal}(K/F)$. We define $I = (H \times H) \rtimes \mathbb{Z}/2\mathbb{Z}$, with $\mathbb{Z}/2\mathbb{Z}$ acting on $H \times H$ by exchanging the factors. The group I acts on G by

$$(h_1, h_2, \epsilon) \cdot \sigma = h_2 \sigma^\epsilon h_1^{-1},$$

where $\epsilon = \pm 1 \in \mathbb{Z}/2\mathbb{Z}$. We let $\mathrm{Stab}(\sigma) \subset I$ denote the stabilizer of $\sigma \in G$, and we define a homomorphism $\rho_\sigma \colon \mathrm{Stab}(\sigma) \to G$ by

$$\rho_\sigma(h_1, h_2, \epsilon) = \begin{cases} h_1 & \text{if } \epsilon = 1; \\ h_1\sigma & \text{if } \epsilon = -1. \end{cases}$$

Let $G_\sigma = \rho_\sigma(\mathrm{Stab}(\sigma))$ and $K_\sigma = K^{G_\sigma}$. We define for each $\sigma \in G$ a quadratic map $Q_\sigma \colon F \to K_\sigma$ by

$$Q_\sigma(t) = t\sigma^{-1}(t).$$

Theorem 6.1. *A weighted boundary stratum with weights $\{t_1, \ldots, t_n\} \subset F$ is admissible if and only if for each $\sigma \in G \setminus H$, the set $\{Q_\sigma(t_1), \ldots, Q_\sigma(t_n)\} \subset K_\sigma$ satisfies the no-half-space condition. In fact, it is enough to check this for each σ in a set of orbit representatives of G/I.*

The tensor product $K \otimes K$ has the structure of a K-bimodule. Given $\sigma \in G$, we define

$$\Lambda_K^\sigma = \{\lambda \in K \otimes K : x \cdot \lambda = \lambda \cdot \sigma(x) \text{ for all } x \in K\},.$$

generalizing the definition of $\Lambda^1 \in \mathrm{Sym}_{\mathbb{Q}}(F)$ in §5.

The trace pairing $\langle x, y\rangle_K = \mathrm{Tr}_{\mathbb{Q}}^K(xy)$ on K induces a pairing on $K \otimes K$:

$$\langle x_1 \otimes x_2, y_1 \otimes y_2\rangle = \langle x_1, y_1\rangle_K \langle x_2, y_2\rangle_K.$$

Lemma 6.2. *Let $r_1, \ldots, r_g$ be a basis of K over $\mathbb{Q}$ and $s_1, \ldots, s_g$ the dual basis with respect to the trace pairing. The element*

$$\epsilon_\sigma = \sum_{i=1}^{g} r_i \otimes \sigma(s_i) \in K \otimes K.$$

lies in Λ^σ and does not depend on the choice of basis (r_i). Moreover, for any $x \in K_\sigma$ and $t \in F$, we have

$$\langle x\epsilon_\sigma, t \otimes t\rangle = [K : K_\sigma]\langle x, Q_\sigma(t)\rangle_{K_\sigma}.$$

Proof. Identifying $K \otimes K$ with $\mathrm{Hom}_{\mathbb{Q}}(K, K)$ via the trace pairing, Λ^σ corresponds to

$$\{\phi \colon K \to K : \phi(x\lambda) = \sigma(x)\phi(\lambda) \text{ for all } x, \lambda \in K\}.$$

Under this correspondence, ϵ_σ is the map $x \mapsto \sigma(x)$. Thus, $\epsilon_\sigma \in \Lambda^\sigma$ and does not depend on the choice of the r_i.

Now, write $t \in F$ as $t = \sum t_i \sigma(r_i)$ for $t_i \in \mathbb{Q}$. We calculate

$$\begin{aligned}
\langle x\epsilon_\sigma, t \otimes t \rangle &= \left\langle \sum_k x r_k \otimes \sigma(s_k), \sum_{\ell,m} t_\ell t_m \sigma(r_\ell) \otimes \sigma(r_m) \right\rangle \\
&= \sum_{k,\ell,m} t_\ell t_m \langle x r_k, \sigma(r_\ell)\rangle_K \langle \sigma(s_k), \sigma(r_m)\rangle_K \\
&= \sum_{k,\ell} t_k t_\ell \langle x r_k, \sigma(r_\ell)\rangle_K \\
&= \mathrm{Tr}^K_{\mathbb{Q}}(x t \sigma^{-1}(t)) \\
&= [K : K_\sigma] \, \mathrm{Tr}^{K_\sigma}_{\mathbb{Q}}(x Q_\sigma(t)).
\end{aligned}$$

∎

Proof of Theorem 6.1. We first wish to identify $\mathrm{Sym}_{\mathbb{Q}}(F)$ and the orthogonal complement $(\Lambda^1_F)^\perp$ as subspaces of $K \otimes K$. We have the orthogonal decomposition,

$$K \otimes K = \bigoplus_{\sigma \in G} \Lambda^\sigma_K.$$

We can recover $\mathrm{Sym}_{\mathbb{Q}}(F)$ as a subspace of $K \otimes K$ by

$$\mathrm{Sym}_{\mathbb{Q}}(F) = \bigoplus_{\tau \in G/I} \Gamma^\tau,$$

where for each orbit $\tau \in G/I$, we define Γ^τ to be the subspace of $\bigoplus_{\sigma \in \tau} \Lambda^\sigma_K$ fixed pointwise by the action of I. Given any σ in an orbit $\tau \in G/I$, we define the isomorphism $v_\sigma \colon K_\sigma \to \Gamma^\tau$ by

$$v_\sigma(x) = \sum_{\gamma \in I/\mathrm{Stab}(\sigma)} \gamma(x\epsilon_\sigma) = \sum_{\gamma \in I/\mathrm{Stab}(\sigma)} x\epsilon_{\gamma\cdot\sigma}.$$

Choose a set $\sigma_1 = 1, \sigma_2, \ldots, \sigma_n \in G$ of representatives of the orbits G/I. We obtain an isomorphism,

$$v \colon \bigoplus_{i=2}^n K_{\sigma_i} \to (\Lambda^1_F)^\perp \subset \mathrm{Sym}_{\mathbb{Q}}(F),$$

defined by $v(x_i)_{i=2}^n = (v_{\sigma_i}(x_i))_{i=2}^n$. By Lemma 6.2, we have for any $x_i \in K_{\sigma_i}$ and $t \in F$,

$$\langle v(x_i)_{i=2}^n, t \otimes t \rangle = \sum_{i=2}^n q_i \langle x_i, Q_\sigma(t)\rangle_{K_{\sigma_i}}, \tag{6.1}$$

for some positive rationals q_i.

Now, identifying $\mathrm{Ann}(\Lambda^1_F) \subset \mathbf{S}_{\mathbb{Q}}(F)$ with $(\Lambda^1_F)^\perp \subset \mathrm{Sym}_{\mathbb{Q}}(F)$ via the trace pairing, the admissibility condition is that for any $x \in (\Lambda^1_F)^\perp$, if $\langle x, t_i \otimes t_i \rangle \geq 0$ for all i, then $\langle x, t_i \otimes t_i \rangle = 0$ for all i. By (6.1), this is equivalent to the $Q_\sigma(t_i)$ satisfying the no-half-space condition for each i. ■

Cubic fields. We now suppose F is a cubic field. Define a quadratic map $Q\colon F \to F$ by

$$Q(x) = N^F_{\mathbb{Q}}(x)/x.$$

In this case, Theorem 6.1 becomes

Corollary 6.3. *Given a totally real cubic field F, a weighted boundary stratum with weights $\{t_1, \ldots, t_n\} \subset F$ is admissible if and only if $\{Q(t_1), \ldots, Q(t_n)\} \subset F$ satisfies the no-half-space condition.*

Proof. If F is Galois, this follows directly from Theorem 6.1, so suppose F is non-Galois with Galois closure K. We may identify $G = \mathrm{Gal}(K/\mathbb{Q})$ with the symmetric group S_3 with $F = K^{(12)}$. The action of I on G has two orbits, so we need only to check the condition of Theorem 6.1 for a single $\sigma \in G \setminus H$. Take $\sigma = (13)$. We have $(132) \cdot Q_{(12)}(x) = Q(x)$ for all $x \in F$, thus the two conditions coincide. ■

7 Rationality and positivity

In this section, we study in more detail the irreducible strata – that is, those that parameterize irreducible stable curves – in the boundary of the real multiplication locus. Given a basis $\boldsymbol{r} = (r_1, \ldots, r_g)$ of a lattice $\mathcal{I} \subset F$, we write $\mathcal{S}_{\boldsymbol{r}}$ for the associated $\mathcal{I}$-weighted boundary stratum, parameterizing irreducible stable curves having $2g$ nodes with weights $\pm r_1, \ldots, \pm r_g$. We say that $\boldsymbol{r}$ is an *admissible basis* of $\mathcal{I}$ if $\mathcal{S}_{\boldsymbol{r}}$ is an admissible stratum in the sense of §5.

We introduce in this section two additional properties of bases of number fields which we call *rationality* and *positivity*. We show that for totally real cubic fields, rationality and positivity together are equivalent to admissibility. For higher degree fields, the relation between these conditions is not clear. We then show that the rationality and positivity conditions are necessary for an irreducible stratum to intersect the boundary of the real multiplication locus. Finally, we give a geometric interpretation of the rationality and positivity conditions in terms of the geometry of locally symmetric spaces, from which we conclude that any lattice has only finitely many rational and positive bases, up to similarity.

Rationality and positivity. Consider a basis $\boldsymbol{r} = (r_1, \ldots, r_g)$ of a lattice in a totally real number field F. We denote by $(s_i)_{i=1}^g$ the dual basis. We say that $\boldsymbol{r}$ is *rational* if

$$\frac{r_i}{s_i} \Big/ \frac{r_j}{s_j} \in \mathbb{Q} \quad \text{for all } i \neq j.$$

We say that $\boldsymbol{r}$ is *positive* if

$$\frac{r_i}{s_i} \gg 0 \quad \text{for all } i,$$

where $x \gg 0$ means that x is positive under each embedding $F \to \mathbb{R}$.

As an intermediate technical notion we say that $\boldsymbol{r}$ is *weakly positive* if

$$\frac{r_i}{s_i} / \frac{r_j}{s_j} \gg 0 \quad \text{for all } i \neq j.$$

Lemma 7.1. *Every weakly positive and rational basis of F is positive.*

Proof. Suppose (r_i) is a basis of F which is weakly positive and rational but not positive. We define for each j

$$a^{(j)} = \left| \frac{s_1^{(j)}}{r_1^{(j)}} \right|^{1/2},$$

where the upper index (j) indicates the j-th embedding, and for each i the vectors

$$\widetilde{r}_i = (a^{(j)} r_i^{(j)})_{j=1}^g \quad \text{and} \quad \widetilde{s}_i = (s_i^{(j)}/a^{(j)})_{j=1}^g.$$

Note that the bases $(\widetilde{r}_i)$ and $(\widetilde{s}_i)$ are dual with respect to the standard inner product on $\mathbb{R}^n$. For each i, define

$$q_i = \frac{r_i}{s_i} / \frac{r_1}{s_1}.$$

By weak positivity and rationality, each q_i is a positive rational. We then have for each i and j

$$\widetilde{r}_i^{(j)} = \epsilon^{(j)} q_i \widetilde{s}_i^{(j)} \tag{7.1}$$

with each $\epsilon^{(j)} = \pm 1$. Since the basis (r_i) is not positive, we must have $\epsilon^{(j)} = -1$ for some j. Consider the matrices $R = (R_{ij}) = (r_i^{(j)})$ and $S = (S_{ij}) = (s_i^{(j)})$. Let D_ϵ be the diagonal matrix with $\epsilon^{(j)}$ the j^{th} diagonal entry, and define D_q similarly. We then have by (7.1),

$$S = D_q R D_\epsilon,$$

so since $R^t S = I$,

$$R^t D_q R = D_\epsilon^{-1}. \tag{7.2}$$

Thus R can be interpreted as an isomorphism between the indefinite quadratic form given by the matrix D_ϵ^{-1} and the definite quadratic form given by D_q, which is impossible. ■

Proposition 7.2. *A basis (r_1, r_2, r_3) of a cubic field F is admissible if and only if it is both rational and positive.*

Proof. Suppose that the no-half-space condition holds. If the three elements $Q(r_1), Q(r_2), Q(r_3)$ are $\mathbb{Q}$-linearly independent, their convex hull cannot contain zero. Since r_1, r_2, r_3 are a basis of F, the $Q(r_i)$ cannot all be $\mathbb{Q}$-multiples. Hence their $\mathbb{Q}$-span is a plane. Let $v_i = Q(r_i) \times Q(r_{i+1})$, where the $Q(\cdot)$-images are considered as elements of $\mathbb{R}^3$ using the three field embeddings. One calculates that

$$v_i = r_i r_{i+1} s_{i+2} \Delta(r_1, r_2, r_3),$$

where $\Delta(w_1, w_2, w_3) = \det(w_i^{(j)})$ and where we identify $v_i \in F$ with its image in $\mathbb{R}^3$.

The no-half-space-condition implies that the v_i are all proportional as elements of $\mathbb{R}^3$, i.e., $v_i/v_j \in \mathbb{Q}$ when considered as elements of F. This gives the rationality condition.

Moreover, the no-half-space condition implies that the angle between $Q(r_i)$ and $Q(r_{i+1})$ (in $\mathrm{Span}(Q(r_i), i = 1, 2, 3)$) is strictly contained in $(0, \pi)$. Thus the v_i are all pointing in the same direction. Consequently, the rational number $\frac{r_i s_j}{r_j s_i}$ is positive. This is weak positivity and the preceding lemma concludes one implication.

Conversely, suppose that rationality and positivity hold for $\{r_i, r_2, r_3\}$. The first part of the proof read backwards implies that the $Q(r_i)$ lie in a plane. If the no-half-space condition fails, we have that $v_i/v_j \in \mathbb{Q}^+$ but $v_i/v_k \in \mathbb{Q}^-$ for a suitable numbering with $\{i, j, k\} = \{1, 2, 3\}$. This contradicts weak positivity, and hence positivity. ■

Necessity of rationality and positivity. Given an irreducible $\mathcal{I}$-weighted boundary stratum $\mathcal{S}_{\boldsymbol{r}}$ and a real embedding ι of F, recall that $\mathcal{S}_{\boldsymbol{r}}^{\iota} \subset \mathbb{P}\Omega\overline{\mathcal{M}}_g$ is the stratum of irreducible stable forms having $2g$ poles of residues $\pm\iota(r_1), \ldots, \pm\iota(r_g)$.

Theorem 7.3. *Any irreducible stable form in the boundary of $\mathcal{E}_{\mathcal{O}}^{\iota}$ is contained in $\mathcal{S}_{\boldsymbol{r}}^{\iota}$ for some rational and positive basis $\boldsymbol{r}$ of a lattice $\mathcal{I} \subset F$ whose coefficient ring contains $\mathcal{O}$.*

Proof. Consider a family of stable curves $\mathcal{X} \to \Delta$, smooth over Δ^*, the fiber X_0 over 0 irreducible, of geometric genus 0, and with real multiplication by $\mathcal{O}$. We label the vanishing cycles of the fiber X_p over p as $\alpha_1, \ldots, \alpha_g$, and we choose a family of cycles $\beta_1, \ldots, \beta_g$ (with β_i defined only up to Dehn twist around α_i) such that $(\alpha_i, \beta_i)_{i=1}^g$ is a symplectic basis of $H_1(X_p; \mathbb{Z})$. As in §5, we may identify as an $\mathcal{O}$-module the subspace $V_p \subset H_1(X_p; \mathbb{Z})$ spanned by the vanishing cycles with some lattice $\mathcal{I}$ whose coefficient ring contains $\mathcal{O}$. Under this identification, the α_i correspond to some $r_i \in \mathcal{I}$. Choose an ordering $\iota_1 = \iota, \ldots, \iota_g$ of the real embeddings of F. We let $\omega^{(j)} \in \Omega(X_p)$ be the ι_j-eigenform determined by

$$\omega^{(j)}(\alpha_i) = r_i^{(j)}.$$

We must show that the r_i are a rational and positive basis of $\mathcal{I}$.

The plumbing coordinates from §3 provide holomorphic functions $t_i \colon \Delta \to \mathbb{C}$ which parameterize the opening-up of the i^{th} node of X_0. Since X_p is nonsingular for $p \neq 0$, each function t_i is vanishing only at 0. We claim that for some positive integers n_i,

$$\operatorname{Im}\frac{\omega^{(j)}(\beta_i)}{\omega^{(j)}(\alpha_i)} \sim \frac{n_i}{2\pi}\log\frac{1}{|t_i|}, \tag{7.3}$$

meaning that the ratio of the two sides tends to 1 as $p \to 0$.

Denote by $\eta_i \in \Omega(X_p)$ the form with $\eta_i(\alpha_j) = \delta_{ij}$. We then have

$$\omega^{(j)} = \sum_i r_i^{(j)}\eta_i,$$

so after exponentiation, we obtain

$$E\left(\frac{\omega^{(j)}(\beta_i)}{\omega^{(j)}(\alpha_i)}\right) = E(\eta_i(\beta_i))\prod_{k\neq i} E\left(\frac{r_k^{(j)}}{r_i^{(j)}}\eta_k(\beta_i)\right). \tag{7.4}$$

By Corollary 4.2, we have

$$E(\eta_i(\beta_i)) = t_i^{n_i}\phi \quad \text{and} \quad E(\eta_i(\beta_j)) = \psi_j \tag{7.5}$$

for ϕ and ψ_j nonzero holomorphic functions on Δ and n_i a positive integer (equal to the intersection number of Δ with the boundary stratum where α_i has been pinched). Substituting (7.5) into (7.4) and taking logarithms yields

$$\operatorname{Im}\frac{\omega^{(j)}(\beta_i)}{\omega^{(j)}(\alpha_i)} = \frac{n_i}{2\pi}\log\frac{1}{|t_i|} + O(1),$$

from which (7.3) follows.

Since we have identified V_p with $\mathcal{I}$ as $\mathcal{O}$-modules, we also have the $\mathcal{O}$-module isomorphism,

$$H_1(X_p;\mathbb{Z})/V_p \cong \operatorname{Hom}(V_p,\mathbb{Z}) \cong \operatorname{Hom}(\mathcal{I},\mathbb{Z}) \cong \mathcal{I}^\vee,$$

where the first isomorphism arises from the intersection pairing and the last from the trace pairing. Under this isomorphism, the basis $(\beta_i,\dots,\beta_g)$ of $H_1(X;\mathbb{Z})/V_p$ corresponds to the basis $(s_1,\dots,s_g)$ of $\mathcal{I}^\vee$ which is dual to $(r_1,\dots,r_g)$. Thus under the action of real multiplication, we have

$$\frac{r_i}{r_k}\cdot\alpha_k = \alpha_i \quad \text{and} \quad \frac{s_i}{s_k}\cdot\beta_k = \beta_i \pmod{V_p}.$$

From this and (7.3), we then obtain

$$\frac{s_i^{(j)}}{r_i^{(j)}}\Big/\frac{s_k^{(j)}}{r_k^{(j)}} = \operatorname{Im}\frac{\omega^{(j)}(\beta_i)}{\omega^{(j)}(\alpha_i)}\Big/\operatorname{Im}\frac{\omega^{(j)}(\beta_k)}{\omega^{(j)}(\alpha_k)} \sim \log\frac{n_i}{|t_i|}\Big/\log\frac{n_k}{|t_k|}. \tag{7.6}$$

Since the right side of (7.6) is independent of j, so is the left side. Thus $(s_i/r_i)/(s_k/r_k)$ is rational. The right side of (7.6) is also positive for $p \sim 0$ because $t_\ell(0) = 0$ for all ℓ, so $(s_i/r_i)/(s_k/r_k)$ is positive as well. Therefore this basis is both rational and weakly positive. By Lemma 7.1 the basis is then positive. ∎

Finiteness of rational and positive bases. We now give a geometric interpretation of bases of lattices satisfying the rationality and positivity conditions as points of intersection of flats in the locally symmetric space $\mathrm{SL}_g(\mathbb{Z})\backslash\mathrm{SL}_g(\mathbb{R})/\mathrm{SO}_g(\mathbb{R})$. This yields a quick proof that there are only finitely many such bases up to the action of the unit group.

We recall the classical correspondence between similarity classes of lattices in degree g totally real number fields and compact totally geodesic flat tori of dimension $g-1$ in $\mathrm{SL}_g(\mathbb{Z})\backslash\mathrm{SL}_g(\mathbb{R})/\mathrm{SO}_g(\mathbb{R})$. Consider a degree g totally real number field F with an ordering $\iota_1,\ldots,\iota_g$ of the embeddings of F into $\mathbb{R}$. Let $\mathcal{I}$ be a lattice in F, which we regard as point in the space of lattices $\mathrm{SL}_g(\mathbb{Z})\backslash\mathrm{SL}_g(\mathbb{R})$. Let $U(\mathcal{I})\subset\mathcal{O}_{\mathcal{I}}$ be the group of totally positive units ϵ such that $\epsilon\mathcal{I}=\mathcal{I}$. We embed $U(\mathcal{I})$ in the group $D\subset\mathrm{SL}_g(\mathbb{R})$ of positive diagonal matrices by the embeddings ι_i. By Dirichlet's units theorem, $U(\mathcal{I})$ is a lattice in D. Let $T(\mathcal{I})=U(\mathcal{I})\backslash D$, a compact torus. The stabilizer of $\mathcal{I}$ under the right action of D on $\mathrm{SL}_g(\mathbb{Z})\backslash\mathrm{SL}_g(\mathbb{R})$ is $U(\mathcal{I})$, so we obtain an immersion $p_{\mathcal{I}}\colon T(\mathcal{I})\to\mathrm{SL}_g(\mathbb{Z})\backslash\mathrm{SL}_g(\mathbb{R})/\mathrm{SO}_g(\mathbb{R})$ of $T(\mathcal{I})$ as a compact flat in $\mathrm{SL}_g(\mathbb{Z})\backslash\mathrm{SL}_g(\mathbb{R})/\mathrm{SO}_g(\mathbb{R})$. Since similar lattices lie on the same D-orbit, this associates a compact flat to each similarity class of lattices.

