LINEAR MANIFOLDS IN THE MODULI SPACE OF ONE-FORMS

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Abstract. We study closures of $\mathrm{GL}_2^+(\mathbb{R})$-orbits in the total space ΩM_g of the Hodge bundle over the moduli space of curves under the assumption that they are algebraic manifolds.

We show that, in the generic stratum, such manifolds are the whole stratum, the hyperelliptic locus or parameterize curves whose Jacobian has additional endomorphisms. This follows from a cohomological description of the tangent bundle to ΩM_g. For non-generic strata similar results can be shown by a case-by-case inspection.

We also propose to study a notion of 'linear manifold' that comprises Teichmüller curves, Hilbert modular surfaces and the ball quotients of Deligne and Mostow. Moreover, we give an explanation for the difference between Hilbert modular surfaces and Hilbert modular threefolds with respect to this notion of linearity.

Introduction

The total space ΩM_g of the Hodge bundle over M_g or equivalently the space of pairs (X, ω) of a Riemann surface X and a holomorphic one-form $\omega \in \Gamma(X, \Omega^1_X)$ admits a linear structure. That is, integration of one-forms along a basis of the first homology relative to zeroes of ω gives locally a map to from ΩM_g to $\mathbb{C}^N$ and different choices of the basis yields a linear coordinate change, in fact defined over $\mathbb{R}$. We review in Section 1 equivalent ways of interpreting a linear structure in terms of connections or local systems.

Due to the linear coordinate changes defined over $\mathbb{R}$, there is a natural action of $\mathrm{GL}_2^+(\mathbb{R})$ on ΩM_g, or rather the complement ΩM_g^* of the zero section. A guiding question consists in the classification of the closures of these orbits. The situation seems to have several similarities to Ratner's theorem on orbit closures of homogeneous manifolds. Indeed for $g = 2$ the classification has been achieved by McMullen ([McM07]).

Here, to analyze what happens for $g \geq 3$, we propose to split the orbit closure problem into three subquestions. First, to show that orbit closures are indeed complex manifolds. Second, to show that these complex manifolds are in fact algebraic. Third, to classify these algebraic manifolds. As Kontsevich observed, orbit closures that are complex manifolds are submanifolds of ΩM_g, or more precisely of its strata, that inherit a linear structure defined over $\mathbb{R}$. See Section 1 for details. The converse also holds: If a submanifold of a stratum inherits a linear structure defined over $\mathbb{R}$, it is $\mathrm{GL}_2^+(\mathbb{R})$-invariant. Thus, the third problem may be translated into a purely algebro-geometric problem, interesting independently of the orbit closure question. The manifolds with linear structure include both Hilbert modular surfaces ([McM07]) and, as we will show, ball quotients ([DeMo86]).

The purpose of the present paper is to stress the role of the hyperelliptic locus in this classification problem. One of our main results is:

Date: November 19, 2007.

Theorem 4.1. Suppose that B is an algebraic submanifold of the generic stratum $S = \Omega M_g(1, \ldots, 1)$ with linear structure. Then there are three possibilities:

(i) B is a connected component of S.
(ii) $g \geq 3$ and B is the preimage in S of the hyperelliptic locus in M_g.
(iii) B parameterizes curves with a Jacobian whose endomorphism ring is strictly larger than $\mathbb{Z}$.

For non-generic strata a similar statement holds true, maybe up to some exceptional linear submanifolds in strata whose projection to M_g has small fiber dimension. We do a case by case analysis for $g = 3$ in Section 4.

In order to prove Theorem 4.1 we describe in Section 2 the tangent bundle cohomologically. This has been done pointwise already in [HuMa79] and yields, locally, the period coordinates used throughout when dealing with the $\mathrm{GL}_2^+(\mathbb{R})$-action. The advantage of the global viewpoint consists of keeping track of the twist by $\mathcal{O}(1)$ on ΩM_g, or rather on a projective completion. We thus obtain some information on the foliation of ΩM_g by constant absolute periods via the multiplication map of one-forms. From this viewpoint, the hyperelliptic locus figures among the above list, precisely because the multiplication map on one-forms fails to be surjective there for $g \geq 3$.

This paper grew out of the attempt to understand the difference, discovered in [McM03], between the linearity of (the eigenform bundle) over a Hilbert modular surface and the non-linearity of the Hilbert modular threefold parameterizing abelian varieties of dimension three with real multiplication by $\mathbb{Z}[\zeta_7 + \zeta_7^{-1}]$.