Let $\mathrm{Rec}\subset\mathrm{SL}_g(\mathbb{Z})\backslash\mathrm{SL}_g(\mathbb{R})/\mathrm{SO}_g(\mathbb{R})$ be the locus of lattices in $\mathbb{R}^g$ which have a basis of orthogonal vectors, a closed (but not compact) flat isometric to $\mathbb{R}^g/C_g$, where $C_g\subset\mathrm{SO}_g(\mathbb{R})$ is the group of symmetries of the cube.

Theorem 7.4. *For each lattice $\mathcal{I}$ in a degree g totally real number field F, the flat $p_{\mathcal{I}}(T(\mathcal{I}))$ intersects* Rec *transversely. There is a natural bijection between the set $p_{\mathcal{I}}^{-1}(\mathrm{Rec})$ and the set of rational and positive bases of $\mathcal{I}$, up to the action of $U(\mathcal{I})$, changing signs, and reordering.*

Note that we do not claim that the intersection is non-empty. Indeed this is note always the cases, as indicated in Figure 4 for $D=39699$.

Proof. Let $\widetilde{\mathrm{Rec}}\subset\mathrm{SL}_g(\mathbb{R})/\mathrm{SO}_g(\mathbb{R})$ be the image of the diagonal orbit of the standard basis of $\mathbb{R}^g$, a lift of Rec to $\mathrm{SL}_g(\mathbb{R})/\mathrm{SO}_g(\mathbb{R})$.

Lifts of $T(\mathcal{I})$ to $\mathrm{SL}_g(\mathbb{R})/\mathrm{SO}_g(\mathbb{R})$ correspond to oriented bases of $\mathcal{I}$ up to the action of the unit group by associating the flat $(r_i^{(j)})\cdot D\cdot\mathrm{SO}_g(\mathbb{R})$ to the ordered basis $r_1,\ldots,r_g$. Points of $p_{\mathcal{I}}^{-1}(\mathrm{Rec})$ correspond bijectively (up to the action of the group $C_g\subset\mathrm{SL}_g(\mathbb{Z})$ of symmetries of the cube) to intersection points of $p_{\mathcal{I}}(T(\mathcal{I}))$ with Rec. Note that if a lift $\mathcal{F}$ intersects $\widetilde{\mathrm{Rec}}$, then so does the lift $\gamma\cdot\mathcal{F}$ for any $\gamma\in C_g$. These intersection points correspond to the same point in $p_{\mathcal{I}}^{-1}(\mathrm{Rec})$, and on the level of bases, replacing $\mathcal{F}$ with $\gamma\cdot F$ corresponds to reordering and changing signs in the basis (r_i).

We must show that $(r_i^{(j)})\cdot D\cdot\mathrm{SO}_g(\mathbb{R})$ intersects $\widetilde{\mathrm{Rec}}$ if and only if (r_i) is rational and positive. Note that the rationality and positivity conditions make sense for bases of $\mathbb{R}^n$, with the j^{th} embedding $r_i^{(j)}$ interpreted as the j^{th} coordinate of the vector r_i. A vector is regarded as rational if all of its coordinates are equal, totally positive if all of its coordinates are positive, and so on. With this interpretation, an orthogonal basis $(r_1,\ldots,r_n)$ of $\mathbb{R}^n$ is rational and positive, since the basis is orthogonal if and only if each dual vector s_i is a positive multiple of the corresponding r_i. The rationality and positivity conditions are invariant under the action of D, thus any basis (r_i) whose D-orbit meets $\widetilde{\mathrm{Rec}}$ is rational and positive.

Now suppose the basis (r_i) of $\mathcal{I}$ is rational and positive. Let (s_i) be the dual basis. For each j, let $a^{(j)} = \sqrt{s_1^{(j)}/r_1^{(j)}}$. Let A be the diagonal matrix $(a^{(1)}, \dots, a^{(g)})$, and let

$$(\widetilde{r}_i^{(j)}) = (a^{(j)} r_i^{(j)}) \quad \text{and} \quad (\widetilde{s}_i^{(j)}) = (s_i^{(j)}/a^{(j)}).$$

Note that $(\widetilde{s}_i^{(j)})$ is the dual basis to $(\widetilde{r}_i^{(j)})$ in $\mathbb{R}^g$, and A is the unique diagonal matrix in $\mathrm{SL}_g(\mathbb{R})$ for which the vectors $(\widetilde{r}_1^{(j)})$ and $(\widetilde{s}_1^{(j)})$ are positively proportional. If the positivity and rationality conditions are satisfied, we have

$$\frac{\widetilde{s}_i^{(j)}}{\widetilde{r}_i^{(j)}} = \frac{1}{(a^{(j)})^2} \cdot \frac{s_i^{(j)}}{r_i^{(j)}} = \frac{q_i}{(a^{(j)})^2} \cdot \frac{s_1^{(j)}}{r_1^{(j)}} = q_i$$

for some positive $q_i \in \mathbb{Q}$. Since each $\widetilde{s}_i$ is proportional to $\widetilde{r}_i$, the basis $(\widetilde{r}_i)$ of $\mathbb{R}^g$ is rectangular, so it is the unique intersection point of $(r_i^{(j)}) \cdot D \cdot \mathrm{SO}_g(\mathbb{R})$ and $\widetilde{\mathrm{Rec}}$. Otherwise for some i the vectors $(\widetilde{r}_i^{(j)})$ and $(\widetilde{s}_i^{(j)})$ are not proportional, so the flats are disjoint. Since we saw that there was at most one intersection point between each lift of the two flats, these intersection points are transverse. ■

Corollary 7.5. *The set of bases of $\mathcal{I}$ satisfying the rationality and positivity conditions is finite, up to the action of $U(\mathcal{I})$.*

Proof. Since $T(\mathcal{I})$ is compact, there are at most finitely many intersection points with Rec by transversality. ■

8 Boundary of the eigenform locus: Sufficiency for genus three

In this section we specialize to genus three. We prove that the boundaries of $\mathcal{RM}_{\mathcal{O}}$ and $\mathcal{E}_{\mathcal{O}}^{\iota}$ are indeed the union of the components described in Theorem 5.2. Moreover, we show how to derive these subvarieties explicitly from the weights of a boundary stratum.

Boundary strata in genus three. The topological type of a geometric genus zero stable curve (or a weighted boundary stratum) can be encoded by a graph where each vertex represents an irreducible component and an edge joining two vertices (or possibly joining a vertex to itself) represents a node at the intersection of those two components. There are fifteen topological types of arithmetic genus three, geometric genus zero stable curves, shown in Figure 1.

We will refer to a stable curve represented by the j^{th} graph in the i^{th} row of Figure 1 as a *type* (i, j) *stable curve*.

An $\mathcal{I}$-weighted stable curve can be represented by a directed graph with a weight $r \in \mathcal{I}$ attached to each edge e (contrary to standard practice, we allow edges which join a vertex to itself). The cusp on the component represented by the terminal vertex of e has weight r, and the other cusp has weight $-r$.

It will be convenient to have a compact notation for boundary strata without separating curves, the only ones which will be important in the sequel. For all but one of these strata the components of the corresponding stable curves can be arranged in chain or one loop. We code those boundary strata in the following way: we write $[m_i]$ for a genus zero component of the stable curve with m_i marked points. We write $\times^{a_i}$ for the number of intersection points with the subsequent curve. The possible patterns for curve systems without separating curves include $[6]$, $[m_1] \times^a [m_2]$, $[m_1] \times^{a_1} [m_2] \times^{a_2} [m_3]$ or $[m_1] \times^{a_1} [m_2] \times^{a_2} [m_3] \times^{a_3}$. In the last pattern, a_3 is the number of nodes joining the last and the first component. For example, a $[5] \times^3 [3]$ boundary stratum is represented by graph $(2, 2)$ in Figure 1 and a $[4] \times^2 [3] \times^1 [3] \times^2$ boundary stratum is represented by graph $(3, 1)$.

Boundary strata of type $[6]$ parameterize irreducible stable curves with three nonseparating nodes, often called "trinodal curves."

Theorem 8.1. *Consider an order $\mathcal{O}$ in a totally real cubic number field F, a real embedding ι of F, and a cusp packet $(\mathcal{I}, E) \in \mathcal{C}(\mathcal{O})$. The closure in $\mathbb{P}\Omega\overline{\mathcal{M}}_g$ of the cusp of $\mathcal{E}^{\iota}_{\mathcal{O}}$ associated to $(\mathcal{I}, E)$ is equal to the union over all admissible $\mathcal{I}$-weighted boundary strata $\mathcal{S}$ of the varieties $\mathcal{S}^{\iota}(E)$.*

The closure of the corresponding cusp of $\mathcal{RM}_{\mathcal{O}}$ in $\overline{\mathcal{M}}_g$ is equal to the union over all $\mathcal{I}$-weighted boundary strata $\mathcal{S}$ of the images of the $\mathcal{S}(E)$ under the forgetful map to $\overline{\mathcal{M}}_g$.

After some preliminary discussion, we prove Theorem 8.1 at the end of this section.

Since the intersection of two algebraic subvarieties of $\overline{\mathcal{M}}_3$ has a finite number of components, we obtain the following generalization for genus three of Corollary 7.5.

Corollary 8.2. *Given a lattice $\mathcal{I}$ in a cubic number field F, the number of $\mathcal{I}$-weighted admissible boundary strata up to similarity is finite.*

We will discuss in Appendix A various aspects concerning enumerating and counting this set of admissible weighted boundary strata.

In order to make Theorem 8.1 completely explicit, we will now give coordinates on some weighted boundary strata in terms of cross-ratios and give explicit equations cutting out the subvarieties $\mathcal{S}(E)$.

We say that a weighted boundary stratum $\mathcal{S}_1$ is a *degeneration* of $\mathcal{S}_2$, if $\mathcal{S}_1$ is obtained by pinching a collection of curves on a surface represented by $\mathcal{S}_2$. We also say that $\mathcal{S}_2$ is an *undegeneration* of $\mathcal{S}_1$ in this situation.

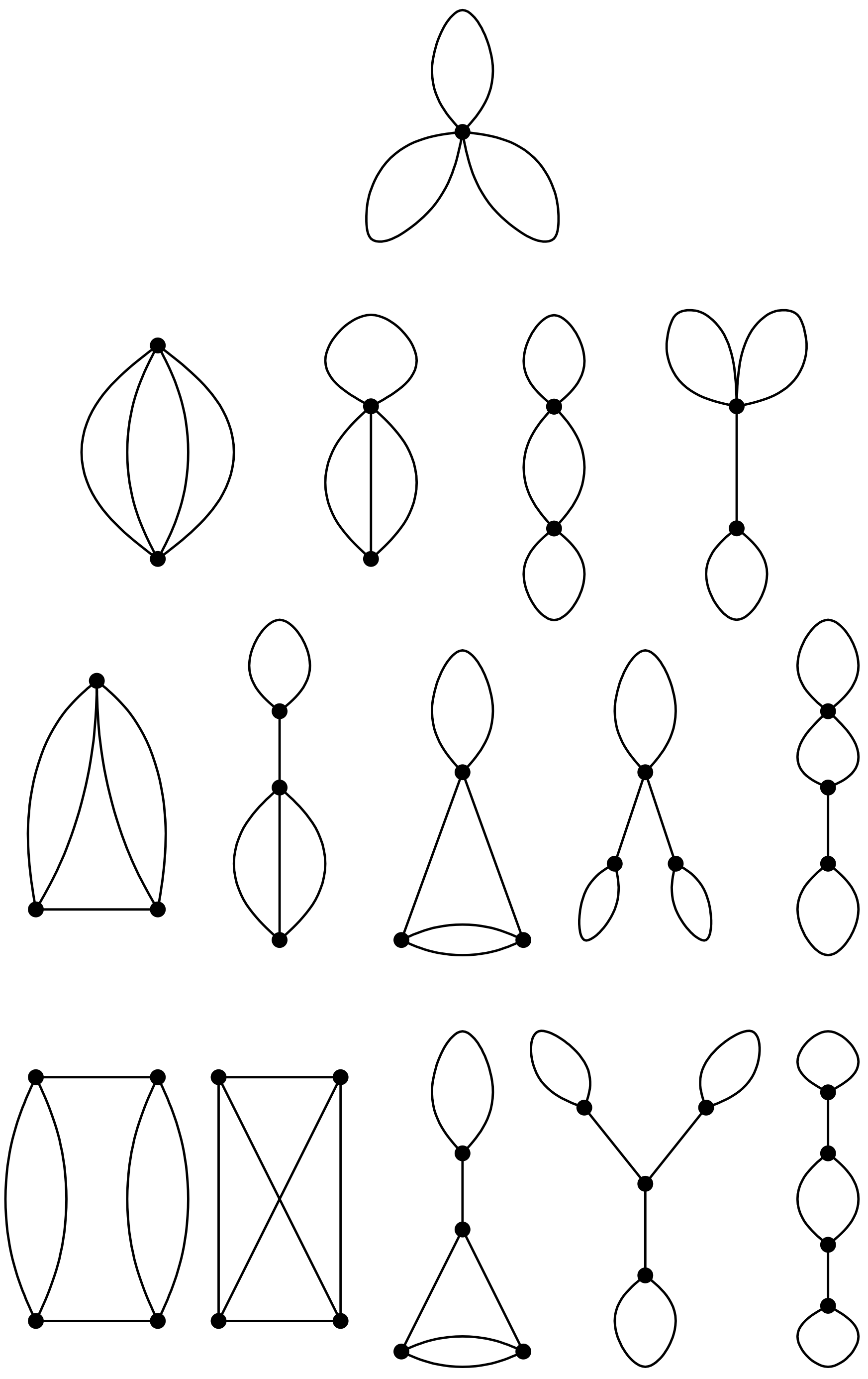

Figure 1: Genus three, geometric genus zero stable curves

Irreducible strata. Consider an irreducible stratum (that is, type [6] if we are in genus three) $\mathcal{S}_{\boldsymbol{r}}$. A weighted stable curve parameterized by $\mathcal{S}_{\boldsymbol{r}}$ is determined by $2g$ distinct points $p_1, \dots, p_g$ and $p_{-1}, \dots, p_{-g}$ on $\mathbb{P}^1$ with weights r_i at p_i and $-r_i$ at p_{-i}, so $\mathcal{S}_{\boldsymbol{r}} \cong \mathcal{M}_{0,2g}$. For $j \neq k$ we define the cross-ratio morphisms $R_{[jk]} \colon \mathcal{S}_{\boldsymbol{r}} \to \mathbb{C} \setminus \{0,1\}$ by

$$R_{[jk]} = [p_j, p_{-j}, p_{-k}, p_k]^{-1}. \tag{8.1}$$

where for $z_1, \dots z_4 \in \mathbb{C}$,

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

Take $(s_1, \dots, s_g)$ to be the dual basis of $\mathcal{I}^\vee$ (with respect to the trace pairing) to $(r_1, \dots, r_g)$. We can now make the cross-ratio map CR defined in (5.3) more explicit.

Proposition 8.3. *The elements $s_j \otimes s_k$ for $j \neq k$ form a basis of $N(\mathcal{S}_{\boldsymbol{r}})$. Moreover we have $\Psi(s_j \otimes s_k) = R_{[jk]}$ as functions on $\mathcal{S}_{\boldsymbol{r}}$.*

Proof. That $s_j \otimes s_k$ belongs to $N(\mathcal{S}_{\boldsymbol{r}})$ follows from the definition of the dual basis with respect to the trace pairing. They are obviously linearly independent and thus the set of $s_j \otimes s_k$ for $j \neq k$ is a basis by a dimension count.

We normalize a point $P = (p_{-g}, \dots, p_g)$ of $\mathcal{S}_{\boldsymbol{r}}$ by a Möbius transformation so that $p_j = 0$, $p_{-j} = \infty$ and $p_-k = 1$. By definition of $\Psi(s_j \otimes s_k)$ we must choose the stable one-form ω on $\mathbb{P}^1$ with residue $\pm \operatorname{Tr}(s_j r_m)/2\pi i$ at the point $p_{\pm m}$, i.e. we have to choose $\omega = dz/2\pi i z$. We then integrate this form over the path whose intersection with the loop around the node at $p_{\pm m}$ is $\operatorname{Tr}(s_k r_m)$. On $\mathbb{P}^1$, this is a path γ joining $p_-k = 1$ to p_k. We then have

$$\Psi(s_j \otimes s_k)(P) = e^{2\pi i \int_\gamma \omega} = p_k = R_{[jk]}(P). \qquad \blacksquare$$

Corollary 8.4. *For $g = 3$, after identifying* $\operatorname{Hom}(N(\mathcal{S}) \cap \mathbf{S}_{\mathbb{Z}}(\mathcal{I}^\vee), \mathbb{C}^*)$ *with* $(\mathbb{C}^*)^3$ *via the basis* $(s_1 \otimes s_2, s_2 \otimes s_3, s_3 \otimes s_1)$ *of* $N(\mathcal{S}) \cap \mathbf{S}_{\mathbb{Z}}(\mathcal{I}^\vee)$*, the map* CR *becomes*

$$\mathrm{CR} = (R_{[12]}, R_{[23]}, R_{[31]}) \colon \mathcal{S}_{\boldsymbol{r}} \to (\mathbb{C} \setminus \{0,1\})^3.$$

The map CR *is a degree two branched cover which identifies orbits of the involution $i \colon \mathcal{S}_{\boldsymbol{r}} \to \mathcal{S}_{\boldsymbol{r}}$ which exchanges each pair p_ℓ and $p_{-\ell}$.*

Proof. That CR is of this form follows immediately from the definition of CR and Proposition 8.3.

That the map $\mathrm{CR} = (R_{[12]}, R_{[13]}, R_{[23]})$ is degree two onto its image can be checked by fixing three of the p_i and solving for the rest. Interchanging each p_ℓ and $p_{-\ell}$ leaves each cross-ratio $R_{[jk]}$ invariant, so CR is the quotient map by this involution. $\blacksquare$

Type $[4]\times^4[4]$ **strata.** Consider an $\mathcal{I}$-weighted stable curve X of type $[4]\times^4[4]$ having weights $r_1,\ldots,r_4 \in \mathcal{I}$ with $\sum r_i = 0$, and let $\mathcal{S}$ be the corresponding $\mathcal{I}$-weighted boundary stratum. We name $u_1,\ldots,u_4$ the four points on one irreducible component with weight $r_1,\ldots,r_4$ and name $v_1,\ldots,v_4$ the opposite points on the other irreducible component. We define the cross-ratios,

$$R_u = [u_1, u_2, u_3, u_4] \quad \text{and} \quad R_v = [v_1, v_2, v_3, v_4].$$

The stratum $\mathcal{S}$ is isomorphic to $\mathcal{M}_{0,4} \times \mathcal{M}_{0,4}$ with R_u and R_v coordinates on the first and second factors.

Type $[4]\times^2[4]$ **strata.** Now consider the $\mathcal{I}$-weighted stable curve shown in Figure 2 with distinct weights $r_1, r_2, r_3 \in \mathcal{I}$, and let $\mathcal{S}$ be the corresponding $\mathcal{I}$-weighted boundary stratum. We label by p_1, p_{-1}, p_2, p_{-2} the points on one irreducible component with weights $r_1, -r_1, r_2, -r_2$ and label by q_1, q_{-1}, q_2, q_{-2} the points on the other irreducible component with weights $r_3, -r_3, -r_2, r_2$. The stratum $\mathcal{S}$ is isomorphic to $\mathcal{M}_{0,4} \times \mathcal{M}_{0,4}$ with coordinates

$$R_1 = [q_1, q_{-1}, q_{-2}, q_2] \quad \text{and} \quad R_3 = [p_1, p_{-1}, p_{-2}, p_2]. \tag{8.2}$$

The stratum $\mathcal{S}$ arises as a degeneration of the irreducible weighted boundary stratum with weights r_1, r_2, r_3 by pinching a curve around the points of weights $r_1, -r_1, r_2$. As this curve is pinched, the cross-ratio $R_{[1,3]}$ tends to 1.

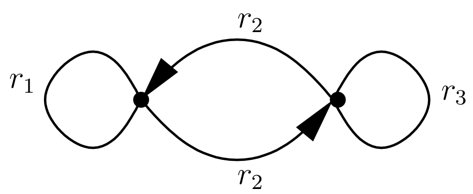

Figure 2: Type $[4]\times^2[4]$ $\mathcal{I}$-weighted stable curve

Calculation of $\mathcal{S}(E)$. We will write R_i for $R_{[jk]}$ where $\{i,j,k\} = \{1,2,3\}$ and we let (s_1, s_2, s_3) be the dual basis to (r_1, r_2, r_3).

Whether $\mathcal{S}(E) = \mathcal{S}$ is the content of the next theorem. To measure the difference we introduce the following notion. Given an $\mathcal{I}$-weighted boundary stratum $\mathcal{S}$, we let $\operatorname{Span}(\mathcal{S}) \subset \mathbb{Q}^3$ denote the $\mathbb{Q}$-span of $\{Q(r) : r \in \operatorname{Weight}(\mathcal{S})\}$, and let $\operatorname{codim}(\operatorname{Span}(\mathcal{S}))$ denote the codimension of $\operatorname{Span}(\mathcal{S})$ in $\mathbb{Q}^3$.

Theorem 8.5. *The locus $\mathcal{S}(E)$ is determined as a subvariety of $\mathcal{S}$ is the various cases as follows.*

- *Case* $[6]$*: For a boundary stratum of type* $[6]$*, we use the cross-ratio coordinates R_1, R_2, R_3 defined in Proposition 8.3. Then the subvariety $\mathcal{S}(E)$*

of the admissible boundary stratum $\mathcal{S}_{(r_1,r_2,r_3)}$ is given by the cross-ratio equation

$$\prod_{i=1}^{3} R_i^{a_i} = \zeta, \tag{8.3}$$

where the a_i are the unique (up to sign) relatively prime integers such that $a_i = tb_i$ for some $t \in F$, and

$$b_i = N_{\mathbb{Q}}^{F}(r_i) \left(\frac{s_i}{r_i}\right)^2, \tag{8.4}$$

and where ζ is the root of unity $\zeta = e^{2\pi i u}$ with

$$u = \langle E, \sigma \rangle, \tag{8.5}$$

where

$$\sigma = \sum_{i=1}^{3} a_i s_{i+1} \otimes s_{i+2}. \tag{8.6}$$

Here we interpret the extension class E as an element of $\mathrm{Sym}_{\mathbb{Q}}(F)$ using (5.4)

- *Case* $[4] \times^2 [4]$*: The subvariety $\mathcal{S}(E)$ of the admissible boundary stratum with weights $\{r_1, r_2, r_3, r_4 = -r_2\}$ is given, using the cross-ratio coordinates defined above, by*

  $$R_1^{a_1} R_3^{a_3} = \zeta, \tag{8.7}$$

  where the a_i and ζ are calculated from $\{r_1, r_2, r_3\}$ as in the preceding case [6]*.*
- *Case* $[4] \times^4 [4]$*: There are two possibilities. If* $\mathrm{codim}(\mathrm{Span}(\mathcal{S})) = 0$*, then $\mathcal{S}(E)$ is the whole stratum. If* $\mathrm{codim}(\mathrm{Span}(\mathcal{S})) = 1$*, then $\mathcal{S}$ is a degeneration of an admissible irreducible weighted boundary stratum $\mathcal{S}_{(r_1,r_2,r_3)}$ with the property that exponents a_i defined above satisfy $\sum_{i=1}^{3} a_i = 0$. Moreover, $\mathcal{S}(E)$ is cut out by the equation*

  $$(R_u R_v)^{a_1} \cdot \left(\frac{R_u}{1 - R_u} \frac{R_v}{1 - R_v}\right)^{a_3} = \zeta, \tag{8.8}$$

  where ζ is as in the case [6]*.*

This is a complete list of the cases of boundary strata without separating curves, where for some admissible boundary stratum $\mathcal{S}$, we can have $\mathcal{S}(E) \subsetneq \mathcal{S}$.

We will refer to the equations stated in the above theorem as the *cross-ratio equations* and to the exponents a_i in (8.3) as *cross-ratio exponents*

For comparison, we sketch the analogous result for genus two from [Bai07] in this notation. There are just two strata, the one-dimensional stratum [4] consisting of irreducible 2-nodal curves and the zero-dimensional stratum $[3] \times^3 [3]$ consisting of stable curves with two components intersecting at three nodes. In the stratum [4] a basis $\{r_1, r_2\}$ is admissible, if $N_{\mathbb{Q}}^{F}(r_1/r_2) < 0$. In the stratum $[3] \times^3 [3]$ the residues are of the form $\{r_1, r_2, -r_1 - r_2\}$ and again the corresponding boundary stratum is admissible if $N_{\mathbb{Q}}^F(r_1/r_2) < 0$. There is no cross-ratio equation in this case. The boundary of the real multiplication locus is, given an admissible boundary stratum, the full stratum of the Deligne-Mumford compactification.

The following lemmas determine the possibilities for $\operatorname{codim}(\operatorname{Span}(\mathcal{S}))$.

Lemma 8.6. *Suppose that the $\mathbb{Q}$-span of r_1, r_2, r_3 is two-dimensional and the $\mathbb{Q}$-linear dependence is given by $b_1r_1 + b_2r_2 + b_3r_3 = 0$ with all $b_i \in \mathbb{Q} \setminus \{0\}$. Then $Q(r_1), Q(r_2)$, and $Q(r_3)$ are $\mathbb{Q}$-linearly independent.*

Proof. Embedding F in $\mathbb{R}^3$ by its three real embeddings, the map Q becomes

$$Q(x, y, z) = (yz, xz, xy),$$

which we regard as a degree two map $Q\colon \mathbb{P}^2(\mathbb{R}) \to \mathbb{P}^2(\mathbb{R})$. Suppose the $Q(r_i)$ are $\mathbb{Q}$-linearly dependent. They then lie on a line $L \subset \mathbb{P}^2(\mathbb{R})$ cut out by an equation $a_1X + a_2Y + a_3Z = 0$ with each $a_i \in \mathbb{Q}$. Each coefficient a_i of this equation must be nonzero, for if (say) a_3 were zero, then r_2r_3 and r_1r_3 are $\mathbb{Q}$-linearly dependent, hence r_2 and r_1 are $\mathbb{Q}$-linearly dependent, contradicting the hypothesis.

The inverse image $Q^{-1}(L)$ is a nonsingular conic, so it intersects any line in at most two points. Thus if the r_i were $\mathbb{Q}$-linearly dependent, they could not map to L. ■

Lemma 8.7. *If the stratum $\mathcal{S}$ is irreducible or if it is of type* $[4] \times^2 [4]$, *then* $\operatorname{codim}(\operatorname{Span}(\mathcal{S})) = 1$. *If it is of type* $[4] \times^4 [4]$, *then either* $\operatorname{codim}(\operatorname{Span}(\mathcal{S})) = 0$ *or* $\operatorname{codim}(\operatorname{Span}(\mathcal{S})) = 1$. *In all of the remaining cases,* $\operatorname{codim}(\operatorname{Span}(\mathcal{S})) = 0$.

Proof. Since the set of weights contains a $\mathbb{Q}$-basis of F, $\operatorname{codim}(\operatorname{Span}(\mathcal{S}))$ is at most 1. Suppose a stable curve parameterized by the stratum $\mathcal{S}$ contains a component isomorphic to a thrice-punctured $\mathbb{P}^1$. The weights of this component satisfy the hypothesis of Lemma 8.6 since they sum to zero and their $\mathbb{Q}$-span is not one-dimensional by Proposition 5.5. Consequently, this lemma implies that $\operatorname{codim}(\operatorname{Span}(\mathcal{S})) = 0$ for those strata.

The only remaining cases are the irreducible stratum and strata of type $[4] \times^2 [4]$. In either case there are only three distinct weights. We only need to remark that three vectors cannot span $\mathbb{R}^3$ and contain 0 in its convex hull at the same time. ■

We will show in Example 2 of Appendix A that this is a complete list of constraints, i.e. all the codimensions of strata not excluded by Lemma 8.7 do occur.

Lemma 8.8. *Suppose that $\{P_i\}_{i=1}^k$ are k points in $\mathbb{R}^n$, $k \geq n+2$, whose $\mathbb{R}^+$-span is all of $\mathbb{R}^n$ and such that no n of the P_i are contained in a subspace of dimension $n-1$. Then there are $n+1$ points among the P_i, whose $\mathbb{R}^+$-span also is all of $\mathbb{R}^n$.*

Proof. Given $k \geq n+2$ points P_i in $\mathbb{R}^n$ whose convex hull contains zero, we must show that there are $k-1$ among them whose convex hull still contains zero. The hypothesis on the span of subsets of n elements will then imply that these vectors span $\mathbb{R}^n$, and the claim follows from induction on k.

Consider the linear map f that assigns to $x \in \mathbb{R}^k$ the sum $f(x) = \sum_{i=1}^k x_i P_i$. The hypothesis implies that $K = \mathrm{Ker}(f)$ contains $w = (w_1, \ldots, w_k)$ with $\sum_{i=1}^k w_i = 1$ and $w_i > 0$. Since $\dim(K) \geq 2$ there is also $0 \neq y \in K$ with $\sum y_i = 0$. The affine space $w + \lambda y$ has to intersect the coordinate hyperplanes at some point different from zero. This point yields a convex combination of zero with at most $k-1$ summands. ■

Proof of Theorem 8.5. We start with case [6]. Recall that $\mathcal{S}(E) \subset \mathcal{S}$ is the subvariety cut out by the equations

$$\Psi(a) = e^{-2\pi i \langle E, a\rangle}, \tag{8.9}$$

as a ranges in $N(\mathcal{S}) \cap \mathrm{Ann}(\Lambda^1) \cap \mathbf{S}_{\mathbb{Z}}(\mathcal{I}^\vee)$. By Lemma 8.7 and equation (5.6), this is a rank-one $\mathbb{Z}$-module, so by Proposition 8.3, it is generated by $\sum_{i=1}^3 a_i s_{i+1} \otimes s_{i+2}$ for some relatively prime integers a_i, and the equation (8.9) is simply (8.3) with ζ as in (8.5). To find the a_i, we will find some rationals b_i with $\sum b_i s_{i+1} \otimes s_{i+2} \in \mathrm{Ann}(\Lambda_1)$, and the a_i will be a primitive integral multiple.

If $b_i \in \mathbb{Q}$, then $\sum b_i s_{i+1} \otimes s_{i+2} \in \mathrm{Ann}(\Lambda^1)$ if and only if

$$\mathrm{Tr}\left(\sum_{i=1}^3 b_i s_{i+1} s_{i+2} x\right) = \left\langle \sum_{i=1}^3 b_i s_{i+1} \otimes s_{i+2}, \sum_{j=1}^3 r_j \otimes s_j x \right\rangle = 0$$

for all $x \in F$, thus if and only if $\sum b_i s_{i+1} s_{i+2} = 0$.

If we let $\widetilde{b}_i = N(r_i)\frac{s_i}{r_i}$ and take c_i satisfying $\sum c_i / r_i = 0$ then we have

$$\sum_{i=1}^3 \widetilde{b}_i \frac{c_i}{N(r_i)} s_{i+1} s_{i+2} = 0.$$

From Lemma 8.9 below, we deduce that $(\widetilde{b}_i \frac{c_i}{N(r_i)})_{i=1}^3$ is proportional to the b_i as in the statement. Thus the exponents in the cross-ratio equation are proportional to the b_i as claimed.

We next treat the case of a stratum $\mathcal{S}$ of type $[4] \times^4 [4]$. As explained above along with the cross-ratio coordinates, this case is a degeneration of a boundary stratum of type [6]. Since $\mathrm{Span}(\mathcal{S})$ here is the same as for $\mathcal{S}_{(r_1, r_2, r_3)}$ we obtain the same equation, only the cross-ratio R_2 is equal to one identically.

It remains to treat the case of a boundary stratum $\mathcal{S}$ of type $[4] \times^4 [4]$ in the case $\dim(\mathrm{Span}(\mathcal{S})) = 2$. Lemma 8.8 implies that $\mathcal{S}$ is a degeneration of some admissible stratum of type [6], say $\mathcal{S}_{(r_1, r_2, r_3)}$ given a suitable numbering of the weights.

Next we show that $\sum a_i = 0$. Admissibility implies that (8.10) below holds for some $c_i \in \mathbb{Q}$. The hypothesis on the dimension of the span implies the equation (8.10) and

$$\frac{1}{r_1+r_2+r_3} = \frac{e_1}{r_1} + \frac{e_2}{r_2}$$

for some $e_1, e_2 \in \mathbb{Q}$. We may moreover rescale such that $r_1 = 1$ and solve the system for cubic equations killing r_2 and r_3 respectively. These equations must be the minimal polynomials of r_2 and r_3. We obtain

$$N_{\mathbb{Q}}^F(r_2) = -\frac{c_2 e_2}{c_1 e_1} \quad \text{and} \quad N_{\mathbb{Q}}^F(r_3) = \frac{c_3^2 e_2}{c_2 c_1 e_1 - c_1^2 e_2}.$$

Using the Corollary 8.10 to the calculations in case [6] below, we only need to check that $\sum c_i^2/N_{\mathbb{Q}}^F(r_i) = 0$, which is obvious.

We may normalize the degeneration from the boundary stratum $\mathcal{S}_{(r_1,r_2,r_3)}$ to $\mathcal{S}$ as follows. Let $p_1 = 0$, $p_2 = 1$, $p_3 = \infty$ and let the p_{-i} all converge to the same point μ, that is, $p_{-i} = \mu + \lambda_i t$ with $t \to 0$. Then

$$R_u = \frac{\mu - 1}{\mu}, \quad R_v = \frac{\lambda_1 - \lambda_3}{\lambda_2 - \lambda_3}$$

and in the limit as $t \to 0$

$$R_1/R_2 = \frac{\mu-1}{\mu}\frac{\lambda_1-\lambda_3}{\lambda_2-\lambda_3}, \quad R_3/R_2 = (1-\mu)\frac{\lambda_1-\lambda_3}{\lambda_1-\lambda_2}.$$

Thus the cross-ratio equation

$$(R_1/R_2)^{a_1} \cdot (R_3/R_2)^{a_3} = \zeta$$

for $\mathcal{S}_{(r_1,r_2,r_3)}$ becomes

$$(R_u R_v)^{a_1} \cdot \left(\frac{R_u}{1-R_u}\frac{R_v}{1-R_v}\right)^{a_3} = \zeta,$$

as we claimed.

The last statement in an immediate consequence of Lemma 8.7. ■

We give here the lemma needed above and as corollary a second version of calculating the exponents of the cross-ratio equation. Using the no-half-space condition, there are rational coefficients c_i such that

$$\frac{c_1}{r_1} + \frac{c_2}{r_2} + \frac{c_3}{r_3} = 0. \tag{8.10}$$

Lemma 8.9. *If the r_i and c_i are as in* (8.10), *then the triple* (c_1, c_2, c_3) *is proportional to* $(N(r_i)s_i/r_i)_{i=1}^3$.

Proof. Note that the triple $(N(r_i)s_i/r_i)_{i=1}^3$ is (up to a factor r_1/s_1) integral by rationality. It thus suffices to check that

$$\sum_{i=1}^3 \left(N(r_i)\frac{s_i}{r_i}\right) \cdot \frac{1}{r_i} = 0.$$

We have

$$
\begin{aligned}
\sum_{i=1}^{3}\left(N(r_i)\frac{s_i}{r_i^2}\right)\cdot\frac{r_1}{s_1} &= \sum_{i=1}^{3} r_i^{(2)} r_i^{(3)} \frac{s_i^{(1)}}{r_i^{(1)}}\frac{r_1^{(1)}}{s_1^{(1)}} \\
&= \sum_{i=1}^{3} r_i^{(2)} r_i^{(3)} \frac{s_i^{(2)}}{r_i^{(2)}}\frac{r_1^{(2)}}{s_1^{(2)}} \quad \text{(by rationality)} \\
&= \frac{r_1^{(2)}}{s_1^{(2)}}\sum_{i=1}^{3} s_i^{(2)} r_i^{(3)} \qquad (8.11)
\end{aligned}
$$

Consider the 3 by 3 matrices $R = (r_i^{(j)})$ and $S = (s_i^{(j)})$. Since the bases (r_i) and (s_i) are dual, we have $RS^t = I$. Thus $S^t R = I$ as well, and (8.11) is 0. ∎

Corollary 8.10. *The exponents a_i appearing in the cross-ratio equation* (8.3) *are the unique (up to sign) relatively prime integers with $a_i = tb'_i$ for some $t \in F$ and*

$$b'_i = c_i^2 / N_{\mathbb{Q}}^F(r_i).$$

Period coordinates. In preparation for the proof of Theorem 8.1, we now define local coordinates around certain Lagrangian boundary strata $\mathcal{S} \subset \overline{\mathcal{M}}_3(L)$ in terms of exponentials of entries of period matrices.

Let $\mathcal{S} \subset \overline{\mathcal{M}}_3(L)$ be a Lagrangian boundary stratum obtained by pinching curves $\gamma_1, \ldots, \gamma_m$ on Σ_3. We say that such a boundary stratum is *nice* if the complement of any two of the γ_i is connected. There are five topological types of nice boundary strata in $\overline{\mathcal{M}}_3(L)$, representing stable curves of type $(1,1)$, $(2,1)$, $(2,2)$, $(3,1)$ and $(4,2)$.

Let $\alpha_i \in L \subset H_1(\Sigma_3;\mathbb{Z})$ denote the homology class of γ_i after choosing an orientation.

Lemma 8.11. *If $\mathcal{S} \subset \overline{\mathcal{M}}_3(L)$ is a nice boundary stratum, then there are elements $\sigma_1, \ldots, \sigma_n \in \mathrm{Hom}(L,\mathbb{Z})$ such that*

$$\langle \sigma_i, \alpha_j \otimes \alpha_j \rangle = \delta_{ij}. \qquad (8.12)$$

Proof. We represent a curve in $\mathcal{S}$ by a directed graph G with the edges weighted by elements of L. A closed circuit c in G determines a functional $\beta_c \in \mathrm{Hom}(L,\mathbb{Z})$ defined as follows. If e is an edge with weight γ, then $\beta_c(\gamma) = n$, where n is the number of times c crosses e in the forward direction minus the number of times c crosses e in the reverse direction.

Each of the graphs in Figure 1 representing nice boundary strata has the property that for each edge e there are two circuits c and d which pass through e once and have no other edge in common. For each edge f, write $\rho(f) = w \otimes w$, where w is the weight of f. Then the functional $\beta_c \otimes \beta_d$ maps $\rho(e)$ to 1 and $\rho(f)$ for any other edge f to 0. ∎

Choose $\sigma_1, \ldots, \sigma_m \in \mathbf{S}(\mathrm{Hom}(L, \mathbb{Z}))$ as in the lemma, and choose a basis $\tau_1, \ldots, \tau_n$ of the annihilator $N(\mathcal{S}) \subset \mathbf{S}(\mathrm{Hom}(L, \mathbb{Z}))$ of $\{\alpha_i \otimes \alpha_i\}_{i=1}^m$.

Let $U \subset \overline{\mathcal{M}}_3(L)$ be the open subset consisting of $\mathcal{M}_3(L)$, $\mathcal{S}$, and any intermediate boundary stratum obtained by pinching some subset of the curves $\{\gamma_i\}$. We consider the map $\Xi\colon U \to \mathbb{C}^m \times (\mathbb{C}^*)^n$ defined by

$$\Xi = (\Psi(\sigma_1), \ldots, \Psi(\sigma_m), \Psi(\tau_1), \ldots, \Psi(\tau_n)),$$

sending $\mathcal{S}$ to $(0, \ldots, 0) \times (\mathbb{C}^*)^n$.

Any automorphism T of L induces an automorphism ϕ_T of $\overline{\mathcal{M}}_g(L)$ defined by replacing the marking ρ of the marked surface (X, ρ) with $\rho \circ T$. Let $\iota\colon L \to L$ be the negation homomorphism $\phi(\alpha) = -\alpha$. We define $\overline{\mathcal{M}}_g'(L)$ to be the quotient of $\overline{\mathcal{M}}_g(L)$ by the involution ϕ_ι.

Each of the meromorphic functions $\Psi(\alpha)$ on $\overline{\mathcal{M}}_g(L)$ is constant on orbits of ϕ_ι and so defines a meromorphic function $\Psi'(\alpha)$ on $\overline{\mathcal{M}}_g'(L)$. If $\mathcal{S}$ is fixed by ϕ_ι, then so is U, and the map Ξ then factors through to a map $\Xi'\colon U' \to \mathbb{C}^m \times (\mathbb{C}^*)^n$, where $U' = U/\phi_\iota$.

Lemma 8.12. *Consider a nice boundary stratum $\mathcal{S} \subset \overline{\mathcal{M}}_3(L)$. If $\mathcal{S}$ is not fixed by π_ι, then for any basis $(\tau_1, \ldots, \tau_n)$ of $N(\mathcal{S})$, the functions $\Psi(\tau_1), \ldots, \Psi(\tau_n)$ form a system of local coordinates on $\mathcal{S}$. If $\mathcal{S}$ is fixed by ϕ_ι, then for any basis $(\tau_1, \ldots, \tau_n)$ of $N(\mathcal{S})$, the functions $\Psi'(\tau_1), \ldots, \Psi'(\tau_n)$ form a system of local coordinates on $\mathcal{S}/\phi_\iota$.*

Proof. It is enough to produce a single basis of $N(\mathcal{S})$ which yields a system of local coordinates, since the coordinate systems defined by any two bases are related by an automorphism of the algebraic torus $(\mathbb{C}^*)^n$.

Any type [6] stratum $\mathcal{S}$ is fixed by ϕ_ι. Corollary 8.4 implies that the functions $\Psi'(s_i \otimes s_j)$ for $i \neq j$ identify $\mathcal{S}/\phi_\iota$ with an open subset of $(\mathbb{C}^*)^3$ (the involution ϕ_ω was called i in that Corollary), and so they give a system of local coordinates on $\mathcal{S}/\phi_\iota$.

Any type $[4] \times^4 [4]$ stratum is also fixed by ϕ_ι. We use the notation for these strata from p. 43. Under the identification of $\mathcal{S}$ with $\mathcal{M}_{0,4} \times \mathcal{M}_{0,4}$ the map ϕ_ι is just the involution exchanging the two factors.

Let $\{s_1, \ldots, s_3\}$ be a basis of F dual to $\{r_1, \ldots, r_3\}$. Let $\tau_1 = (s_2 - s_1) \otimes s_3$ and $\tau_2 = (s_3 - s_1) \otimes s_2$. From the definition of Ψ,

$$\Psi'(\tau_1) = R_u R_v \quad \text{and} \quad \Psi'(\tau_2) = (1 - R_u)(1 - R_v),$$

a system of local coordinates on $\mathcal{M}_{0,4} \times \mathcal{M}_{0,4}/\phi_\iota$.

The remaining cases are strata not fixed by ϕ_ι. We leave these simpler cases to the reader. ■

Proposition 8.13. *Consider a nice L-weighted boundary stratum $\mathcal{S}$ in $\overline{\mathcal{M}}_3(L)$. If $\mathcal{S}$ is not fixed by ϕ_ι, then the map Ξ is locally biholomorphic on a neighborhood of $\mathcal{S}$. Otherwise Ξ' is locally biholomorphic on a neighborhood of $\mathcal{S}/\phi_\iota$. In either case, the map Ξ is open.*

Proof. Suppose $\mathcal{S}$ is not fixed by the involution. Centered at an arbitrary point of $\mathcal{S}$, we choose plumbing coordinates $t_1, \ldots, t_m$, $s_1, \ldots, s_n$, as in §3, so that each divisor D_i where γ_i has been pinched is cut out by $t_i = 0$. We must show that the Jacobian of Ξ at $(\mathbf{0}, \mathbf{0})$ is nonzero. The functions $\Psi(\sigma_i)$ vanish to order one on D_i and zero on D_j for $j \neq i$. We then have $\frac{\partial \Psi(\sigma_i)}{\partial t_j}(\mathbf{0}, \mathbf{0}) = 0$ if $i \neq j$, $\frac{\partial \Psi(\sigma_i)}{\partial s_j}(\mathbf{0}, \mathbf{0}) = 0$ for all i and j, and $\frac{\partial \Psi(\sigma_i)}{\partial t_i}(\mathbf{0}, \mathbf{0}) \neq 0$ for all i. Thus, to show that the Jacobian of Ξ at $(\mathbf{0}, \mathbf{0})$ is nonzero, it suffices to show that the matrix $(\frac{\partial \Psi(\tau_i)}{\partial s_j}(\mathbf{0}, \mathbf{0}))$ is invertible. In other words, we must show that the functions $\Psi(s_j)$ locally define a system of local coordinates on $\mathcal{S}$. This is the content of Lemma 8.12.