For Hilbert modular threefolds we show that there are two obstacles to linearity (see Theorem 5.5). The first one is related to properties of the multiplication map. It is already used in [McM03]. We show that that this phenomenon can arise only at the hyperelliptic locus. Geometrically it says that the normal vector to the hyperelliptic locus coincides with one of the three natural foliations.

The second one is related to the intersection of the eigenlocus over Hilbert modular threefold with the foliation by constant absolute periods. Neither of the problems appears in genus $g = 2$ and this observation enables us to reprove linearity in genus $g = 2$ (Proposition 5.2, [McM03] Theorem 7.1).

It seems very likely that the generic part of a Hilbert modular threefold is never linear. But a proof of this might first need to be able to decide in which stratum a Hilbert modular threefold lies generically, depending on the endomorphism ring. At present, this question remains open.

In Section 6 we examine which manifolds we get if we replace linearity defined over $\mathbb{R}$ by just linearity. As remarked above, the ball quotients of Deligne and Mostow fit into this picture. But contrary to the case of linear transition functions defined over $\mathbb{R}$, pathological cases and manifolds with linear structure unrelated to uniformization occur as well. We propose a definition of a linear manifold that includes precisely those manifolds with a linear structure related to the uniformization of the manifold. The condition we demand additionally is the existence of a compactification to which the linear structure "extends" in terms of a surjection of some tangent bundle to a Deligne extension of a local system.

Already in the case of 2-dimensional linear manifolds the definition has surprising consequences. Using the cohomological description we quickly reprove in Section 7 the classification result of McMullen in genus 2, of course under the general assumption of algebraicity.

But there are, even in genus 2, two-dimensional linear manifolds that are not canonical lifts of Teichmüller curves, since the linear structure is not defined over $\mathbb{R}$.

In Section 8 we apply the same classification techniques to the hyperelliptic locus $\mathcal{H}$ inside the odd spin component of the stratum $\Omega M_3(2,2)$. This locus is maybe the most simple, besides $g=2$, to study $\mathrm{GL}_2^+(\mathbb{R})$-orbit closures. See also [HLM06]. One consequence is:

Corollary 8.2. Let Δ be a Teichmüller disc generated by a pair $(X_0,\omega_0) \in \mathcal{H}$ which is stabilized by a pseudo-Anosov diffeomorphism with trace field of degree 3 over $\mathbb{Q}$. If the closure of the orbit $\mathrm{GL}_2^+(\mathbb{R}) \cdot (X_0,\omega_0)$ is an algebraic manifold, then it is either the canonical lift of a Teichmüller curve to $\mathcal{H}$ or it is as big as possible, i.e.

$$\overline{\mathrm{GL}_2^+(\mathbb{R}) \cdot (X_0,\omega_0)} = \mathcal{H}.$$

Acknowledgements. The author had many fruitful discussions with Eckart Viehweg on linear manifolds and Hilbert modular varieties. The author thanks him and Hélène Esnault a lot for their help and the participants of the conference 'Outer space and Teichmüller space', CIRM (2007), and Curt McMullen for their comments. The author also thanks the referees for their comments.

1. Manifolds with affine structures and linear structures

Affine and linear structures

An *affine structure* on a complex manifold M (say of dimension d) is an atlas of charts $(U_i, \varphi_i : U_i \to \mathbb{C}^d)$ all whose transition functions $\varphi_j \circ \varphi_i^{-1}$ are affine maps. For convenience we suppose that the atlas is maximal for that property.

We recall below that it is equivalent to have a flat connection $\nabla : \Omega_M \to \Omega_M \otimes \Omega_M$ on M which is symmetric, i.e. sends closed forms to $\mathrm{Sym}^2\Omega_M$. Another equivalent definition is that there is a $\mathbb{C}$-local system $\mathbb{L}$ on M such that $T_M = \mathbb{L} \otimes_{\mathbb{C}} \mathcal{O}_M$ which is torsion-free, i.e. such that the corresponding connection ∇^{T_M} satisfies

$$\nabla^{T_M}_X Y - \nabla^{T_M}_Y X = [X,Y]$$

for any local sections X, Y of T_M.

To see that the definitions are equivalent, start with a manifold M with an affine structure. The $\mathbb{C}$-valued functions that are locally affine form a local system Aff_M of rank $d+1$ that contains the constant functions. The quotient $\mathrm{Aff}_M/\mathbb{C}$ is a local system whose associated vector bundle is the cotangent bundle. The symmetry of this connection is checked locally.