The case where $\mathcal{S}$ is fixed is nearly identical. Note that since the quotient mapping $\overline{\mathcal{M}}_3(L) \to \overline{\mathcal{M}}'_3(L)$ is unbranched along the boundary divisors, the order of vanishing of any $\Psi'(a)$ along D_i is also given by the formula of Theorem 4.1.

The last statement follows, since any quotient map – in particular, the canonical map $\mathcal{M}_3(L) \to \mathcal{M}'_3(L)$ – is open. ■

Closures of algebraic tori. The period coordinates above reduce the problem of computing the boundary of the eigenform locus to computing the closures of algebraic tori $T \subset (\mathbb{C}^*)^n \subset \mathbb{C}^n$, which we now consider.

Consider the algebraic torus $T = (\mathbb{C}^*)^k \times (\mathbb{C}^*)^\ell \subset \mathbb{C}^k \times (\mathbb{C}^*)^\ell$. We identify the character group $\chi(T)$ with $\mathbb{Z}^k \oplus \mathbb{Z}^\ell$ by assigning to $(\boldsymbol{a}, \boldsymbol{b}) = (a_1, \ldots, a_k, b_1, \ldots, b_\ell)$ the character $\lambda_{(\boldsymbol{a},\boldsymbol{b})} \colon T \to \mathbb{C}^*$ defined by

$$\lambda_{(\boldsymbol{a},\boldsymbol{b})}(\boldsymbol{z}, \boldsymbol{w}) = z_1^{a_1} \cdots z_k^{a_k} w_1^{b_1} \cdots w_\ell^{b_\ell}$$

Given a subgroup L of $\chi(T)$ with $\chi(T)/L$ torsion-free and a homomorphism $\phi \colon L \to \mathbb{C}^*$, we define $T_{L,\phi}$ to be the subvariety of T cut out by the monomial equations

$$\lambda_{(\boldsymbol{a},\boldsymbol{b})}(\boldsymbol{z}, \boldsymbol{w}) = \phi(\boldsymbol{a}, \boldsymbol{b}) \tag{8.13}$$

for each $(\boldsymbol{a}, \boldsymbol{b}) \in L$, a translate of a subtorus of T.

Let $\Delta = \{\mathbf{0}\} \times (\mathbb{C}^*)^\ell$. We define

$$C = \{(\boldsymbol{a}, \boldsymbol{b}) \in \chi(T) : a_i \geq 0 \text{ for } 1 \leq i \leq k\}, \quad \text{and}$$
$$N = \{\mathbf{0}\} \oplus \mathbb{Z}^\ell \subset \chi(T).$$

Let $\Delta_{L,\phi}$ be the subvariety of Δ cut out by the monomial equations (8.13) for $(\boldsymbol{a}, \boldsymbol{b}) \in L \cap N$.

Theorem 8.14. *The closure $\overline{T}_{L,\phi} \cap \Delta$ is nonempty if and only if $L \cap C \subset N$, in which case we have $\overline{T}_{L,\phi} \cap \Delta = \Delta_{L,\phi}$.*

Proof. Suppose $(\boldsymbol{a}, \boldsymbol{b})$ is a nonzero element of $(L \cap C) \setminus N$. The equation (8.13) is then satisfied on $T_{L,\phi}$, but $\lambda_{(\boldsymbol{a},\boldsymbol{b})}(\boldsymbol{z}, \boldsymbol{w}) \equiv 0$ on Δ, so Δ and $\overline{T}_{L,\phi}$ must be disjoint.

Conversely, suppose $L \cap C \subset N$. Then the orthogonal projection $p(L)$ of L onto the $\mathbb{Z}^k$ factor of $\chi(T)$ satisfies $p(L) \cap C = 0$. Theorem 15.7 of [Rom92] states that given a subspace V of $\mathbb{R}^n$ with $V \cap \{\boldsymbol{x} \in \mathbb{R}^n : x_i \geq 0 \text{ for all } i\} = 0$, there is a vector $\boldsymbol{y} \in V^\perp$ with each coordinate positive. Thus we may find an integral $\boldsymbol{c} \in p(L)^\perp \subset \mathbb{Z}^k$ with positive coordinates.

Note that the curve parameterized by

$$f(w) = (d_1 w^{c_1}, \ldots, d_k w^{c_k}, e_1, \ldots, e_\ell)$$

lies in $T_{L,\phi}$ if and only if for each $(\boldsymbol{a}, \boldsymbol{b}) \in L$, the equation

$$d_1^{a_1} \ldots d_k^{a_k} e_1^{b_1} \ldots e_\ell^{b_\ell} = \phi(\boldsymbol{a}, \boldsymbol{b}) \tag{8.14}$$

is satisfied, in which case $(0, \ldots, 0, e_1, \ldots, e_\ell) \in \overline{T}_{L,\phi} \cap \Delta$.

Choose some $(\boldsymbol{0}, \boldsymbol{e}) \in \Delta_{L,\phi}$, and let $(\boldsymbol{a}_i, \boldsymbol{b}_i) = (a_{i1}, \ldots . a_{ik}, b_{i1}, \ldots, b_{i\ell})$ for $1 \leq i \leq \dim(L)$ be a basis of L with $a_{ij} = 0$ for $i \leq \dim(L \cap N)$. We must find $g_1, \ldots, g_k$ satisfying the equations

$$a_{i1} g_1 + \cdots + a_{ik} g_k + b_{i1} \log e_1 + \cdots + b_{i\ell} \log e_\ell = \log \phi(\boldsymbol{a}_i, \boldsymbol{b}_i). \tag{8.15}$$

The first $\dim(L \cap N)$ equations don't involve the g_i and are satisfied automatically because $(\boldsymbol{0}, \boldsymbol{e}) \in \Delta_{L,\phi}$ as long as the values of log were chosen correctly. The vectors $\boldsymbol{a}_{\dim(L \cap N)+1}, \ldots, \boldsymbol{a}_{\dim(L)}$ are linearly independent, so the matrix (a_{ij}) (with $\dim(L \cap N) < i \leq \dim(L)$ and $1 \leq j \leq k$) has maximal rank. Thus we can solve (8.15) for the g_i. Setting $d_i = e^{g_i}$, (8.14) is satisfied. ■

Proof of Theorem 8.1. It suffices to show that for any cusp packet $(\mathcal{I}, E)$ and admissible $\mathcal{I}$-weighted boundary stratum $\mathcal{S} \subset \overline{\mathcal{M}}_3(\mathcal{I})$ the variety $\mathcal{S}(E)$ lies in the closure of $\mathcal{RM}_{\mathcal{O}}(\mathcal{I}, E)$.

For nice boundary strata, the map Ξ of Proposition 8.13 reduces the computation of the closure of $\mathcal{RM}_{\mathcal{O}}$ to the computation of the closure of an algebraic torus in $\mathbb{C}^n$ (since under an open mapping, the inverse image of the closure of a set is equal to the closure of the inverse image), which is done in Theorem 8.14. It is easily checked that the condition of this theorem is equivalent to the admissibility condition. This handles admissible boundary strata of type $(1,1), (2,1), (2,2), (3,1)$, and $(4,2)$ in Figure 1.

Admissible boundary strata which are in the boundary of a nice admissible boundary stratum $\mathcal{S}$ with $\operatorname{codim}(\mathcal{S}) = 0$ are then automatically in the closure of $\mathcal{RM}_{\mathcal{O}}(\mathcal{I}, E)$. It follows from Lemma 8.8 that any admissible boundary stratum $\mathcal{S}$ with $\operatorname{codim}(\mathcal{S}) = 0$ is in the boundary of such a nice admissible stratum, since some collection of nodes can be unpinched to obtain a stratum of type $[4] \times^4 [4]$ or $[5] \times^3 [3]$ where the cone condition still holds. This handles admissible boundary strata of type $(3,2), (3,3), (4,1)$, and $(4,3)$.

It remains to consider admissible boundary strata of type $(2,3)$, $(2,4)$, $(3,4)$, $(3,5)$, $(4,4)$, and $(4,5)$. Any such boundary stratum in the closure of a unique irreducible Lagrangian boundary stratum $\mathcal{S}$. The weights of $\mathcal{S}$ define the equation

$$\Psi(\sigma) = u, \tag{8.16}$$

with u and σ as in (8.5) and (8.6). Let $V \subset \overline{\mathcal{M}}_3(\mathcal{I})$ be the subvariety cut out by this equation. For any stratum $\mathcal{S}' \subset \overline{\mathcal{S}}$, we have $\mathcal{S}'(E) = \mathcal{S}' \cap V$ by the definition of $\mathcal{S}'(E)$, so we must show for any such $\mathcal{S}'$ that $\mathcal{S}' \cap V \subset \overline{\mathcal{RM}}_{\mathcal{O}}(\mathcal{I}, E)$. Since we have already handled irreducible boundary strata, we know that $V \cap \mathcal{S} = \mathcal{RM}_{\mathcal{O}}(\mathcal{I}, E) \cap \mathcal{S}$. It follows that $\overline{V \cap \mathcal{S}} \subset \overline{\mathcal{RM}}_{\mathcal{O}}(\mathcal{I}, E)$. If $\overline{\mathcal{S}} \cap V$ were irreducible, it would follow that $\overline{\mathcal{S}} \cap V = \overline{\mathcal{S} \cap V}$, and we would be done.

We see the irreducibility of $\overline{\mathcal{S}} \cap V$ as follows. Since V is codimension-one and $\mathcal{S} \cap V$ is irreducible, as is easily seen from the form of the cross-ratio equation (8.3), $\overline{\mathcal{S}} \cap V$ could only fail to be irreducible if a two-dimensional stratum in the boundary of $\mathcal{S}$ were contained in V. Such a stratum must be of type $(2, 3)$ (that is, $[4] \times^2 [4]$) or $(2, 4)$ in Figure 1. The restriction of the equation (8.16) to a type $[4] \times^2 [4]$ stratum is the cross-ratio equation (8.7) which is not satisfied on an entire stratum. Similarly, a type $(2, 4)$ stratum is isomorphic to $\mathcal{M}_{0,5}$, and the equation (8.16) reduces to the equation $R = u$, where R is a cross-ratio of four marked points and u is a root of unity. This equation is not satisfied on the entire stratum. ■

9 Existence of an admissible basis

In this section we construct, for any totally real cubic number field F with ring of integers $\mathcal{O}_F$, an $\mathcal{O}_F$-ideal with an admissible basis. This will be used in the Section 11 to show $\mathrm{GL}_2^+(\mathbb{R})$-noninvariance of eigenform loci.

Lemma 9.1. *For any cubic number field F, there is some fractional $\mathcal{O}_F$-ideal $\mathcal{I}$ with basis $\{1, \alpha, \alpha^2\}$.*

Proof. Given $\alpha \in F \setminus \mathbb{Q}$, let $\mathcal{I}_\alpha \subset F$ be the lattice $\langle 1, \alpha, \alpha^2 \rangle$. If $aX^3 + bX^2 + cX + d \in \mathbb{Z}[X]$ is the minimal polynomial of α, one checks that

$$R = \langle 1, a\alpha, a\alpha^2 + b\alpha \rangle \quad \text{satisfies} \quad R \cdot \mathcal{I}_\alpha \subset \mathcal{I}_\alpha.$$

We must arrange that $R = \mathcal{O}_F$. Let $\{1, \mu, \nu\}$ be a basis of $\mathcal{O}_F$. Associated to this basis is the index form, an integral binary cubic form which is defined by

$$C(x, y)^2 = \operatorname{disc}(x\nu - y\mu) / \operatorname{disc}(F)$$

for $x, y \in \mathbb{Q}$ (see [Coh00, Proposition 8.2.1]), where $\operatorname{disc}(\alpha)$ is the discriminant of the lattice $\mathcal{I}_\alpha$. If we choose α to be a root of $C(x, 1)$, then $R = \mathcal{O}_F$ by [Coh00, Proposition 8.2.3]. ■

Proposition 9.2. *Given a totally real cubic field F, there is an $\mathcal{O}_F$-ideal $\mathcal{I}$ with an admissible basis.*

Proof. Let $\mathcal{I}$ be a fractional ideal with basis $\{1, \alpha, \alpha^2\}$ which is provided by Lemma 9.1. The basis given by $r_1 = \alpha$, $r_2 = (1 - \alpha)$, and $r_3 = \alpha(\alpha - 1)$ satisfies the equation

$$\frac{1}{N^F_{\mathbb{Q}}(r_1)} \frac{N^F_{\mathbb{Q}}(r_1)}{r_1} + \frac{1}{N^F_{\mathbb{Q}}(r_2)} \frac{N^F_{\mathbb{Q}}(r_2)}{r_2} + \frac{1}{N^F_{\mathbb{Q}}(r_3)} \frac{N^F_{\mathbb{Q}}(r_3)}{r_3} = 0 \qquad (9.1)$$

so

$$\dim \operatorname{Span}\{N_{\mathbb{Q}}^F(r_1)/r_1, N_{\mathbb{Q}}^F(r_2)/r_2, N_{\mathbb{Q}}^F(r_3)/r_3\} = 2.$$

The no-half-space condition is then equivalent to the coefficients of (9.1) having the same sign, that is $N_{\mathbb{Q}}^F(\alpha) < 0$, and $N_{\mathbb{Q}}^F(1-\alpha) < 0$. We are free to replace α with $\alpha' = \alpha - k$ for any $k \in \mathbb{Z}$, since the basis $\{1, \alpha', \alpha'^2\}$ spans the same lattice. Thus the problem is reduced to finding $k \in \mathbb{Z}$ such that $N_{\mathbb{Q}}^F(\alpha+k)$ and $N_{\mathbb{Q}}^F(\alpha+k+1)$ have opposite signs.

Define $P(k) = N_{\mathbb{Q}}^F(\alpha+k)$. Then $P(k) = -C(k)$, where C is the monic minimal polynomial of α. We claim that there are consecutive integers at which P has opposite signs. In fact, this holds for any polynomial P of odd degree with no integral roots, for if P had the same sign at any two consecutive integers, then it must have the same sign at all integers. This is impossible, as the sign of $P(x)$ as $x \to \infty$ is the opposite of $P(x)$ as $x \to -\infty$. ■

Example 9.3. Consider the field $F = \mathbb{Q}[x]/\langle x^3 - x^2 - 10x + 8\rangle$ of discriminant $D = 961$. Its ring of integers $\mathcal{O}_F = \langle 1, x, (x^2+x)/2\rangle$ is not monogenic, i.e. does not have a basis of the form $\{1, \theta, \theta^2\}$ for any θ in F. The class number of $\mathcal{O}_F$ is one, so the above algorithm provides a basis of this form spanning some fractional ideal similar to $\mathcal{O}_F$.

One calculates the index form to be $C(X,1) = 2X^3 - X^2 - 5X + 2$, thus if θ is a root of this polynomial, then $\mathcal{O}_F = \langle 1, 2\theta, 2\theta^2 - \theta\rangle$ and $\mathcal{I} = \langle 1, \theta, \theta^2\rangle$. Here $N(\alpha) = -1$ and $N_{\mathbb{Q}}^F(1-\alpha) = -1$, so the last step of the proof is unnecessary.

Corollary 9.4. *For any field F the closure of the eigenform locus $\mathcal{E}_{\mathcal{O}_F}$ intersects a boundary stratum of type* [6]*, that is, a stratum of trinodal curves.*

We do not know if the class of the ideal class of $\mathcal{I}$ given by Lemma 9.1 always is the class of $\mathcal{O}_F$. Nor do we know if there is always an admissible basis of $\mathcal{O}_F$. Computer experiments using the algorithm described in Appendix A suggest an affirmative answer. This algorithm also produces examples of ideal classes with no such bases.

10 Teichmüller curves and the $\mathrm{GL}_2^+(\mathbb{R})$ action

In preparation for the next sections, we recall the well-known action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\Omega\mathcal{M}_g$ and the basic properties of Teichmüller curves in $\mathcal{M}_g$.

Translation surfaces. A Riemann surface X equipped with a nonzero holomorphic one-form ω is otherwise known as a *translation surface*. The form ω defines a metric $|\omega|$ on $X \setminus Z(\omega)$, where $Z(\omega)$ is the set of zeros of ω, assigning to a vector v the length $|\omega(v)|$. The metric $|\omega|$ has cone singularities at the zeros of ω.

The form ω defines an atlas of charts $\{\phi_\alpha \colon U_\alpha \to \mathbb{C}\}$ covering $X \setminus Z(\omega)$, where $\phi_\alpha(z) = \int_p^z \omega$ for some choice of basepoint $p \in U_\alpha$. The transition functions of this atlas are translations of $\mathbb{C}$, and the form ω is recovered by $\omega|_{U_\alpha} = \phi_\alpha^{-1}(dz)$.

Any translation-invariant geometric structure on $\mathbb{C}$ can then be pulled back to X via this atlas. In particular, for any slope $\theta \in \mathbb{R} \cup \{\infty\}$ there is a foliation $\mathcal{F}_\theta$ of X by geodesics of slope θ.

$\mathrm{GL}_2^+(\mathbb{R})$ **action.** We can now regard $\Omega\mathcal{M}_g$ as the moduli space of genus g translation surfaces. $\mathrm{GL}_2^+(\mathbb{R})$ acts on $\Omega\mathcal{M}_g$ as follows. We identify $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way so that a matrix $A \in \mathrm{GL}_2^+(\mathbb{R})$ determines a $\mathbb{R}$-linear automorphism of $\mathbb{C}$. Replacing the atlas of charts $\{\phi_\alpha : U_\alpha \to \mathbb{C}\}$ defined above with $\{A \circ \phi_\alpha : U_\alpha \to \mathbb{C}\}$ yields a new atlas with transition functions also translations of $\mathbb{C}$. Pulling back the complex structure of $\mathbb{C}$ and the one-form dz via this atlas defines a new translation surface $A \cdot (X, \omega)$.

Strata. Given a partition $n_1, \ldots, n_r$ of $2g-2$, there is the stratum

$$\Omega\mathcal{M}_g(n_1, \ldots, n_r) \subset \Omega\mathcal{M}_g$$

of forms with exactly r zeros of orders given by the n_i. This stratification is preserved by the $\mathrm{GL}_2(\mathbb{R})$-action.

Veech surfaces and Teichmüller curves. We define the affine automorphism group of a translation surface (X, ω) to be the group $\mathrm{Aff}^+(X, \omega)$ of orientation preserving, locally affine homeomorphisms of (X, ω). There is a homeomorphism

$$D \colon \mathrm{Aff}^+(X, \omega) \to \mathrm{SL}_2(\mathbb{R}),$$

sending a map A to its derivative DA in a local translation chart. We define $\mathrm{SL}(X, \omega) = D(\mathrm{Aff}^+(X, \omega)) \subset \mathrm{SL}_2(\mathbb{R})$. The group $\mathrm{SL}(X, \omega)$ is known as the *Veech group* of (X, ω).

The surface (X, ω) is said to be *Veech* if $\mathrm{SL}(X, \omega)$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$. The group $\mathrm{SL}(X, \omega)$ coincides with the stabilizer of (X, ω) under the $\mathrm{GL}_2^+(\mathbb{R})$-action. Thus (X, ω) is Veech if and only if $\mathrm{GL}_2^+(\mathbb{R}) \cdot (X, \omega) \subset \Omega\mathcal{M}_g$ descends to an immersed finite volume Riemann surface (orbifold) in $\mathcal{M}_g$. An immersed finite volume Riemann surface arising in this way is called a *Teichmüller curve* and is necessarily isometrically immersed with respect to the Teichmüller metric.

A Teichmüller curve can also be regarded as an embedded smooth curve in $\mathbb{P}\Omega\mathcal{M}_g$.

Periodicity. A *saddle connection* on a translation surface (X, ω) is an embedded geodesic segment connecting two zeros of ω.

The foliation $\mathcal{F}_\theta$ of slope θ is said to be *periodic* if every leaf of $\mathcal{F}_\theta$ is either closed (i.e. a circle) or a saddle connection. In this case, we call θ a periodic direction. A periodic direction θ yields a decomposition of (X, ω) into finitely many maximal cylinders foliated by closed geodesics of slope θ. The complement of these cylinders is a finite collection of saddle connections.

Veech proved the following strong periodicity property of Veech surfaces.

Theorem 10.1 ([Vee89])**.** *Suppose (X, ω) is a Veech surface with either a closed geodesic or a saddle connection of slope θ. Then the foliation $\mathcal{F}_\theta$ is periodic and the moduli of the cylinders in the direction θ are commensurable.*

Given a Veech surface (X, ω) generating a Teichmüller curve $C \subset \mathbb{P}\Omega\mathcal{M}_g$, there is a natural bijection between the cusps of C and the periodic directions on (X, ω), up to the action of $\mathrm{SL}(X, \omega)$. The cusp associated to a periodic direction θ is the limit of the geodesic $A_t R \cdot (X, \omega)$, where $R \subset \mathrm{SO}_2(\mathbb{R})$ is a rotation which makes θ horizontal, and

$$A_t = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix}.$$

The stable form in $\mathbb{P}\Omega\overline{\mathcal{M}}_g$ which is the limit of this cusp is obtained by cutting each cylinder of slope θ along a closed geodesic and gluing a half-infinite cylinder to each resulting boundary component (see [Mas75]). These infinite cylinders are the poles of the resulting stable form, and the two poles resulting from a single infinite cylinder are glued to form a node.

A periodic direction θ of a Veech surface (X, ω) generating a Teichmüller curve C is *irreducible* if the complement of the cylinders of $\mathcal{F}_\theta$ is a connected union of saddle connections. Equivalently, a periodic direction is irreducible if the stable curve at the cusp corresponding to the limit of geodesic in that direction is irreducible. An irreducible periodic direction always has g cylinders, where g is the genus of X. We also say, for short, that a *cusp is irreducible* if the stable curve it parameterizes is irreducible.

Lemma 10.2. *Every Veech surface (X, ω) having at most two zeros has an irreducible periodic direction.*

Proof. If (X, ω) has only a single zero, then every periodic direction is irreducible.

If (X, ω) has two zeros, take a saddle connection I joining them. Such a saddle connection can be obtained by straightening any path joining the two zeros to a geodesic path. The direction determined by I is periodic by Theorem 10.1, and this direction is irreducible as the graph of saddle connections is connected. ∎

Algebraic primitivity. The trace field of a Veech surface (X, ω) is the field $\mathbb{Q}(\mathrm{Tr}\, A : A \in \mathrm{SL}(X, \omega))$. The trace field of (X, ω) is a number field which is totally real (see [Möl06b] or [HL06]) whose degree is at most the genus of X (see [McM03]). A Veech surface (X, ω) is said to be *algebraically primitive* if the degree of its trace field is equal to the genus of X.

Our finiteness theorem for algebraically primitive Teichmüller curves will require the following facts.

Theorem 10.3 ([Möl06b, Möl06a])**.** *Suppose (X, ω) is an algebraically primitive Veech surface. We then have*

- $\mathrm{GL}_2^+(\mathbb{R}) \cdot (X,\omega)$ *lies in the locus of eigenforms for real multiplication by the trace field of* (X,ω).
- *For any two distinct zeros* p *and* q *of* ω *the divisor* $p-q$*, regarded as a point in* $\mathrm{Jac}(X)$*, is torsion.*

The following lemma shows that the heights of cylinders in an irreducible periodic direction of an algebraically primitive Veech surface can be recovered from knowledge of their widths.

Lemma 10.4. *Suppose* $(X,\omega) \in \Omega\mathcal{M}_g$ *is an eigenform for real multiplication by a totally real field* F *of degree* g*, and suppose the horizontal direction of* (X,ω) *is periodic and irreducible. Then the vector* $(r_i)_{i=1}^g$ *of widths of the* g *horizontal cylinders is a real multiple of a basis of* F *over* $\mathbb{Q}$*, and the corresponding vector* $(s_i)_{i=1}^g$ *of heights of these cylinders is a real multiple of the dual basis of* F *over* $\mathbb{Q}$ *with respect to the trace pairing.*

Proof. Let $M \subset H_1(X;\mathbb{Q})$ be the g-dimensional subspace generated by the core curves of the horizontal cylinders, and let $N = H_1(X;\mathbb{Q})/M$. Real multiplication gives both M and N the structure of one-dimensional F-vector spaces, so we may choose isomorphisms of F-vector spaces $\phi\colon M \to F$ and $\psi\colon N \to F$. Since ω is an eigenform, there are constants $c, d \in \mathbb{R}$ and an embedding $\iota\colon F \to \mathbb{R}$ such that

$$\int_\alpha \omega = c\iota(\phi(\alpha)) \quad \text{and} \quad \mathrm{Im}\int_\beta \omega = d\iota(\psi(\beta)) \tag{10.1}$$

for all $\alpha \in M$ and $\beta \in N$.

The intersection pairing between M and N yields a perfect pairing $\langle\,,\,\rangle\colon F \times F \to \mathbb{Q}$ which is compatible with the action of F in the sense that $\langle \lambda x, y\rangle = \langle x, \lambda y\rangle$ for all $\lambda \in F$. A second such pairing is given by $(x,y) = \mathrm{Tr}(xy)$. Since the space of all such perfect pairings is a one-dimensional F-vector space, there is a $\lambda \in F$ such that

$$\langle x, \lambda y\rangle = \mathrm{Tr}(xy) \tag{10.2}$$

for all $x, y \in F$.

Let $\alpha_i \in M$ be the class of a core curve of the ith horizontal cylinder C_i, let $r_i = \phi(\alpha_i)$, and let the s_i be the dual basis of F to the r_i. Choose $\beta_i \in H_1(X;\mathbb{Q})$ such that $\beta_i \equiv \psi^{-1}(\lambda s_i) \pmod{M}$. By (10.2), the β_i are dual to the α_i with respect to the intersection pairing. It follows that β_i crosses C_i once and no other cylinder, so the height of C_i is $\mathrm{Im}\int_{\beta_i}\omega$. By (10.1), we have

$$\int_{\alpha_i} \omega = c\iota(r_i) \quad \text{and} \quad \mathrm{Im}\int_{\beta_i} \omega = d\iota(\lambda)\iota(s_i). \qquad \blacksquare$$

As a consequence of this lemma we see that for an algebraically primitive Teichmüller curve and an irreducible cusp, the well-known commensurability of moduli is equivalent to rationality. With this in mind, one may restate Theorem 5.2 and Theorem 8.1 by saying that even without the presence of Teichmüller curves the generalization 'commensurable moduli' is a necessary (and for $g = 3$ together with positivity also sufficient) condition for irreducible cusps to lie in the eigenform locus.

11 $\mathrm{GL}_2^+(\mathbb{R})$ non-invariance

In this section we show that the $\mathrm{GL}_2^+(\mathbb{R})$ action on $\Omega\mathcal{M}_g$ admits a continuous extension to the Deligne-Mumford compactification. We deduce from this and the previous sections that the eigenform locus for real multiplication by the ring of integers in any totally real cubic field is *not* invariant under the action of $\mathrm{GL}_2^+(\mathbb{R})$. McMullen proved non-invariance in [McM03] for the maximal order in $\mathbb{Q}(\cos(2\pi/7))$ using the existence of a curve with a special automorphism group.

$\mathrm{GL}_2^+(\mathbb{R})$-action on $\Omega\overline{\mathcal{M}}_g$. The definition of the $\mathrm{GL}_2^+(\mathbb{R})$ action on Abelian differentials works just as well for stable Abelian differentials (X, ω), regarding ω as a holomorphic one-form on the punctured Riemann surface X. The opposite-residue condition is preserved by linearity of the $\mathrm{GL}_2^+(\mathbb{R})$-action on $\mathbb{R}^2$: If α and $-\alpha$ are simple loops around a pair of opposite nodes p and q then

$$A \cdot \mathrm{res}_p(\omega) = A \cdot \int_\alpha \omega = -A \cdot \int_{-\alpha} \omega - A \cdot \mathrm{res}_q(\omega).$$

Thus we obtain an action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\Omega\overline{\mathcal{M}}_g$ and $\Omega\overline{\mathcal{T}}(\Sigma_g)$.

Proposition 11.1. *The action of* $\mathrm{GL}_2^+(\mathbb{R})$ *on* $\Omega\overline{\mathcal{M}}_g$ *is continuous.*

Proof. We show that the action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\Omega\overline{\mathcal{T}}(\Sigma_g)$ is continuous. As the $\mathrm{GL}_2^+(\mathbb{R})$-action on $\Omega\overline{\mathcal{T}}$ commutes with the action by the mapping class group, this action then descends to a continuous action on $\Omega\overline{\mathcal{M}}_g$.

We claim that under the action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\Omega\overline{\mathcal{T}}(\Sigma_g)$ the hyperbolic lengths of simple closed curves vary continuously. Since the topology of $\overline{\mathcal{T}}(\Sigma_g)$ is the smallest topology such that hyperbolic lengths of simple closed curves are continuous functions $\ell_\gamma \colon \overline{\mathcal{T}}(\Sigma_g) \to \mathbb{R}_+ \cup \{\infty\}$, it follows that under this action, the underlying Riemann surfaces are varying continuously.

That the length of a simple closed curve γ varies continuously follows easily from considering the annular covering of X corresponding to $\langle\gamma\rangle \subset \pi_1(X)$. The modulus of this annulus varies continuously under quasiconformal deformation, and the length of γ is determined by this modulus (see for example [DH93, Proposition 7.2]).

Consider a form $([f \colon \Sigma_g \to X], \omega) \in \Omega\overline{\mathcal{T}}(\Sigma_g)$. Say the collapse f pinches a set of curves S on Σ_g. We may choose a set of curves $\alpha_1, \dots, \alpha_g$ on Σ_g that generate a Lagrangian subspace of $H_1(\Sigma_g, \mathbb{Z})$ and such that each of the α_i is either one of the curves in S or intersects each curve in S trivially. We obtain a trivialization of the bundle $\Omega\overline{\mathcal{T}}(\Sigma_g)$ over a neighborhood of X sending a form η to $(\eta(\alpha_1), \dots, \eta(\alpha_g)) \in \mathbb{C}^g$.

Say $A \cdot (Y, \eta) = (Z, \zeta)$. From the definition of the $\mathrm{GL}_2^+(\mathbb{R})$ action, we have

$$\zeta(\alpha_i) = A \cdot \eta(\alpha_i),$$

with $A \in \mathrm{GL}_2^+(\mathbb{R})$ acting on $\mathbb{C} \cong \mathbb{R}^2$ in the usual way. Thus $\eta(\alpha_i)$ varies continuously under the $\mathrm{GL}_2^+(\mathbb{R})$-action, and so the action on $\Omega\overline{\mathcal{T}}(\Sigma_g)$ is continuous. ■

Four-punctured spheres. Given $r_1, r_2 \in \mathbb{C}$, we let $\mathcal{R}_{(r_1,r_2)} \cong \mathcal{M}_{0,4}$ be the moduli space of pairs (X, ω), where X is the four-punctured sphere $\mathbb{P}^1 \setminus \{p_1, p_{-1}, p_2, p_{-2}\}$. and ω is the unique meromorphic one-form with simple poles at the p_i with residue $r_{\pm i}$ at $p_{\pm i}$. We identify $\mathcal{R}_{(r_1,r_2)}$ with $\mathbb{C} \setminus \{0, 1\}$ via the cross-ratio $R = [p_1, p_{-1}, p_{-2}, p_2]$ and write (X_R, ω_R) for the form associated to the cross-ratio R.

If $r_1, r_2 \in \mathbb{R}$, then the subgroup $P \subset \mathrm{GL}_2^+(\mathbb{R})$ of matrices fixing the vector $(1, 0)$ acts on $\mathcal{R}_{(r_1,r_2)}$, as this is the subgroup of $\mathrm{GL}_2^+(\mathbb{R})$ preserving the residues r_i.

Proposition 11.2. *Suppose $r_1, r_2 \in \mathbb{R}$, and $r_1 \neq \pm r_2$. We then have*

- *The horizontal foliation of each $(X_R, \omega_R) \in \mathcal{R}_{(r_1,r_2)}$ is periodic. Each (X_R, ω_R) has either two or three cylinders (counting the two half-infinite cylinders of width r_i as a single cylinder).*
- *The form ω_R has a double zero for the single value of R,*

$$R = \left(\frac{r_1 - r_2}{r_1 + r_2}\right)^2. \tag{11.1}$$

- *We define $\mathrm{Spine}_{(r_1,r_2)} \subset \mathcal{R}_{(r_1,r_2)}$ to be the locus of two-cylinder forms. $\mathrm{Spine}_{(r_1,r_2)}$ is the locus of singular leaves of a quadratic differential on $\mathcal{R}_{(r_1,r_2)}$. $\mathrm{Spine}_{(r_1,r_2)}$ is homeomorphic to a figure-9, with the three pronged singularity at the unique form (X_R, ω_R) with a double zero. The one-pronged singularity is at $R = 1$, the point in the boundary of $\mathcal{R}_{(r_1,r_2)}$ obtained by pinching the curve separating $p_{\pm 1}$ from $p_{\pm 2}$.*
- Spine_{r_1,r_2} *is the locus of points fixed by the action of P on $\mathcal{R}_{(r_1,r_2)}$.*

Proof. See [Bai10, Proposition 7.3] for the first statement, [Bai07, Proposition 6.10] for the second statement, and [Bai10, Proposition 7.4] for the third statement.

For the final statement, suppose $(X, \omega) \in \mathcal{R}_{(r_1,r_2)}$ is a three-cylinder surface. Then there is a single finite horizontal cylinder $C \subset X$ with a simple zero of ω on the top and bottom boundaries of C. The period $\int_\gamma \omega$ along a curve joining these two zeros has nonzero imaginary part, so it is not fixed by any matrix in P. Thus P does not fix ω.

If $(X, \omega) \in \mathrm{Spine}_{(r_1,r_2)}$, then (X, ω) is obtained by gluing four half-infinite cylinders to graph (the spine of (X, ω). There is an affine automorphism of (X, ω) with derivative P which is the identity on the spine. Thus (X, ω) is stabilized by the action of P. ∎

$\mathrm{GL}_2^+(\mathbb{R})$ **non-invariance.** We write $\Omega\mathcal{E}_{\mathcal{O}}^{\iota} \subset \Omega\mathcal{M}_g$ for the locus of ι-eigenforms (as opposed to its projectivization $\mathcal{E}_{\mathcal{O}}^{\iota}$).

Theorem 11.3. *Let $\mathcal{O}$ be a totally real cubic order and $X \subset \Omega\mathcal{M}_3$ an irreducible component of $\Omega\mathcal{E}_{\mathcal{O}}^{\iota}$. If $\overline{X} \subset \Omega\overline{\mathcal{M}}_3$ has nontrivial intersection with a boundary stratum of type* [6]*, then X is not invariant under the action of* $\mathrm{GL}_2^+(\mathbb{R})$.

Proof. Suppose $\overline{X}$ meets the locus $\mathcal{R}_{(r_1,r_2,r_3)}$ of irreducible stable forms with poles of residues $(\pm r_1, \pm r_2, \pm r_3)$, with (r_1, r_2, r_3) an admissible basis of $\iota(F)$. In the boundary of $\mathcal{R}_{(r_1,r_2,r_3)}$ is a stratum $\mathcal{R}'$ of type $[4] \times^2 [4]$ parameterizing forms with two nodes of residue $\pm r_2$, one of residue $\pm r_1$, and one of residue $\pm r_3$. We identify $\mathcal{R}'$ with $\mathcal{R}_{(r_1,r_2)} \times \mathcal{R}_{(r_3,r_2)} \cong \mathcal{M}_{0,4} \times \mathcal{M}_{0,4}$, with cross-ratio coordinates R_1 on $\mathcal{R}_{(r_1,r_2)}$ and R_3 on $\mathcal{R}_{(r_3,r_2)}$ as in the previous paragraph.

By Theorems 8.5 and 8.1, $\overline{X} \cap (\mathcal{R}_{(r_1,r_2)} \times \mathcal{R}_{(r_3,r_2)})$ contains an irreducible component V cut out by the equation

$$R_1^{a_1} R_3^{a_3} = \zeta \tag{11.2}$$

for some root of unity ζ. We suppose X is $\mathrm{GL}_2^+(\mathbb{R})$-invariant, in which case V is invariant under $P \subset \mathrm{GL}_2^+(\mathbb{R})$ by Proposition 11.1.

We define $i, \psi_j \colon \mathbb{C} \to \mathbb{C}$ by $\psi_j(z) = z^{a_i}$, and $i(z) = \zeta/z$. Since the spine in $\mathcal{R}_{(r_i,r_2)}$ is the locus fixed by the action of $P \subset \mathrm{GL}_2^+(\mathbb{R})$ by Proposition 11.2, if V is preserved by this action, we must have

$$\psi_3^{-1} i \psi_1(\mathrm{Spine}_{(r_1,r_2)}) \subset \mathrm{Spine}_{(r_3,r_2)} .$$

Moreover, since the ψ_j and i are local homeomorphisms, for p a one or three-pronged singularity of $\mathrm{Spine}_{(r_1,r_2)}$, we must have that $\psi_3^{-1} i \psi_1(p)$ consists entirely of one or three-pronged (respectively) singularities of $\mathrm{Spine}_{(r_3,r_2)}$. Since each spine has only one singularity of each type, we must have $a_3 = \pm 1$. By switching the roles of r_1 and r_3, we must also have $a_1 = \pm 1$. As the one-pronged singularity of each spine is located at $R_j = 1$, we must have $\zeta = 1$, or else $\psi_3^{-1} i \psi_1(1) \neq 1$.

It remains to consider the case where $a_j = \pm 1$ and $\zeta = 1$. Given the location of the three-pronged singularities (11.1) and the cross-ratio equation (11.2), we obtain

$$\left(\frac{r_1 - r_2}{r_1 + r_2}\right) \left(\frac{r_3 - r_2}{r_3 + r_2}\right)^{\pm 1} = 1,$$

which implies

$$\frac{r_1}{r_2} = \pm \frac{r_3}{r_2}.$$

This contradicts the requirement that (r_1, r_2, r_3) is a basis of F. ∎

Corollary 11.4. *If $\mathcal{O}_F$ is the maximal order in a totally real cubic number field F, then the eigenform locus $\Omega\mathcal{E}^\iota_{\mathcal{O}_F}$ is not invariant under the action of* $\mathrm{GL}_2^+(\mathbb{R})$.

Proof. If $\mathcal{O}_F$ is a maximal totally real cubic order, Proposition 9.2 provides an admissible basis of some ideal in $\mathcal{O}$. By Theorem 8.1, the eigenform locus $\mathcal{E}_{\mathcal{O}_F}$ then intersects the corresponding irreducible boundary stratum, so $\mathcal{E}_{\mathcal{O}_F}$ is not invariant by Theorem 11.3. ∎

It should be true also for nonmaximal orders $\mathcal{O}$ that no irreducible component of $\Omega\mathcal{E}^\iota_{\mathcal{O}}$ is $\mathrm{GL}_2^+(\mathbb{R})$-invariant. To achieve this using our approach one needs to have information about which symplectic extensions of $\mathcal{O}$-modules arise from cusps of a given irreducible component X of $\mathcal{E}_{\mathcal{O}}$. This seems like a quite delicate number theoretic question.

12 Intersecting the eigenform locus with strata

Given the results of the previous section, one might now ask whether the intersection of the eigenform locus with lower-dimensional strata or the hyperelliptic locus is $\mathrm{GL}_2^+(\mathbb{R})$-invariant. Refined versions of the proof of Theorem 11.3 are likely to give negative answers to this question as well, provided that the intersection has large enough dimension so that the degeneration techniques can still be applied.

The most basic dimension question is, whether the eigenform locus lies generically in the principal stratum $\Omega\mathcal{M}_3(1,1,1,1)$, i.e. the stratum of maximal dimension. Motivation for this question is the following coarse heuristics. Almost all primitive Teichmüller curves in genus two are obtained by intersecting the eigenform locus with the minimal stratum $\Omega\mathcal{M}_2(2)$. In genus three, the stratum $\Omega\mathcal{M}_3(4)$ (of minimal dimension) has codimension three in the principal stratum. Hence if the the eigenform locus $\mathcal{E}_\mathcal{O}$ lies generically in the principal stratum, then the expected dimension for its intersection (in $\mathbb{P}\Omega\mathcal{M}_3$) with $\mathbb{P}\Omega\mathcal{M}_3(4)$ is zero - too small for a Teichmüller curve. On the other hand, components of $\mathcal{E}_\mathcal{O}$ that do not lie generically in the principal stratum are a potential source of Teichmüller curves. We show that such components do not exist.

Theorem 12.1. *For any given order $\mathcal{O}$ in a totally real cubic number field each component of the eigenform locus $\Omega\mathcal{E}_\mathcal{O}$ lies generically in the principal stratum.*

The theorem will follow from an intersection property of the real multiplication locus with small strata.

Lemma 12.2. *Given a weighted admissible boundary stratum $\mathcal{S}$ of type $[4]\times^2[4]$ there is a weighted admissible boundary stratum $\mathcal{S}'$ of type $[3]\times^2[3]\times^1[3]\times^2[3]\times^1$ which is a degeneration of $\mathcal{S}$.*

Proof. Let $\pm r_1, \pm r_2$ be the weights in one component of curves parameterized by $\mathcal{S}$ and let $\pm r_2, \pm r_3$ be the weights in the other component. Admissibility implies that the $\mathbb{Q}^+$-span of $Q(r_1), Q(r_2), Q(r_3)$ is a half-plane H in $\mathbb{R}^3$. In each of the two component we can pinch further curves. They necessarily carry the weights $\pm(r_1 \pm r_2)$ resp. $\pm(r_2 \pm r_3)$, the signs depending on the choice of the curve. By Lemma 8.6 we know that $Q(r_1 \pm r_2)$ does not lie in H. In the Galois closure of F we calculate

$$Q(r_1 \pm r_2) = Q(r_1) + Q(r_2) \pm (r_1^\sigma r_2^{\sigma^2} + r_2^\sigma r_1^{\sigma^2}).$$

Consequently the two choices of the sign lead to Q-images on different sides of H. To produce $\mathcal{S}'$ is thus suffices to pinch some curve that acquires the weight $r_2 + r_3$ and also to pinch a curve on the other component acquiring the weight $r_1 \pm r_2$ with the sign chosen such that $Q(r_2 + r_3)$ and $Q(r_1 \pm r_2)$ lie on opposite sides of H. ∎

Lemma 12.3. *For any given order $\mathcal{O}$ in a totally real cubic number field each cusp of the eigenform locus $\mathcal{E}_\mathcal{O}$ has non-empty intersection with a boundary* *stratum parameterizing stable curves without separating curves and all of whose components are thrice punctured projective lines (i.e. a pants decomposition without separating curves).*

Proof. Since the boundary of the locus of $\mathcal{RM}_{\mathcal{O}}$ is obtained by intersecting with a divisor of $\overline{\mathcal{M}}_3$, every boundary stratum is contained in the closure of a two-dimensional boundary stratum of $\mathcal{RM}_{\mathcal{O}}$. Suppose this two-dimensional stratum is an admissible weighted boundary stratum $\mathcal{S}$ with $\dim(\mathrm{Span}(\mathcal{S})) = 3$. Case distinction and dimension count shows that $\mathcal{S}$ does not contain any separating curves. Any degeneration of $\mathcal{S}$ is again admissible. Thus in this case it suffices to pinch enough non-separating curves to obtain a pants decomposition.