Conversely starting with a connection ∇, consider the functions whose total differential is flat for ∇. Locally, a complement of the constant functions defines a chart. These charts fit to an affine structure if the connection is symmetric.

The equivalence between ∇ being symmetric and ∇^{T_M} being torsion free is immediate from the definition of the dual connection. We will drop the superscript T_M from the connection ∇ in the sequel.

A *linear structure* (sometimes also called *radiant affine structure*) is defined by an atlas as above, whose transition functions are linear. This is equivalent to the monodromy group of $\mathbb{L}$ being equal to the *holonomy group*, the monodromy group of Aff_M. Again equivalently, $\mathbb{C}$ has a complement in Aff_M. A manifold M has a *linear structure defined over* $\mathbb{R}$ or *is locally defined by* $\mathbb{R}$*-linear equations* if the transition maps are in $\mathrm{GL}_d(\mathbb{R}) \subset \mathrm{GL}_d(\mathbb{C})$.

We will reserve the expression 'linear (sub)manifold' for a manifold with linear structure plus additional conditions related to the boundary, see Section 6.

Lemma 1.1. *If $B \subset M$ is a submanifold of an affine manifold M and the connection ∇ restricts to a connection $\nabla_B : \Omega_B \to \Omega_B \otimes \Omega_B$, then B has also an affine structure. If moreover M has a linear structure, then B has a linear structure.*

Proof. Since ∇_B is obtained by restriction from ∇, it is automatically flat and torsion free. The second claim is obvious. $\square$

Infinitesimal $\mathrm{GL}_2^+(\mathbb{R})$-action

A complex manifold M with a linear structure defined over $\mathbb{R}$ automatically has a natural infinitesimal $\mathrm{GL}_2^+(\mathbb{R})$-action. I.e. at each point $x \in M$ there is a neighborhood of $\mathrm{Id} \in \mathrm{GL}_2^+(\mathbb{R})$ that acts on a neighborhood of x in the following way: On the linear charts $\varphi_i(x) \in \mathbb{C}^d = \mathbb{R}^d \otimes_{\mathbb{R}} \mathbb{C}$ the group $\mathrm{GL}_2^+(\mathbb{R})$ acts linearly on $\mathbb{C} \cong \mathbb{R}^2$. This action commutes with the transition functions, since they are defined over $\mathbb{R}$, they act on the first factor of the tensor product and since $\mathbb{R}$, diagonally embedded in $\mathrm{GL}_d(\mathbb{R})$ resp. $\mathrm{GL}_2^+(\mathbb{R})$, is central in both cases. This argument also implies that the action, if defined, does not depend on the chart chosen.
One cannot hope to globalize this action without further knowledge on M. For example one could have removed a closed subset, but not an orbit, from a manifold where the global action was defined.

The Hodge bundle over M_g and its strata.

Let ΩM_g denote (the total space of) the Hodge bundle over the moduli space M_g of curves of genus g and let ΩM_g^* denote the complement of the zero section. Points in ΩM_g^* correspond to pairs (X, ω) of a curve X of genus g and a non-zero holomorphic one-form on X.
There is a natural $\mathrm{GL}_2^+(\mathbb{R})$-action on ΩM_g^* defined by post-composing the local charts given by integrating ω with the linear map. See [MaTa02] or [Zo06] for recent surveys. This action respects the strata $\Omega M_g(k_1, \ldots, k_r)$ consisting of pairs (X, ω) such that the zeroes of ω are of the form $\sum_{i=1}^r k_i P_i$ for disjoint points P_i on X. The connected components of the strata have been completely determined in [KoZo03].

Period map.

Let $S_{T_g} \subset \Omega T_g$ be the pullback of a stratum S the universal bundle to the Teichmüller space T_g. On a simply connected open subset U of S_{T_g} choose a basis of the relative homology. Integrating the one-from along this basis gives a map, called the *period map*, that attaches to a pair (X, ω) a point in $\mathbb{C}^N$ for $N = 2g - 1 + r$. It is well-known that the period map is a local biholomorphism ([Ve86]). One may deduce this also from the considerations in Section 2. The mapping class group acts linearly on the relative periods. Consequently, the period map provides the stratum with a linear structure.

The mapping class group maps absolute periods to absolute periods. Hence, forgetting relative periods gives locally a map $\mathbb{C}^N \to \mathbb{C}^{2g}$, equivariant with respect to the action of the mapping class group. Thus, the strata have a natural foliation by leaves of complex dimension $r - 1$, called the *foliation by constant absolute periods*.