The only case of an admissible weighted boundary stratum $\mathcal{S}$ that gives a two-dimensional component of $\partial\mathcal{RM}_{\mathcal{O}}$ and with the property $\dim(\mathrm{Span}(\mathcal{S})) = 2$ is the stratum of type [6]. We can degenerate this to a stratum of type $[4] \times^2 [4]$ without changing admissibility. Now Lemma 12.2 concludes the proof. ■

Proof of Theorem 12.1. By Lemma 12.3, there exists a stable form on the boundary of each component of $\mathcal{E}_{\mathcal{O}}$ with each of the four irreducible components a thrice punctured sphere. This form must then have four simple zeros, one in each irreducible component. Since the eigenform over a degenerate curve has simple zeros, so does the eigenform over a general curve. ■

13 Finiteness for the stratum $\Omega\mathcal{M}_3(3, 1)$

The aim of this section is to prove the following finiteness result for Teichmüller curves using the cross-ratio equation and the torsion condition of Theorem 10.3. This stratum contains one of the two known algebraically primitive Teichmüller curves in genus three, the billiard table $T(2, 3, 4)$ whose unique irreducible cusp in $\Omega\overline{\mathcal{M}}_3$ is described in Example 13.8 below.

Theorem 13.1. *There are only finitely many algebraically primitive Teichmüller curves in the stratum $\Omega\mathcal{M}_3(3, 1)$.*

This theorem will follow from the following finiteness theorem for cusps.

Theorem 13.2. *There are only finitely many points in $\mathbb{P}\Omega\overline{\mathcal{M}}_3(3, 1)$ which are limits of irreducible cusps of algebraically primitive Teichmüller curves in $\mathbb{P}\Omega\mathcal{M}_3(3, 1)$.*

Heights. The proof of Theorem 13.2 will require some facts about heights of subvarieties of $\mathbb{P}^n(\overline{\mathbb{Q}})$ which we summarize here. Unless stated otherwise, proofs can be found in [HS00].

Consider a number field K and a point $P = (x_0 : \ldots : x_n) \in \mathbb{P}^n(K)$. The *absolute logarithmic Weil height* of P is

$$h(P) = \frac{1}{[K : \mathbb{Q}]} \log \prod_{v \in M_K} \max\{\|x_0\|_v, \ldots, \|x_n\|_v\},$$

where M_K is the set of places of K, and $\|\cdot\|_v$ is the normalized absolute value at v. The height $h(P)$ is unchanged under passing to an extension of K, so h is a well-defined function $h\colon \mathbb{P}^n(\overline{\mathbb{Q}}) \to [0,\infty)$.

There is a more general notion of the height of a subvariety V of $\mathbb{P}^n(\overline{\mathbb{Q}})$. The precise definition is not important for us; see [HS00, p. 446]. We write $h(V) \in [0,\infty)$ for the height of V.

We will require the following properties of heights:

- (Northcott's Theorem) A collection of points in $\mathbb{P}^n(\overline{\mathbb{Q}})$ with uniformly bounded height and degree is finite.
- The height of a hypersurface $V \subset \mathbb{P}^n(\overline{\mathbb{Q}})$ cut out by a polynomial f is equal to the height of the vector of coefficients of f.
- (Arithmetic Bézout Theorem [Phi95]) If X and Y are irreducible projective subvarieties of $\mathbb{P}^n(\overline{\mathbb{Q}})$ with $Z_1, \dots, Z_n$ the irreducible components of $X \cap Y$, then for some constant C,

$$\sum_{i=1}^{n} h(Z_i) \leq \deg(X)h(Y) + \deg(Y)h(X) + C\deg(X)\deg(Y).$$

- The height of a zero-dimensional subvariety of $\mathbb{P}^n(\overline{\mathbb{Q}})$ is the sum of the heights of its individual points.
- [HS00, Theorem B.2.5] Given a degree d rational map $\phi\colon \mathbb{P}^n \to \mathbb{P}^m$ defined over $\overline{\mathbb{Q}}$ with indeterminacy locus Z, we have for any $P \in \mathbb{P}^n(\overline{\mathbb{Q}}) \setminus Z$

$$h(\phi(P)) \leq dh(P) + O(1). \tag{13.1}$$

Finally, there is the important theorem of Bombieri-Masser-Zannier [BMZ99] on intersections of curves with algebraic subgroups of the torus $\mathbb{G}_m^n$. We define $\mathcal{H}_k \subset \mathbb{G}_m^n$ to be the union of all algebraic subgroups of dimension at most k.

Theorem 13.3. *Let $C \subset \mathbb{G}_m^n$ be a curve defined over $\overline{\mathbb{Q}}$ which is not contained in a translate of a subtorus. Then $C \cap \mathcal{H}_{n-1}$ is a set of bounded height, and $C \cap \mathcal{H}_{n-2}$ is finite.*

The $\mathcal{H}_0$ case was proved in [Lau84]. An effective version of this theorem was proved in [Hab08].

Finiteness of cusps. We now begin working towards a weaker version of Theorem 13.2, namely that there are up to scaling finitely possible triples of widths of cylinders of irreducible periodic directions (i.e. of residues) of algebraically primitive Veech surfaces in $\Omega\mathcal{M}_3(3,1)$.

We first introduce some notation which will be used throughout the next two paragraphs. Consider the moduli space $\mathcal{M}_{0,8}$ of eight distinct labeled points in $\mathbb{P}^1$. We label these points $p, q, x_1, x_2, x_3, y_1, y_2, y_3$. Given a point $P \in \mathcal{M}_{0,8}$, there is a unique (up to scale) meromorphic one-form ω_P with a threefold zero at p, a simple zero at q, and a simple pole at each x_i or y_i. We will usually make the normalization that $p = 0$ and $q = \infty$, and write

$$\omega_P = \frac{z^3 dz}{\prod_{i=1}^{3}(z - x_i)(z - y_i)}. \tag{13.2}$$

Under this normalization, $\mathcal{M}_{0,8}$ is naturally identified with an open subset of $\mathbb{P}^5$ via $P \mapsto (x_1 : \ldots : y_3)$. We use this identification to define the Weil height h on $\mathcal{M}_{0,8}$. We define $S(3,1) \subset \mathcal{M}_{0,8}$ to be the locus of P such that ω_P satisfies the opposite-residue condition $\operatorname{Res}_{x_i} \omega_P = -\operatorname{Res}_{y_i} \omega_P$ for each i. The variety $S(3,1)$ is locally parameterized by the projective 4-tuple consisting of the three residues and one relative period, so $S(3,1)$ is three-dimensional.

We define the cross-ratio morphisms $Q_i \colon S(3,1) \to \mathbb{G}_m$ and $R_i \colon S(3,1) \to \mathbb{G}_m$ by

$$Q_i = [p, q, y_i, x_i] \quad \text{and} \quad R_i = [x_{i+1}, y_{i+1}, y_{i+2}, x_{i+2}]^{-1},$$

with indices taken mod 3. In the standard normalization of (13.2), $Q_i = y_i/x_i$. We define $Q, \mathrm{CR} \colon S(3,1) \to \mathbb{G}_m^3$ by $Q = (Q_1, Q_2, Q_3)$ and $\mathrm{CR} = (R_1, R_2, R_3)$. We define $\operatorname{Res} \colon S(3,1) \to \mathbb{P}^2$ by $\operatorname{Res}(P) = (\operatorname{Res}_{x_i} \omega_P)_{i=1}^3$. Finally, given $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{G}_m^3$, we define $S_\zeta(3,1) \subset S(3,1)$ to be the locus where $Q_i = \zeta_i$ for each i. The situation is summarized in the following diagram.

$$\begin{tikzcd}
\scriptstyle 5-\mathrm{dim} & \scriptstyle 3-\mathrm{dim} & \\
\mathcal{M}_{0,8} \arrow[d, "Q"] \arrow[dr, "CR"] & S(3,1) \arrow[l, hook'] \arrow[r, "\mathrm{Res}"] & \mathbb{P}^2 \\
\mathbb{G}_m^3 & \mathbb{G}_m^3 &
\end{tikzcd}$$

Lemma 13.4. *Any irreducible stable form $(X, \omega) \in \mathbb{P}\Omega\overline{\mathcal{M}}_3(3,1)$ which is a limit of a cusp of an algebraically primitive Teichmüller curve $C \subset \mathbb{P}\Omega\mathcal{M}_3(3,1)$ is equal to ω_P for some $P \in S_{(\zeta_1,\zeta_2,\zeta_3)}(3,1) \cap \mathrm{CR}^{-1}(T)$ with the ζ_i nonidentity roots of unity and $T \subset \mathbb{G}_m^3$ a proper algebraic subgroup. Moreover, if we normalize the components $(r_1 : r_2 : r_3)$ of $\operatorname{Res}(P)$ such that $r_1 \in \mathbb{Q}$, then $\{r_1, r_2, r_3\}$ is a basis of some totally real cubic number field.*

Proof. The procedure how to obtain the limit stable surface while flowing along the Teichmüller geodesic flow in a periodic direction is described in [Mas75]. One should cut open the surface along the core curves of the cylinders and glue in annuli of larger and larger moduli and, in the limit, glue in a pair of discs joined at the node. In this picture the one-form is a multiple of dz/z, since annuli are flat cylinders in the corresponding metric. The residue of the one-form is $2\pi i$ times the integral along the core curve of any of the annuli and thus equal to the width of the respective cylinder.

Consequently, the limit of an irreducible cusp of C is an irreducible stable form with two zeros of order 3 and 1, and 6 poles whose residues (up to sign and constant multiple) are the widths of the 3 horizontal cylinders of (X, ω). Since a form generating C is an eigenform for real multiplication by Theorem 10.3 and the residues r_i are widths of cylinders, they are a basis of the trace field by Lemma 10.4.

That the ζ_i are roots of unity follows from the torsion condition of Theorem 10.3. By Abel's theorem ([GH78, Second version, p. 235]), there is an n such that for each $(Y, \eta) \in C$ we may find a degree n meromorphic function $Y \to \mathbb{P}^1$ with a single pole of order n at one zero of η and a zero of order n at the other zero of η. Taking a limit of such functions (this is justified in [Möl08, p. 75]), we obtain a meromorphic function $f\colon X \to \mathbb{P}^1$ with a single zero at p and a single pole at q. In the normalization of (13.2), such a function must be of the form $f(z) = z^n$. Since x_i and y_i are identified, we must have $x_i^n = y_i^n$, as desired.

That $\mathrm{CR}(P)$ lies on an algebraic subgroup follows directly from Theorems 5.2 and 8.5. ■

Lemma 13.5. *Let ζ_i be roots of unity, all of them different from one. If the ζ_i are not all cube roots of unity, then $S_{(\zeta_1,\zeta_2,\zeta_3)}(3,1)$ is zero-dimensional. Otherwise $S_{(\zeta_1,\zeta_2,\zeta_3)}(3,1)$ has a single one dimensional component, a line in $\mathcal{M}_{0,8}$. Specifically, if $\zeta_i = e^{2\pi i/3}$ for all i, this component is the line L cut out by the equation,*

$$x_1 + x_2 + x_3 = 0,$$

under the normalization $p = 0$ and $q = \infty$.

Proof. $S_{(\zeta_1,\zeta_2,\zeta_3)}(3,1)$ is cut out by the equations $y_i = \zeta_i x_i$ and

$$D_i = \zeta_i^3 \prod_{j \neq i}(x_i - x_j)(x_i - \zeta_j x_j) - \prod_{j \neq i}(\zeta_i x_i - x_j)(\zeta_i x_i - \zeta_j x_j). \tag{13.3}$$

Suppose that $S_{(\zeta_1,\zeta_2,\zeta_3)}(3,1)$ has a positive dimensional component, and suppose first that (say) ζ_1 is not a cube root of unity. Then there is a homogeneous polynomial P of some degree $d < 4$ which divides D_k for all k. Expanding D_k, we obtain

$$D_k = x_k^4(\zeta_k^3 - \zeta_k^4) + \cdots + \zeta_{k+1} x_{k+1}^2 \zeta_{k+2} x_{k+2}^2 (\zeta_k^3 - 1),$$

with indices taken mod 3. Because each D_k contains x_k^4 with non-zero coefficient, each monomial x_k^d appears in P with non-zero coefficient. We have

$$P(0, x_2, x_3) = \alpha_2 x_2^d + \alpha_3 x_3^d + \ldots \quad | \quad D_1(0, x_2, x_3) = \zeta_2 x_2^2 \zeta_3 x_3^2 (\zeta_1^3 - 1).$$

This is not possible since the α_i are nonzero and $\zeta_1^3 \neq 1$.

Now suppose that $\zeta_i = e^{2\pi i/3}$ for all i. A simple computation shows that $P = x_1 + x_2 + x_3$ divides each D_k, so L is a component of $S_{(\zeta_1,\zeta_2,\zeta_3)}(3,1)$. An argument as above shows that the quotients D_k/P have no common factor, so L is the only one-dimensional component.

Finally, suppose the ζ_i are arbitrary cube roots of unity. Replacing some of the cube roots of unity $e^{2\pi i/3}$ with their complex conjugates amounts to swapping the corresponding x_i and y_i. Thus the new $S_{(\zeta_1,\zeta_2,\zeta_3)}(3,1)$ is simply a rotation of the old one. ∎

Lemma 13.6. *No one-dimensional component of any $S_{(\zeta_1,\zeta_2,\zeta_3)}(3,1)$ lies in $CR^{-1}(T)$ for T any algebraic subgroup of $\mathbb{G}_m^3$.*

Proof. By Lemma 13.5, we need only to show that the equation

$$R_1^{a_1}R_2^{a_2}R_3^{a_3} = \zeta \tag{13.4}$$

is not satisfied identically on the line L cut out by $x_1 + x_2 + x_3 = 0$. We may assume without loss of generality that $a_1 \neq 0$. Normalizing so that $x_1 = 1$ and setting $x_3 = -1 - x_2$, the left hand side of (13.4) becomes a rational function R in the single variable x_2 which must be identically equal to ζ. The factor $(2x_2 + 1)$ lies in the numerator of R_1 and appears nowhere else in R. Since $\mathbb{C}[x]$ is a unique factorization domain, it follows that R is not constant. ∎

Proposition 13.7. *There is a finite number of projectivized triples of real cubic numbers $(r_1 : r_2 : r_3)$ such that for any irreducible periodic direction on any $(X,\omega) \in \Omega\mathcal{M}_3(3,1)$ generating an algebraically primitive Teichmüller curve, the projectivized widths of the cylinders in that direction is one of the triples $(r_1 : r_2 : r_3)$.*

In particular, there are only a finite number of trace fields F of algebraically primitive Teichmüller curves in $\Omega\mathcal{M}_3(3,1)$.

Proof. By Northcott's Theorem, we need only to give a uniform bound for the heights of the triples $(r_1 : r_2 : r_3)$ of widths of cylinders, or equivalently of residues of limiting irreducible stable forms satisfying the conditions of Lemma 13.4.

Let $T_i(\zeta_1,\zeta_2,\zeta_3) \subset \mathbb{P}^5$ be the subvariety cut out by the polynomial D_i of (13.3). Since $\|\zeta\|_v = 1$ for any root of unity ζ and place v, it follows directly from the definition of the Weil height that there is a uniform bound on the heights of the $T_i(\zeta_1,\zeta_2,\zeta_3)$, independent of the root of unity. Since $S_{(\zeta_1,\zeta_2,\zeta_3)}(3,1)$ is the intersection of the $T_i(\zeta_1,\zeta_2,\zeta_3)$ and the hypersurfaces defined by $x_i - \zeta_i y_i$ (which have height 0), it follows from the Arithmetic Bézout theorem that the varieties $S_{(\zeta_1,\zeta_2,\zeta_3)}(3,1)$ have uniformly bounded height. Thus the zero dimensional components of the $S_{(\zeta_1,\zeta_2,\zeta_3)}(3,1)$ have uniformly bounded height as well. By (13.1), the heights of these points increase by a bounded factor under the rational map Res. Thus the residue triples arising from the zero dimensional components of the $S_{(\zeta_1,\zeta_2,\zeta_3)}(3,1)$ have uniformly bounded heights.

By Lemma 13.5, it only remains to bound the heights of the residue triples arising from the line $L \subset \mathcal{M}_{0,8}$ cut out by the equations $x_1 + x_2 + x_3 = 0$ and $x_i - \theta y_i$ for each i, where $\theta = e^{2\pi i/3}$. Suppose a point $P \in L$ is a cusp of an algebraically primitive Teichmüller curve. By Lemma 13.4, $\mathrm{Res}(P)$ must be defined over a cubic number field, and $\mathrm{CR}(P)$ must lie in $\mathcal{H}_2$. Let $L' \subset L$ be the set of points satisfying these two conditions. If $\mathrm{Res}(P)$ lies in $\mathbb{P}^2(F)$ for some cubic number field F, then P is defined over $F(\theta)$. Thus L' and $\mathrm{CR}(L')$ consist of points of degree at most 9. By Lemma 13.6, $\mathrm{CR}(L)$ is not contained in a translate of a subtorus of $\mathbb{G}_m^3$. Thus Theorem 13.3 applies, and we conclude that $\mathrm{CR}(L) \cap \mathcal{H}_2$ is a set of points of bounded height. Therefore $\mathrm{CR}(L')$ is finite by Northcott's theorem. The map CR is finite on L by Lemma 13.6, so L' and thus $\mathrm{Res}(L')$ are finite as well. Thus there are at most finitely many residue triples arising from L as desired. ■

Remark. All of the estimates in the preceding propositions, in particular Theorem 13.3 and the height estimates are effective. It is thus possible in principle to give a complete list of triples (r_1, r_2, r_3) that may appear in Proposition 13.7. Unfortunately the available bounds are so bad that this is currently not feasible.

Example 13.8. There is one known example of an algebraically primitive Teichmüller curve in $\Omega\mathcal{M}_3(3,1)$, discovered in [KS00]. It is the surface (X, ω) obtained by unfolding the $(2,3,4)$ triangle, shown in Figure 7 of [KS00]. The trace field of (X, ω) is the field $K = \mathbb{Q}[v]/P(v)$ of discriminant 81, where $P(v) = v^3 - 3v + 1$ has a solution $v = 2\cos(2\pi/9)$. The vertical direction is of type $[5] \times^3 [3]$, and the circumferences of the vertical cylinders are

$$\begin{aligned}
w_1 &= 2\cos(3\pi/9) &&= 1\\
w_2 &= -2(\cos(3\pi/9) + \cos(8\pi/9)) &&= v^2 + v - 1\\
w_3 &= 2(\cos(2\pi/9) + \cos(3\pi/9) + \cos(8\pi/9)) &&= -v^2 - 3\\
w_4 &= 2\cos(4\pi/9) &&= v^2 - 2.
\end{aligned}$$

One can check that the w_i form an admissible basis for a lattice in K.

The horizontal direction is irreducible periodic, with cylinder widths,

$$\begin{aligned}
r_1 &= -(2w_1 + w_2 + w_3 + w_4) &&= -v^2 - v\\
r_2 &= w_1 + w_2 + w_3 &&= v + 1\\
r_3 &= -(3w_1 + 3w_2 + 2w_3 + w_4) &&= -2v^2 - 3v + 2.
\end{aligned}$$

In fact, this is the unique irreducible cusp of the Teichmüller curve generated by (X, ω). This cusp lies on the line L of Lemma 13.5, as we will now show. The irreducible cusp (X_0, ω_0) is of the form

$$\omega_0 = C\frac{z^3 dz}{\prod(z - x_i)(z - \zeta_i x_i)} = \sum\left(\frac{r_i}{z - x_i} - \frac{r_i}{z - \zeta_i x_i}\right) \tag{13.5}$$

for some constant C and roots of unity $\zeta_i = e^{2\pi i p_i/q_i}$. To calculate the ζ_i, we consider a relative period. There is a path joining the two zeros of (X, ω) of period $\sum r_i/3$, so the integral of ω_0 along a path γ joining 0 to ∞ must be $\sum(a_i + 1/3)r_i$, for some integers a_i. From (13.5), we calculate

$$\frac{1}{3}\sum r_i = \int_\gamma \omega_0 = \sum r_i \log\zeta_i = \sum r_i \frac{p_i}{q_i},$$

so we must have $\zeta_i = e^{2\pi i/3}$ for each i by the linear independence of the r_i. One then calculates that up to multiplication by an element in $\mathbb{C}^*$ there is a unique triple (x_1, x_2, x_3) so that ω_0 has the residues r_i,

$$x_1 = 1, \quad x_2 = 2 - v^2, \quad \text{and} \quad x_3 = v^2 - 3.$$

Since the sum of the x_i is 0, this cusp lies on the line L.

Theorem 13.2 now follows directly from Proposition 13.7 and the following proposition.

Proposition 13.9. *Given a basis (r_1, r_2, r_3) over $\mathbb{Q}$ of a totally real cubic number field, there are only finitely many limits of cusps of algebraically primitive Teichmüller curves in $\Omega\mathcal{M}_3(3,1)$ having residues (r_1, r_2, r_3).*

Proof. Consider the variety $C = \mathrm{Res}^{-1}(r_1 : r_2 : r_3) \subset S(3,1)$ of forms having residues $\pm r_i$ and two zeros of order 3 and 1. A dimension count shows that C is at least one-dimensional. In fact, C is exactly one-dimensional, as C is locally parameterized by the single relative period of the forms ω_P. Let C_0 be a component of C. We suppose that C_0 contains infinitely many cusps of algebraically primitive Teichmüller curves and derive a contradiction. Consider the image $Q(C_0) \subset (\mathbb{C}^*)^3$. We claim that $Q(C_0)$ is a curve. If not, and $Q(C_0) = (\zeta_1, \zeta_2, \zeta_3)$, then C_0 is a component of $S_{(\zeta_1,\zeta_2,\zeta_3)}$. Then C_0 must be the line L of Lemma 13.5. It is easily checked that Res is not constant along L, so this is impossible. Now since C_0 contains infinitely many cusps of Teichmüller curves, $Q(C_0)$ must contain infinitely many torsion points of $(\mathbb{C}^*)^3$ by Lemma 13.4. From this it follows that $Q(C_0)$ is a translate of a subtorus of $(\mathbb{C}^*)^3$ by a torsion point. This is a consequence of the main result of [Lau84]. It can also be seen by first applying Theorem 13.3 to show that $Q(C_0)$ lies on a subtorus $T \subset (\mathbb{C}^*)^3$, then applying Theorem 13.3 again to T. We now claim that $Q(C_0)$ is in fact a subtorus of $(\mathbb{C}^*)^3$, rather than a translate. To see this, it suffices to show that the identity $(1,1,1)$ is contained in the closure of $Q(C_0)$. Given a form (X, ω) representing a point $P \in C_0$, we may choose a saddle connection joining the two zeros p and q. Following [EMZ03], we may collapse this saddle connection (and possibly simultaneously a homologous saddle connection) to obtain a path in C_0 such that the zeros p and q collide. Under this deformation, each cross-ratio Q_i tends to 1, so $(1,1,1)$ is in the closure, as desired. It remains to show that $Q(C)$ is not a subtorus of $(\mathbb{C}^*)^3$. If this were true, we could find roots of unity ζ_i and a projective triple $(x_1(a) : x_2(a) : x_3(a))$ depending on a parameter a, such that for all $a \in \mathbb{C}$ the differential

$$\omega_\infty = \left(\sum_{i=1}^{3} \frac{r_i}{z - x_i(a)} - \frac{r_i}{z - \zeta_i^a x_i(a)} \right) dz = \frac{p(z)dz}{\prod_i (z - x_i(a))(z - \zeta_i^a x_i(a))}$$

has a triple zero at $z = 0$ and a simple zero at $z = \infty$. The vanishing of the z^4-term of $p(z)$ implies

$$\sum r_i x_i (1 - \zeta_i^a) = 0$$

and the linear term (divided by $x_1 x_2 x_3$) also yields a linear equation. Using the normalization $x_1 = 1$ we may solve the two linear equations for x_2 and x_3. We then take the limit of x_2 and x_3 as $a \to 0$, applying l'Hôpital's rule twice. If we let $\zeta_i = e^{2\pi i q_i}$ for some $q_i \in \mathbb{Q}$, we obtain

$$x_2(0) = \frac{q_3 r_3 - q_1 r_1}{q_2 r_2 - q_3 r_3} \quad \text{and} \quad x_3(0) = \frac{q_2 r_2 - q_1 r_1}{q_3 r_3 - q_2 r_2}. \tag{13.6}$$

Taking the derivative of the z^2-term of $p(z)$ with respect to a at $a = 0$ and making the substitution (13.6), we obtain

$$(q_3 r_3 - r_1 q_1)(q_2 r_2 - q_1 r_1)(q_1 r_1 + q_2 r_2 + q_3 r_3) = 0.$$

The $\mathbb{Q}$-linear independence of the r_i yields the desired contradiction. ■

Finiteness of Teichmüller curves. Theorem 13.1 follows from Theorem 13.2 and the following proposition.

Proposition 13.10. *Suppose that there are at most finitely many limits of irreducible cusps in $\mathbb{P}\Omega\overline{\mathcal{M}}_g$ of algebraically primitive Teichmüller curves in a component of the stratum $\mathbb{P}\Omega\mathcal{M}_g(m,n)$ (resp. in a component of the stratum $\mathbb{P}\Omega\mathcal{M}_g(2g-2)$). Then there are at most finitely many algebraically primitive Teichmüller curves in this component of $\mathbb{P}\Omega\mathcal{M}_g(m,n)$ (resp. in this component of the stratum $\mathbb{P}\Omega\mathcal{M}_g(2g-2)$).*

Proof. Suppose $(X,\omega) \in \Omega\mathcal{M}_g(m,n)$ generates an algebraically primitive Teichmüller curve. Let θ be an irreducible periodic direction on (X,ω), and let I and J each be either a saddle connection or periodic direction of slope θ. Since lengths of saddle connections or circumferences of cylinders of a given slope are unchanged under passing to the corresponding limiting stable form, from finiteness of irreducible cusps we obtain a constant C, depending only on the stratum, so that

$$\frac{1}{C} < \frac{\operatorname{length}(I)}{\operatorname{length}(J)} < C \tag{13.7}$$

for I and J any saddle connections or closed geodesics of the same slope. There is an irreducible periodic direction on (X,ω) by Lemma 10.2. Choose one, and apply a rotation of ω so that it is horizontal. Let $C_1,\dots,C_g$ be the horizontal cylinders of (X,ω). There must be some cylinder C_i having one of the two zeros in its bottom boundary component and the other zero in the top. Take a saddle connection γ contained in C_i and connecting these zeros. Applying the action of a matrix $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, we may take γ to be vertical, whence the vertical direction is irreducible periodic with g cylinders $D_1,\dots,D_g$. By Lemma 10.4, after normalizing by the action of a diagonal element of $\mathrm{GL}_2^+(\mathbb{R})$, we have $w(C_i) = r_i$ and $h(C_i) = s_i$ (where we write $w(C)$ and $h(C)$ for the width and height of the cylinder C) for some basis (r_i) of F (with a chosen real embedding) and dual basis (s_i). By finiteness of cusps, there are only finitely many possibilities for the r_i, and thus the s_i, so we may take them to be fixed. Since the saddle connection γ crosses only one cylinder, its length is bounded by a constant depending only on the stratum. This implies that the $w(D_i)$ are bounded as well by (13.7) and hence takes only finitely many values. Therefore the intersection matrix $(B_{ij}) = (C_i \cdot D_j)$ has bounded entries, and we may take it to be fixed. The widths and heights of the D_j are determined by B, as well as the widths and heights of the C_i, so we may take them to be fixed as well. Now each intersection of C_i and D_j is isometric to a rectangle R_{ij} of width $h(D_j)$ and height $h(C_i)$. Thus the surface (X, ω) may be built by gluing the finite collection of rectangles consisting of B_{ij} copies of R_{ij} for each index (i, j). As there are only finitely many gluing patterns for a finite collection of rectangles, there are only finitely many possibilities for (X, ω).

In the case $\mathbb{P}\Omega\mathcal{M}_g(2g-2)$ the same argument works. It is even simplified by the fact that every direction is irreducible. ■

14 Finiteness conjecture for $\Omega\mathcal{M}_3(4)^{\mathrm{hyp}}$

In this section, we give numerical and theoretical evidence for the following conjecture, which together with Proposition 13.10 implies Conjecture 1.4 for the case of the stratum $\Omega\mathcal{M}_3(4)^{\mathrm{hyp}}$.

Conjecture 14.1. *There are only a finite number of possibilities for the projectivized triples $(r_1 : r_2 : r_3)$ of widths of cylinders of algebraically primitive Teichmüller curves in $\Omega\mathcal{M}_3(4)^{\mathrm{hyp}}$.*

Everything in this section should hold as well for the other component $\Omega\mathcal{M}_3(4)^{\mathrm{odd}}$ of $\Omega\mathcal{M}_3(4)$, but we only consider the hyperelliptic component for simplicity. The hyperelliptic component contains the other of the two known examples of algebraically primitive Teichmüller curves in genus three, Veech's 7-gon. We describe the stable form which is the limit of the unique cusp of this curve in Example 14.4 below. Finally we will give the algorithm for searching any eigenform locus for Teichmüller curves in $\Omega\mathcal{M}_3(4)$ which is used to prove Theorem 1.6.

Finiteness for fixed admissibility coefficients. Recall from (8.10) that if $\mathcal{S}$ is a weighted admissible boundary stratum of type [6], then the weights r_i satisfy $\sum_{i=1}^{3} c_i/r_i = 0$ for some $c_i \in \mathbb{Z}$. We call the triple (c_1, c_2, c_3) of coprime integers the *admissibility coefficients* of the r_i.

Proposition 14.2. *For any fixed triple (c_1, c_2, c_3) there is only a finite number of algebraically primitive Teichmüller curves in $\Omega\mathcal{M}_3(4)^{\mathrm{hyp}}$ that possess a direction whose cylinders have lengths with admissibility coefficients (c_1, c_2, c_3).*

This has as obvious consequence:

Corollary 14.3. *In $\Omega\mathcal{M}_3(4)^{\mathrm{hyp}}$ there is only a finite number of algebraically primitive Teichmüller curves meeting the infinite collection of weighted boundary strata provided by the algorithm in the proof of Proposition 9.2.*

The limiting differential in the hyperelliptic case. We want to make the cross-ratio coordinates more explicit and therefore normalize the hyperelliptic involution on the stable curve X_∞ corresponding to a Teichmüller curve in $\Omega\mathcal{M}_3(4)^{\rm hyp}$. Necessarily, X_∞ is irreducible, and consequently the desingularization of X_∞ is a $\mathbb{P}^1$ with coordinate z, where we may normalize the hyperelliptic involution to be $z \mapsto -z$ and where $z = 0$ is the 4-fold zero. The preimages of the nodes are $\pm x_i$ for $i = 1, 2, 3$, and we will at some points in the sequel use the full threefold transitivity of Möbius transformations to normalize moreover $x_1 = 1$. The differential ω_∞ pulls back on the normalization to

$$\omega_\infty = \sum_{i=1}^{3} \left(\frac{r_i}{z - x_i} - \frac{r_i}{z + x_i} \right) dz = \frac{Cz^4}{\prod_{i=1}^{3}(z^2 - x_i^2)} dz \tag{14.1}$$

for some constant C that can obviously be expressed in the r_i and x_i. Since $x_1x_2x_3 \neq 0$, coefficient comparison yields the two equations

$$\sum_{i=1}^{3} r_i x_{i+1} x_{i+2} = 0 \tag{14.2}$$

$$\sum_{i=1}^{3} r_i x_i (x_{i+1}^2 + x_{i+2}^2) = 0, \tag{14.3}$$

where indices are to be read mod 3. The cross-ratio map CR as defined by Equation 8.1 is given by

$$\mathrm{CR} = (R_1, R_2, R_3), \quad \text{where} \quad R_i = \left(\frac{x_{i+1} - x_{i+2}}{x_{i+1} + x_{i+2}} \right)^2.$$

It will be convenient to use that CR factors as a composition of the squaring map and the rational map $\mathrm{CR}_0 \colon \mathbb{P}^2 \to (\mathbb{C}^*)^3$ defined by $\mathrm{CR}_0 = (R'_1, R'_2, R'_3)$, where

$$R'_i(x_1 : x_2 : x_3) = \frac{x_{i+1} - x_{i+2}}{x_{i+1} + x_{i+2}}.$$

Example 14.4. Veech's 7-gon curve lies in this stratum, and we conjecture it is the only one. Let $F = \mathbb{Q}[v]/\langle v^3 + v^2 - 2v - 1 \rangle$ be the cubic field of discriminant $D = 49$. There is a unique cusp whose cylinder widths are projectively equivalent to

$$r_1 = 1, \quad r_2 = v^2 + v - 2, \quad r_3 = v^2 - 2,$$

with $v = 2\cos(2\pi/7)$. Since

$$\sum_i \frac{1}{r_i} = 0 \quad \text{and} \quad N_{\mathbb{Q}}^F(r_i) = 1$$

for all i, the cross-ratio exponents in (8.3) are all 1. Only one of the three nontrivial solutions to equations (14.2) and (14.3) satisfies the cross-ratio equation $\prod R_i = 1$, namely

$$x_1 = 1, \quad x_2 = -v^2 - v + 1, \quad x_3 = v^2 + v - 2.$$

Note in comparison with Proposition 14.7 below that here the c_i, the $N_{\mathbb{Q}}^F(r_i)$ and also the moduli of the cylinders are all one. That is, all the auxiliary parameters are arithmetically as simple as possible.

Inside the domain of CR_0 the rationality condition $\sum_{i=1}^3 c_i/r_i = 0$ together with the opposite-residue condition, i.e. equations (14.2) and (14.2), defines a curve $Y = Y_{(c_1,c_2,c_3)}$. We want to apply Theorem 13.3 to this curve and now check the necessary hypothesis.

Lemma 14.5. *Let $X \subset (\mathbb{C}^*)^n$ be an irreducible curve whose closure in $\mathbb{C}^n$ contains points $P_1, \ldots, P_n$ where $P_i = (p_{i1}, \ldots, p_{in})$ and where for all i we have $p_{ii} = 0$ while $p_{ij} \neq 0$ for $i \neq j$. Then X is not contained in the translate of an $(n-1)$-dimensional algebraic subtorus in $(\mathbb{C}^*)^n$.*

Proof. Let z_i be coordinates of $\mathbb{C}^n$ and suppose on the contrary that X is contained in such a torus given by the equation $\prod z_i^{b_i} = t$ for some $b_i \in \mathbb{Z}$ not all zero and $t \in \mathbb{C}^*$. This equation holds on X, thus on its closure. Plugging in P_i implies $b_i = 0$. Using all the P_i we obtain the contradiction that all of the b_i are zero. ∎

Corollary 14.6. *The curve $\mathrm{CR}_0(Y)$ does not lie in a translate of an algebraic subtorus in $(\mathbb{C}^*)^3$.*

Proof. Normalizing $x_1 = 1$ and applying the degeneration $x_2 \to 0$ to $\mathrm{CR}_0(Y)$ we obtain the limit point $(1, 0, 1) \in \mathbb{C}^3$. Permuting coordinates, we obtain a limit point where any single coordinate vanishes, so we may apply Lemma 14.5 after verifying irreducibility.

A computer algebra system with an algorithm for computing Weierstrass normal form (e.g. MAPLE, using [vH]) exhibits a birational map from $Y_{(c_1,c_2,c_3)}$ to the curve

$$\widetilde{Y}: \quad y^2 = c_1^2x^6 - 3c_1^2x^5 + 3c_1^2x^4 + (c_2^2 - c_3^2 - c_1^2)x^3 + 3c_3^2x^2 - 3c_3^2x + c_3^2.$$

A straightforward calculation shows that the right hand side is not a perfect square for any (c_1, c_2, c_3). Consequently, $\widetilde{Y}$ is irreducible and thus also $Y_{(c_1,c_2,c_3)}$. ∎

Proof of Proposition 14.2. The preceding lemma allows us to apply Theorem 13.3. As a consequence, the height of any point $(R_1, R_2, R_3) \in \mathrm{CR}_0(Y)$ that lies on an algebraic subtorus is bounded. This applies in particular to the torus given by the cross-ratio equation. More precisely, since the degree of Y is independent of the c_i we deduce from [Hab08, Theorem 1] (applied to $X = Y$ and $p = (R_1, R_2, R_3)$) that there is a constant C_1 such that

$$h((R_1, R_2, R_3)) \leq C_1(1 + h(c_1 : c_2 : c_3)). \tag{14.4}$$

Moreover, the R_i lie in a field of degree at most three over F as can be checked solving (14.2) and (14.3). Consequently, by Northcott's theorem, there is only a finite number of possible R_i lying on $\mathrm{CR}(Y)$ and satisfying the cross-ratio equation. ∎

Unlikely cancellations. We now show that if the finiteness conjecture fails, then there has to be a sequence of Teichmüller curves with the admissibility coefficients c_i becoming more and more complicated simultaneously for all the directions on the generating flat surface, but meanwhile there are miraculously enormous cancellations making the cross-ratio exponents much smaller than the c_i.

Proposition 14.7. *Suppose Conjecture 14.1 fails for* $\Omega\mathcal{M}_3(4)^{\mathrm{hyp}}$. *Then there exists a sequence of Teichmüller curves* $\{C_n\}_{n\in\mathbb{N}}$ *generated by flat surfaces* (X_n, ω_n) *such that for every periodic direction* θ *on the* X_n

(i) the residues $r_{i,n,\theta}$ *have admissibility coefficients* $(c_{1,n,\theta}, c_{2,n,\theta}, c_{3,n,\theta})$ *with the height lower bound*

$$h(c_{1,n,\theta}, c_{2,n,\theta}, c_{3,n,\theta}) \geq n,$$

and on the other hand

(ii) the cross-ratio exponents have upper bound

$$|a_i| \leq C_2(1 + h(c_{1,n,\theta}, c_{2,n,\theta}, c_{3,n,\theta}))^2$$

for some constant C_2 *independent of* n *and* θ.

Note that in ii) the height on the right is logarithmic in the the c_i, whereas on the left of the inequality we have the usual absolute value.

As preparation we examine the image $Z \subset (\mathbb{C}^*)^3$ of $\Omega\mathcal{M}_3(4)^{\mathrm{hyp}}$ under CR.

Lemma 14.8. *There is no translate of an algebraic subtorus of* $(\mathbb{C}^*)^3$ *contained in* Z.

Proof. It suffices to prove the claim for the image Z_0 of Z under CR_0. The variety Z_0 is cut out by the equation

$$\frac{1}{R_1'R_2'} + \frac{1}{R_1'R_3'} + \frac{1}{R_2'R_3'} + 1 = 0. \tag{14.5}$$

This variety does not contain the image of $y \mapsto (\alpha_1 y^{n_1}, \alpha_2 y^{n_2}, \alpha_3 y^{n_3})$ for any nonzero α_i and integers n_i, as substituting the $\alpha_i y^{n_i}$ into the left hand side of (14.5) always yields a nonzero Laurent series in y. ∎

Proof of Proposition 14.7. The existence of a sequence satisfying (i) follows from Proposition 14.2. That this sequence moreover satisfies (ii) follows from a close examination of the proof of [Hab08, Theorem 1]. We fix θ and n and drop these indices. We write $\underline{c} = (c_1 : c_2 : c_3)$. We follow the notation in loc. cit. The idea of Habegger is to use the geometry of numbers to construct a subtorus H_u of $(\mathbb{C}^*)^3$ determined by a triple $u = (u_1, u_2, u_3)$ of integers depending on a parameter T – the precise dependence is explained in (14.6) below – such that for a point $p = (R_1, R_2, R_3)$ in the intersection of $W = \mathrm{CR}_0(Y_{\underline{c}})$ and H_u the following holds:

$$h(pH_u) \leq C_3(T^{-1/2}(h(p)+1)+T) \quad \text{and} \quad \deg(H_u) \leq C_4 T$$

for some constants C_i (Lemma 5 of loc. cit.). An application of the arithmetic Bézout theorem yields

$$h(p) \leq C_5 h(pH_u) + C_6 \deg(H_u) h(W) + C_7 \deg(H_u).$$

Choosing T large enough, controlled by $\deg(W)$ and the constants C_i (i.e. independently of $h(W)$) makes the contribution of $T^{-1/2}h(p)$ to the right hand side become inessential and proves the height bound

$$h(p) \leq C_8(1 + h(W)) \leq C_9(1 + h(\underline{c}))$$

We need more precisely Lemma 1 and Lemma 3 of loc. cit. which construct the u. Together they show that there exists u with

$$|u| \leq T \quad \text{and} \quad h(p^u) \leq C_{10} T^{-1/2} h(p). \tag{14.6}$$

Together with the previous estimate this yields

$$h(p^u) \leq C_{11} T^{-1/2}(1 + h(\underline{c})),$$

where C_{10} and C_{11} depend only on the dimensions of the varieties in question, not on $h(\underline{c})$. Since p lies in a field of bounded degree over F, choosing $T > C_{12}(1 + h(\underline{c}))^2$, with C_{12} independent of $h(\underline{c})$, suffices by Northcott's theorem to conclude that $h(p^u) = 0$.

We now have two cases. In the first case u and the cross-ratio exponents (a_1, a_2, a_3) are proportional. In this case, (ii) holds by $|u| \leq T$ and the primitivity of the triple (a_1, a_2, a_3). The second case is that they are not proportional, i.e. p lies on a torus of codimension two. Then we can apply [Hab08, Theorem 1] to Z since the hypotheses are met by Lemma 14.8. The conclusion of this theorem together with Northcott's Theorem is that the second case can happen only a finite number of times. ■

A computer search for Teichmüller curves. We now describe the algorithm underlying Theorem 1.6 given in the introduction.

We first claim that for given discriminant D it is possible to list all the admissible triples (r_1, r_2, r_3) for all lattices $\mathcal{I}$ with coefficient ring $\mathcal{O}_{\mathcal{I}}$ of discriminant D. To do so, one has to first list all orders of discriminant D. Belabas [Bel97] gives an algorithm which enumerates all cubic fields of discriminant less than a given bound. Given a number field F of discriminant at most D, enumerating all orders in F of discriminant D is a finite search through all sub-$\mathbb{Z}$-modules $\mathcal{O}$ of the maximal order $\mathcal{O}_F$ of bounded index. To list all $\mathcal{O}$-ideals is a finite search through all $\mathbb{Z}$-modules containing $\mathcal{O}$ up to an index bound depending on D. Such a bound appears in the usual proofs of the finiteness of class numbers, e.g. [BS66, Theorem 2.6.3]. (We do not claim that this is an efficient algorithm). Given a lattice $\mathcal{I}$ in a cubic field, an algorithm to find all admissible bases of $\mathcal{I}$ is described in Appendix A. In practice we have restricted the search to maximal orders, since maximal orders have been tabulated and representative elements of the ideal classes are easily computed by Pari.

Fixing a cubic order $\mathcal{O}$, if there is a Teichmüller curve in $\mathcal{E}_\mathcal{O} \cap \mathbb{P}\Omega\mathcal{M}_3(4)^{\mathrm{hyp}}$, then it has a cusp whose limiting stable form ω_∞ is of the form (14.1), with the triple (r_i) in the finite list constructed above. Normalizing $x_1 = 1$, equations (14.2) and (14.3) reduce to a single cubic polynomial in x_2. Solving this cubic polynomial for x_2 (for each triple of r_i) and verifying that none of the solutions satisfies the cross-ratio equation allows us to verify that there are no Teichmüller curves in $\mathcal{E}_\mathcal{O} \cap \mathbb{P}\Omega\mathcal{M}_3(4)^{\mathrm{hyp}}$. Applying this algorithm to the 1778 fields of discriminant less than 40000 yields Theorem 1.6.