Our starting point is the following observation. Folklore attributes the application of it to the period coordinates of a $\mathrm{GL}_2^+(\mathbb{R})$-invariant submanifold to Kontsevich:

Proposition 1.2. *A* $\mathrm{GL}_2^+(\mathbb{R})$*-invariant* B *closed analytic subspace in* $\mathbb{C}^N$ *is linear and defined locally by* $\mathbb{R}$*-linear equations.*

Proof: Suppose that $B \subset \mathbb{C}^N$ is a hypersurface given by a power series f with complex coefficients in the variables $z_1 = x_1 + iy_1, \ldots, z_N = x_N + iy_N$. We claim that f is a homogeneous polynomial of some degree m. To show this assume the contrary. Then for some $\lambda \neq 1$, the power series f and $f(\lambda z_1, \ldots, \lambda z_N)$ are linearly independent. This contradicts that B is a hypersurface and invariant under the diagonal subgroup in $\mathrm{GL}_2^+(\mathbb{R})$.

The natural action of $\mathrm{GL}_2^+(\mathbb{R})$ on points of $\mathbb{C}^N \cong \mathbb{R}^N \otimes_{\mathbb{R}} \mathbb{R}^2$ by the linear action on the second factor corresponds on the level of polynomial defining closed subspaces to the action

$$g \cdot f := f \circ g^{-1} := f((\tilde{a}x_1 + \tilde{b}y_1) + i(\tilde{c}x_1 + \tilde{d}y_1), \ldots, (\tilde{a}x_N + \tilde{b}y_N) + i(\tilde{c}x_N + \tilde{d}y_N)),$$

where $g^{-1} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}$. Since f is a homogeneous polynomial of degree m, the $\mathbb{R}$-vector space

$$V = \left\langle g \cdot f,\, g \in \mathrm{GL}_2^+(\mathbb{R}) \right\rangle$$

of polynomials with complex coefficients contains a vector of weight m (as a $\mathrm{GL}_2^+(\mathbb{R})$-module). Hence $\dim_{\mathbb{R}} V \geq m+1$. We claim that this implies that that the real codimension of B in $\mathbb{C}^N$ is at least $m+1$. Since B is a complex hypersurface we conclude that f is linear and moreover that we can choose f to have real coefficients.

To prove the claim, suppose that the zero set in $\mathbb{R}^{2N}$ of the polynomials in V in the variables x_i and y_i has codimension less than $m+1$. Tensoring with $\mathbb{C}$, we conclude that the zero set in $\mathbb{C}^{2N}$ of the polynomials in V has codimension less than $m+1$ and we obtain a contradiction.

We use the hypersurface case to argue by induction on both N and $\mathrm{codim}_{\mathbb{C}^N}(B)$. The map $\varphi : \mathbb{C}^N \to \mathbb{C}^{N_1}$ by forgetting the last coordinate is $\mathrm{GL}_2^+(\mathbb{R})$-equivariant. Hence the image $\varphi(B)$ has a linear structure defined over $\mathbb{R}$ by induction hypothesis. If we let W be the preimage $\varphi(B)$ then $W \cong \mathbb{C}^{N'}$ for some N'. Moreover, the embedding $\mathbb{C}^{N'} \to \mathbb{C}^N$ is given by $\mathbb{R}$-linear equations. On the other hand, B either equals W or a hypersurface in W and we can again apply the induction hypothesis. $\square$

We emphasize that for a submanifold B in a manifold S with linear structure (usually a stratum) we say B *has a linear structure* as shorthand for 'B inherits a linear structure from S' or 'B is defined locally by linear equations in period coordinates'. That is, we are never interested in intrinsic linear structures on B.

The converse of Proposition 1.2 holds in the strata of ΩM_g.

Proposition 1.3. *Let* $B \subset \Omega M_g(k_1, \ldots, k_r)$ *be a closed, linear submanifold in some stratum. If* B *is defined by* $\mathbb{R}$*-linear equations then* B *is* $\mathrm{GL}_2^+(\mathbb{R})$*-invariant.*

Proof. On $\Omega M_g(k_1, \ldots, k_r)$ we have an action of $\mathrm{GL}_2^+(\mathbb{R})$, not only of a neighborhood of $\mathrm{Id} \in \mathrm{GL}_2^+(\mathbb{R})$. By the hypothesis this action respects B. $\square$

Remark 1.4. We a priori work in the category of complex analytic manifolds. Some of the stronger classification results below only work under the hypothesis that these manifolds are algebraic. Dividing by the $\mathbb{C}^*$-action on ΩM_g^* one observes that these two kinds of objects coincide, if one can show that the closures of objects in question (linear manifolds in strata) in some projective compactification are still manifolds.

Orbifolds.
The moduli space of curves M_g should be considered as a complex orbifold in the sequel. To avoid technicalities and since all our results will be independent of passing to a finite unramified cover of M_g we fix a suitable one (say $M_g^{[n]}$ with some level-$[n]$-structure), which is a smooth manifold and over which the universal family of curves $f : X \to M_g$ exists. We nevertheless keep the symbol M_g for simplicity. We also denote the pullback of the universal family to any manifold over M_g by f, hoping not to create confusion.

However with this notation the Torelli map $t^{[n]} : M_g^{[n]} \to A_g^{[n]}$ is no longer injective. Instead it is a $2:1$ covering ramified precisely over the hyperelliptic locus, see diagram (9).

Variation of Hodge structures
We will mainly be concerned with variations of Hodge structures (VHS) of weight one on an algebraic manifold B, sometimes completed by a normal crossing divisor S to $\overline{B}$. A weight one VHS consists of a $\mathbb{Z}$-local system $\mathbb{V}$ plus a filtration of vector bundles $\mathbb{V}^{(1,0)} \subset (\mathbb{V} \otimes \mathcal{O}_B)_{\mathrm{ext}} =: \mathcal{V}$, where ext denotes the Deligne extension ([De70]). The only compatibility condition of the filtration we make use of here is $\mathbb{V}^{(0,1)} := \mathcal{V}/\mathbb{V}^{(1,0)} \cong (\mathbb{V}^{(1,0)})^\vee$. With this data one defines the Higgs field to be the composition

$$\theta : \mathbb{V}^{(1,0)} \to \mathcal{V} \to \mathcal{V} \otimes \Omega^1_B(\log S) \to \mathcal{V}/\mathbb{V}^{(1,0)} \otimes \Omega^1_B(\log S).$$

If the VHS comes from a family of smooth curves $f : X \to B$, i.e. if $\mathbb{V} = R^1 f_* \mathbb{Z}$ and $\mathbb{V}^{(1,0)} = f_* \omega_{X/B}$, we can use duality to obtain from θ a map

$$\mathrm{Sym}^2(\mathbb{V}^{(1,0)}) \to \Omega^1_B(\log S),$$

which is well-known to factor through $f_* \omega^2_{X/B}$. The dual of this factorization is usually called the *Kodaira-Spencer map.*

2. Cohomological description of several tangent bundles

The goal of this technical section is to describe the (co)tangent bundle to strata in ΩM_g by hypercohomology sheaves. It will be important to do this in the relative setting not just pointwise. Only in that way we can keep track of the twist by $\mathcal{O}_{\mathbb{P}}(1)$. Although this bundle admits a trivialization over the strata, this twist helps to identify certain bundle maps as induced by multiplication maps on one-forms. Properties of the multiplication map will then be exploited in the subsequent sections.

2.1. **Tangent bundle to the completed one-form bundle.** Our first aim is to describe the tangent bundle to ΩM_g. For technical reasons we complete this bundle to a projective bundle, rather than taking the projectivisation $\mathbb{P}\Omega M_g$. The reason is that the latter has no longer a linear structure. We start fixing some conventions.

For a vector bundle $\mathcal{E}$ on Y let $\mathbb{P}$ be the projective bundle $\mathrm{Proj}(\mathrm{Sym}(\mathcal{E}^\vee))$ with projection $p : \mathbb{P} \to Y$. There is canonical map $\mathcal{O}_{\mathbb{P}}(-1) \to p^*\mathcal{E}$ and the Euler exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}} \to p^*\mathcal{E}(1) \to T_{\mathbb{P}/Y} \to 0.$$

Suppose that $\mathcal{E} = \mathcal{E}' \oplus \mathcal{O}_Y$. Then the hyperplane at infinity is defined by applying Proj to the inclusion $\mathcal{O}_Y \to \mathcal{E}' \oplus \mathcal{O}_Y$ on the second summand. The exact sequence

$$0 \to \Omega^1_{\mathbb{P}/Y} \to \Omega^1_{\mathbb{P}/Y}(\log \infty) \to \mathcal{O}_\infty \to 0 \tag{1}$$

together with the Euler sequence implies that $\Omega^1_{\mathbb{P}/Y}(\log\infty) \cong p^*\mathcal{E}'^\vee(-1)$.