A Boundary strata in genus three: Algorithms, examples, counting

In this appendix, we describe an algorithm for enumerating all boundary strata of a given eigenform locus $\mathcal{E}^\iota_\mathcal{O}$, and we give some examples and counts of admissible boundary strata obtained from this algorithm.

Enumerating admissible $\mathcal{I}$-weighted strata from one example. Given a lattice $\mathcal{I}$ in a totally real cubic field, define a graph $\mathcal{G}(\mathcal{I})$ as follows. The vertices of $\mathcal{G}(\mathcal{I})$ are the two-dimensional admissible $\mathcal{I}$-weighted boundary strata, up to similarity. Two vertices are connected by an edge if the corresponding strata have a common degeneration which is a one-dimensional boundary stratum.

Proposition A.1. *$\mathcal{G}(\mathcal{I})$ is connected.*

Proof. By Theorem 8.1, the vertices of $\mathcal{G}(\mathcal{I})$ correspond to the two-dimensional boundary components of some cusp of some eigenform locus $\mathcal{E}^\iota_\mathcal{O}$. Thus it suffices to show that the boundary in $\mathbb{P}\Omega\overline{\mathcal{M}}_3$ of each cusp of $\mathcal{E}^\iota_\mathcal{O}$ is connected.

Consider the normalization $Y^\iota_\mathcal{O}$ of $\overline{\mathcal{E}}^\iota_\mathcal{O}$. By normality, the canonical morphism $\mathcal{E}^\iota_\mathcal{O} \to X_\mathcal{O}$ extends to a morphism $p\colon Y^\iota_\mathcal{O} \to \widehat{X}_\mathcal{O}$ (see [Bai07, Theorem 8.10] for related arguments). Since $\widehat{X}_\mathcal{O}$ is normal, $p^{-1}(c)$ is connected by Zariski's Main Theorem. The image of $p^{-1}(c)$ in $\overline{\mathcal{E}}^\iota_\mathcal{O}$ is then connected, as desired. ■

It is a simple matter to enumerate all admissible $\mathcal{I}$-weighted boundary strata adjacent to a given one: It suffices to perform all the (finitely many) possible degenerations (as defined in Section 8) of the presently found boundary strata and check which of them are admissible $\mathcal{I}$-weighted. Then one tries all the possible undegenerations and so on, until this process adds no more admissible $\mathcal{I}$-weighted boundary strata to the known list. So Proposition A.1 allows us to enumerate all two dimensional $\mathcal{I}$-weighted boundary strata starting from a single one. Lower dimensional boundary strata can be easily enumerated from the two-dimensional ones.

Producing *one* admissible $\mathcal{I}$-weighted boundary stratum. We now describe an algorithm which locates a single admissible $\mathcal{I}$-weighted boundary stratum. In practice this algorithm is fast and always succeeds, though we do not prove this. The algorithm of Proposition 9.2 also works for lattices of the form $\langle 1, x, x^2 \rangle$, but not every lattice is similar to one of this form.

For an $\mathcal{I}$-weighted boundary stratum $\mathcal{S}$ let $\mathrm{Cone}(\mathcal{S}) \subset \mathbb{R}^3$ be the $\mathbb{R}^+$-cone spanned by $\{Q(w) : w \in \mathrm{Weight}(\mathcal{S})\}$, considered as a subset of $\mathbb{R}^3$ via the three field embeddings of F. There are various possible shapes of this cone, which we call its *type*. It could be all of $\mathbb{R}^3$, for short type (A), it could be a half-space (H), a proper cone of dimension three strictly contained in a half-space (C), a two-dimensional subspace (S), or a 2-dimensional cone ("fan") in a subspace (F).

The idea of the algorithm is to simply start with any irreducible stratum $\mathcal{S}$ and then to apply a sequence of degenerations and undegenerations to $\mathcal{S}$, at each stage trying to increase, or at least not decrease, the size of $\mathrm{Cone}(\mathcal{S})$.

Algorithm A.2. *Given a lattice $\mathcal{I}$, compute an admissible $\mathcal{I}$-weighted boundary stratum $\mathcal{S}$.*

(i) Initialize $\mathcal{S}$ to be the irreducible boundary stratum with weights given by any $\mathbb{Z}$-basis of $\mathcal{I}$.

(ii) While $\mathrm{Cone}(\mathcal{S})$ *is neither of type (A), (H), nor (S):*

- *(Superfluous curves) If $\mathcal{S}$ has a node n which lies on the boundary of two distinct irreducible components with $Q(\mathrm{wt}(n))$ in the interior of $\mathrm{Cone}(\mathcal{S})$, then let $\mathcal{S}_1$ be obtained from S by undegenerating n.*
- *(Try to degenerate) Else*
  - •• *Loop through all degenerations $\mathcal{S}_1$ of $\mathcal{S}$ and check if $\mathcal{S}_1$ contains a node n with $Q(\mathrm{wt}(n)) \notin \mathrm{Cone}(\mathcal{S})$.*
  - •• *(Got stuck) If no such degeneration was found, the algorithm is stuck. Start again at (i) with a random new choice of initial basis.*
- *Let $\mathcal{S} = \mathcal{S}_1$.*

(iii) If the type of $\mathrm{Cone}(\mathcal{S})$ *is (H), first undegenerate $\mathcal{S}$ until $\mathcal{S}$ contains only 4 elements still spanning a half-space and then undegenerate the new $\mathcal{S}$ by removing the node n with the property that $Q(\mathrm{wt}(n))$ does not lie in the bounding hyperplane of C. (The new $\mathcal{S}$ thus obtained is of type (S)).*

iv) Return $\mathcal{S}$.

As far as we know, it is possible for the algorithm to either get stuck with every choice of initial basis, or to loop infinitely, producing larger and larger cones without ever giving a half-space or the full space. We have never seen this happen, though very rarely it gets stuck and must be restarted with a new initial basis.

Some counts of boundary strata obtained from this algorithm are shown in Figure 4.

It would be interesting to give an algorithm in the spirit of Algorithm A.2 which is guaranteed to always find an admissible boundary stratum.

Example 1: Discriminant 49. Figure 3 presents the outcome of the preceding algorithm for the unique ideal class of the maximal order in the field $F = \mathbb{Q}[x]/\langle x^3 + x^2 - 2x - 1\rangle$ of discriminant 49. There are two two-dimensional boundary strata. Dotted lines join each two-dimensional stratum to its one-dimensional degenerations.

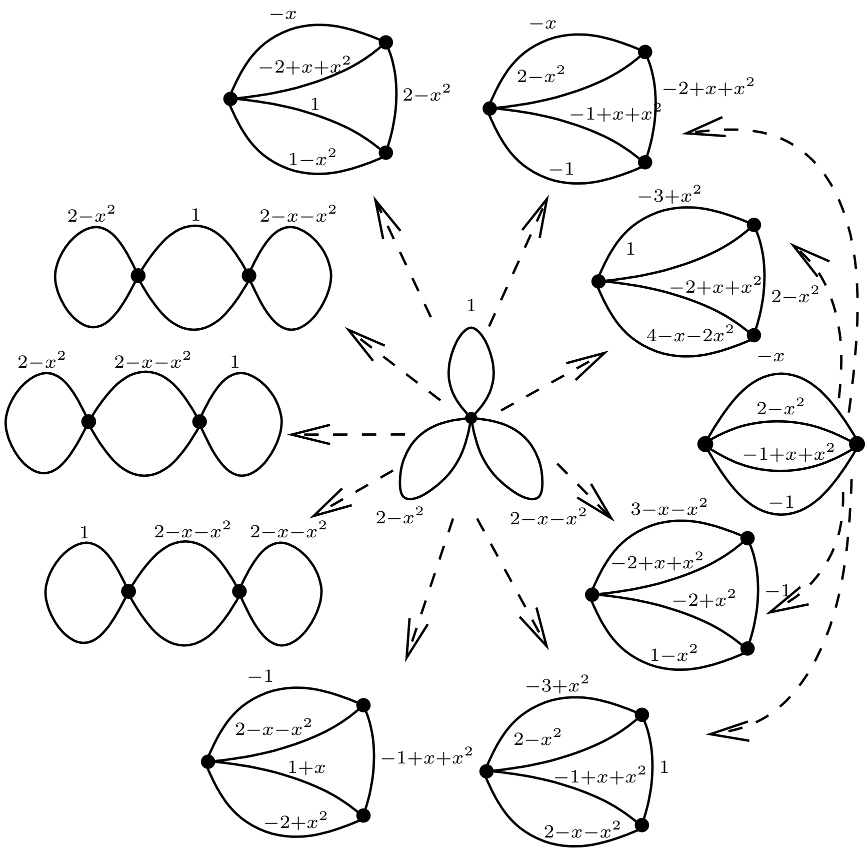

Figure 3: The boundary of the Hilbert modular threefold of discriminant 49.

Example 2: All possible types of admissible strata do occur. We give a list of examples showing that all possible types of boundary strata without separating nodes do occur.

- If the stratum is of type [6], then dim(Span) $= 2$ and $D = 49$ contains an example.
- If the stratum is of type $[5] \times^3 [3]$ then dim(Span) $= 3$. Most cusps contain such an example, for example the unique cusp of the cubic field of discriminant 81.
- If the stratum is of type $[4] \times^4 [4]$ then dim(Span) $= 2$ or dim(Span) $= 3$.

The second case frequently appears, e.g. for $D = 49$. The first case rarely occurs, here is an example: For the field $F = \mathbb{Q}[x]/\langle x^3 - x^2 - 10x + 8\rangle$ with discriminant 961, take the ideal $\mathcal{I} = \mathcal{O}_F$ and the weights $r_1 = 4 - x/2 - x^2/2$, $r_2 = 5 + x/2 - x^2/2$, $r_3 = 1$ and $r_4 = -(r_1 + r_2 + r_3)$.

- If the stratum is of type $[4] \times^2 [4]$ then $\dim(\mathrm{Span}) = 2$. These lie in the boundary of every irreducible stratum, for example in discriminant 49.
- All the remaining possible types of boundary strata without separating nodes have necessarily $\dim(\mathrm{Span}) = 3$ and examples are easily obtained as degenerations of the preceding examples.

Example 3: Ideal classes with no admissible bases. Consider one of the two fields of discriminant 3969, namely $\mathbb{Q}[x]/\langle x^3 - 21x - 35\rangle$. Its ideal class group is of order three. According to a computer search, both of the ideal classes $\mathcal{I}_1 = \langle 7, 7x, x^2 - 14\rangle$ and $\mathcal{I}_2 = \mathcal{I}_1^2 = \langle 7, x, x^2 - 3x - 14\rangle$ do not admit any irreducible boundary strata. But $\mathcal{I}_3 = \mathcal{O}_F = \langle 1, x, x^2 - 3x - 14\rangle$ has a single irreducible boundary stratum given by the weights $r_1 = 1$, $r_2 = x + 3$, $r_3 = x^2 - 2x - 16$.

B Components of the eigenform locus

In this section we show that, in contrast to the quadratic case, that the $\mathcal{E}_{\mathcal{O}}^{\iota} \cong X_{\mathcal{O}}$ is not necessarily connected for cubic orders $\mathcal{O}$.

Recall from §2 that the irreducible components of $X_{\mathcal{O}}$ correspond bijectively to isomorphism classes of proper, rank-two, symplectic $\mathcal{O}$-modules. One example of such a module is $\mathcal{O} \oplus \mathcal{O}^\vee$. We will show that there is such a module M such that for *no* submodule $\mathcal{I}$ of M the sequence

$$0 \to \mathcal{I} \to M \to \mathcal{I}^\vee \to 0,$$

is split, thus M is not isomorphic to $\mathcal{O} \oplus \mathcal{O}^\vee$.

We remark that such examples cannot exist for the ring of integer $\mathcal{O}_F$ since Dedekind domains are projective nor can they exist for $[F : \mathbb{Q}] = 2$ e.g. by structure theorems for rings all whose ideals are generated by two elements [Bas62].

The calculations will be easier to do in the local situation, and if the above sequence was split, it would be also split locally. Choose a totally real cubic number field F and a prime p different from 2 and from 3 such that the residue field k is isomorphic to $\mathbb{F}_{p^3}$. Let K be the completion of F at the prime p. Let R_K be the ring of integers in K and let R be the preimage of the prime field under the surjection $R_K \to k$. We will exhibit an R-module M with the claimed properties. From there it is obvious how to construct a module over $\mathcal{O}$, the preimage of the prime field under $\mathcal{O}_F \to k$, that also has the claimed properties.

For simplicity we suppose moreover that R_K is monogenic, i.e. that $R_K = \mathbb{Z}_p[\theta]/f$ for some cubic polynomial f.

D	$h(D)$	regulator	[6]-components	total 2-dim components
49	1	0.525454	1	2
81	1	0.849287	1	6
148	1	1.662336	3	10
169	1	1.365049	1	14
229	1	2.355454	4	16
257	1	1.974593	2	19
316	1	3.913458	7	26
321	1	2.569259	3	24
......				
961	1	12.195781	19	104
993	1	5.554643	5	69
......				
2597	3	4.795990	5+5+6	51+47+85
......				
3969	3	4.201690	0+0+1	53+57+114
3969	3	12.594188	18+13+18	132 + 144+152
8281	3	15.622299	12 +7+ 12	259 + 224+ 266
8281	3	7.949577	6+6+1	148 + 92+ 179
......				
11884	1	72.746005	79	1008
.....				
20733	5	12.114993	12+21+8+8+12	250+222+138+143+281
.....				
22356	1	49.555997	31	967
22356	1	32.935933	16	751
22356	1	37.348523	23	787
.....				
28165	5	7.935079	4+2+2+4+6	174+125+121+152+337
46548	3	17.990764	6+6+10	289 + 306 + 719
46548	3	21.437334	9+9+16	324 + 337 + 741
.....				
84837	1	129.205864	73	2795
84872	3	60.681694	42+42+54	1121+1064+1373
84889	1	77.482276	32	1913
84893	1	124.912555	85	2610
84905	1	73.229843	27	1723
84925	1	90.776953	37	2112
84945	1	82.760047	50	1879
......				
161249	1	65.942246	16	1882
161753	2	26.530548	10+10	641+ 1084
......				
438492	1	504.944683	228	12265

Figure 4: The number of boundary components for given discriminant D

Lemma B.1. *We have*

$$R_K = R_K^\vee \subset_{p^2} R^\vee \subset_p p^{-1}R_K,$$

where the subscripts denote the index. In fact,

$$R^\vee = \{r \in p^{-1}R_K \,|\, \mathrm{Tr}(pr) \equiv 0 \mod (p)\}.$$

More precisely, there exists a $\mathbb{Z}_p$-basis $\{1, x, y\}$ of R_K which is orthogonal with respect to the trace pairing. Using this basis we have

$$R = \langle 1, px, py \rangle_{\mathbb{Z}}, \quad R_K^\vee = \left\langle 1, \frac{x}{p}, \frac{y}{p} \right\rangle_{\mathbb{Z}}.$$

Proof. The ring $R_K^\vee$ is generated by $\theta^i/f'(\theta)$ for $i = 0, 1, 2$. Since $f'(\theta)$ is a unit in R_K by the hypothesis on the residue field, we obtain $R_K = R_K^\vee$.

Suppose $s \in p^{-1}R_K$. We use that by definition any $y \in R$ is congruent mod (p) to $z \in \mathbb{Z}$. Thus since

$$\mathrm{Tr}(rs) \equiv z\,\mathrm{Tr}(r) \mod (p)$$

we conclude that $r \in R^\vee$ if and only if $\mathrm{Tr}(pr) = 0$ (using $p \neq 3$). ∎

Lemma B.2. *The quotients R_K/pR_K, $R^\vee/pR^\vee$ and R/pR are three-dimensional as $\mathbb{F}_p$ vector spaces but different as R-modules:*

- *R_K/pR_K splits into a direct sum of $\langle 1 \rangle$ and $\langle x, y \rangle$, orthogonal with respect to the trace pairing.*
- *R/pR has the irreducible R-submodule $\langle px, py \rangle$ and the corresponding sequence is not split.*
- *$R^\vee/pR^\vee$, as the dual of the preceding module, has the quotient R-module $\langle x/p, y/p \rangle$, and the corresponding sequence is not split.*

Proof. The structure of R_K/pR_K is obvious. Suppose $1 + p(ax + by)$ generates an R-submodule of R/pR of dimension one over $\mathbb{F}_p$. Multiplying by px we see that this submodule contains also px, We thus obtain a contradiction. ∎

Lemma B.3. *We can calculate* Ext*-groups as follows:*

$$\begin{aligned}
&\mathrm{Ext}^1_R(R^\vee, R) = \mathrm{Hom}_R(R^\vee, R/pR)/\mathrm{Hom}_R(R^\vee, R) \cong \mathbb{F}_p \\
&\mathrm{Ext}^1_R(R^\vee, R_K) = \mathrm{Hom}_R(R^\vee, R_K/pR_K)/\mathrm{Hom}_R(R^\vee, R_K) \cong \mathbb{F}_p \qquad \text{(B.1)} \\
&\mathrm{Ext}^1_R(R^\vee, R^\vee) = 0
\end{aligned}$$

Proof. The short exact sequence of multiplication by p gives a long exact sequence

$$\mathrm{Hom}_R(R^\vee, M) \to \mathrm{Hom}_R(R^\vee, M/pM) \to \mathrm{Ext}^1(R^\vee, M) \to \mathrm{Ext}^1(R^\vee, M),$$

where the last map is induced by multiplication by p. Under the second map the image of $\mathrm{Hom}_R(R^\vee, M/pM)$ is p-torsion and thus $\mathrm{Ext}^1(R^\vee, M)$ is p-torsion as well.

We first deal with the case $M = R$. Obviously p^2R_K is contained in $\mathrm{Hom}(R^\vee, R)$ and we claim they are equal. If such a homomorphism was given by multiplication with an element $s \notin p^2R_K$, take $t = x/p \in R$ where x is as above. Then $ts \notin pR_K$ and its reduction is not in the prime field, since the reductions of $\{1, x, y\}$ are linearly independent over $\mathbb{F}_p$. This contradiction proves the claim.

First we claim that

$$\mathrm{Hom}_R(R^\vee, R/pR) \cong \mathrm{Hom}_{\mathbb{F}_p}(k/\mathbb{F}_p, \mathrm{Ker}(\mathrm{Tr})),$$

where we consider $\mathrm{Ker}(\mathrm{Tr}) \subset k$. A homomorphism from $R^\vee$ to R/pR factors through $R^\vee/pR^\vee$. By Lemma B.2 there are no isomorphisms between them, in fact the classification of quotient resp. submodules in this lemma shows more precisely that such a homomorphism factors through an element in $\mathrm{Hom}_{\mathbb{F}_p}(k/\mathbb{F}_p, \mathrm{Ker}(\mathrm{Tr}))$. Both on the quotient module $\langle x/p, y/p\rangle \cong k/\mathbb{F}_p$ and on the submodule $\langle px, py\rangle \cong \mathrm{Ker}(\mathrm{Tr})$, the ring R acts through its quotient $\mathbb{F}_p$ so that indeed every $\mathbb{F}_p$-homomorphism is an R-homomorphism. Multiplication by p^2R defines a subspace isomorphic to k inside $\mathrm{Hom}_{\mathbb{F}_p}(k/\mathbb{F}_p, \mathrm{Ker}(\mathrm{Tr}))$. This concludes the second isomorphism of the second claim.

Second we look at the case $M = R_K$. Now $\mathrm{Hom}(R^\vee, R_K) \cong pR$ and elements in $\mathrm{Hom}_R(R^\vee, R_K/pR_K)$ factor through $\mathrm{Hom}_{\mathbb{F}_p}(k/\mathbb{F}_p, \mathrm{Ker}(\mathrm{Tr}))$ using the submodule structure of the finite R-modules determined in Lemma B.2.

The last statement follows by the same reasoning. ■

Proposition B.4. *Let $0 \to R \to M \to R^\vee \to 0$ be a symplectic extension corresponding to a non-trivial element in $\mathrm{Ext}^1_R(R^\vee, R)$. Then M is a proper R-module. Moreover, M has a unimodular symplectic structure and the R-action is by self-adjoint endomorphisms. M is not a direct sum of two R-modules of rank one.*

Proof. The trace pairing on R and $R^\vee$ induces a symplectic and unimodular pairing on M. The R-submodule R of M is isotropic for this alternating pairing. Thus if M is an $\widetilde{R}$-module for some ring $\widetilde{R}$ containing M and acting by self-adjoint endomorphisms, then R is also an $\widetilde{R}$-module. This implies $\widetilde{R} = R$, i.e. that M is a proper R-module.

It remains to show that M is not a direct sum. If it is, then $M \cong \mathfrak{a} \oplus \mathfrak{a}^\vee$. If we apply $\mathrm{Hom}(R^\vee, \cdot)$ to the extension defining M, we obtain an exact sequence

$$\mathrm{Hom}_R(R^\vee, R^\vee) \to \mathrm{Ext}^1_R(R^\vee, R) \to \mathrm{Ext}^1_R(R^\vee, M) \to \mathrm{Ext}^1_R(R^\vee, R^\vee).$$

The first map is a non-zero map $R \to \mathbb{F}_p$ by the fact that M was constructed as a non-trivial extension. The hypothesis on M implies that

$$\mathrm{Ext}^1_R(R^\vee, M) = \mathrm{Ext}^1_R(R^\vee, \mathfrak{a}) \oplus \mathrm{Ext}^1_R(R^\vee, \mathfrak{a}^\vee) \cong \mathbb{F}_p.$$

Since $\mathrm{Ext}^1_R(R^\vee, R^\vee) = 0$ it remains to show that at least one of the two groups $\mathrm{Ext}^1_R(R^\vee, \mathfrak{a})$ and $\mathrm{Ext}^1_R(R^\vee, \mathfrak{a}^\vee)$ is non-zero. The Ext-groups don't change if we replace $\mathfrak{a}$ by $p\mathfrak{a}$. Under this equivalence the pair $(\mathfrak{a}, \mathfrak{a}^\vee)$ is either $(R, R^\vee)$, $(R^\vee, R)$ or (R_K, R_K). Thus the claim follows from Lemma B.3. ■

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