We apply the above remarks in the case $Y = M_g$ and $\mathcal{E} = (f_*\Omega^1_{X/M_g} \oplus \mathcal{O}_{M_g})$, where $f : X \to M_g$ is the universal family of curves. Moreover we abuse the letter f also for pullback families, e.g. $f : X_\mathbb{P} \to \mathbb{P}$.

Theorem 2.1. *The tangent bundle of* $\mathbb{P} = \mathbb{P}((f_*\Omega^1_{X/M_g} \oplus \mathcal{O}_{M_g})^\vee)$ *sits in an exact sequence*

$$0 \to T_{\mathbb{P}/M_g}(-\log\infty) \to T_\mathbb{P}(-\log\infty) \to p^*T_{M_g} \to 0.$$

The extension comes form

$$T_\mathbb{P} = R^1 f_*(\varphi : (\Omega^1_{X_\mathbb{P}/\mathbb{P}})^\vee \to f^*\mathcal{O}_\mathbb{P}(1) \otimes \Omega^1_{X_\mathbb{P}/\mathbb{P}})$$

and the identifications

$$T_{\mathbb{P}/M_g}(-\log\infty) = f_*\Omega^1_{X_\mathbb{P}/\mathbb{P}}(1) \textit{ and } p^*T_{M_g} = R^1 f_*(\Omega^1_{X_\mathbb{P}/\mathbb{P}})^\vee$$

The map φ *is given by*

$$(\Omega^1_{X_\mathbb{P}/\mathbb{P}})^\vee \xrightarrow{\text{can}} f^*\mathcal{O}_\mathbb{P}(1) \xrightarrow{\quad d \quad} f^*\mathcal{O}_\mathbb{P}(1) \otimes \Omega^1_{X_\mathbb{P}}(\log f^{-1}(\infty)) \longrightarrow f^*\mathcal{O}_\mathbb{P}(1) \otimes \Omega^1_{X_\mathbb{P}/\mathbb{P}} .$$

Proof. We first show that the claims are true locally at some point $(X^0, \omega^0) \in \mathbb{P}(\mathbb{C})$. In this part we basically follow [HuMa79], who deal with the case of quadratic differentials. See also [We83].
Let X_ε be a deformation of X^0 over $\Delta := \text{Spec } \mathbb{C}[\varepsilon]/\varepsilon^2$ and let $\omega_\varepsilon \in \Gamma(X_\varepsilon, \Omega^1_{X_\varepsilon/\Delta})$ be a deformation of ω^0. We want to associate with these data a class in $H^1(X_0, T_{X^0} \to \Omega^1_{X_0})$, where the map is the Lie derivative, i.e. contracts a tangent vector against ω^0 and then takes the exterior derivative of the resulting function.
Let $\{U_\alpha = \text{Spec } A_\alpha \times \Delta\}$ be a covering of X_ε such that X_ε is given by transition functions $\psi_{\alpha\beta}$ with

$$\psi_{\alpha\beta}(\varepsilon) = \varepsilon, \quad \psi_{\alpha\beta}(f) = f + \varepsilon D_{\alpha\beta}(f).$$

Since $\psi_{\alpha\beta}$ are ring homomorphisms, $D_{\alpha\beta}$ is a $\mathbb{C}$-derivation with values in $A_{\alpha\beta}$. The transition functions of $\Omega^1_{X_\varepsilon/\Delta}$ are given by

$$df + \varepsilon dg \mapsto d(f + \varepsilon D_{\alpha\beta}(f)) + \varepsilon d(g + \varepsilon D_{\alpha\beta}(g)) = df + \varepsilon dg + \varepsilon dD_{\alpha\beta}(f).$$

If we describe the section as $(\omega_\varepsilon)_\alpha = \omega^0_\alpha + \varepsilon ds_\alpha$, the gluing condition of the bundle is

$$ds_\alpha - ds_\beta = dD_{\alpha\beta}(\omega^0_\alpha).$$

Hence the pair $(\{D_{\alpha\beta}\}, \{ds_\alpha\}) \in C^1(U_\alpha, T_X) \oplus C^0(U_\alpha, \Omega^1_{X^0})$ forms a 1-cochain of the desired complex. It is straightforward to see that the pair is indeed a cochain and that different choices modify the cochain only up to a coboundary (see loc. cit.). One obtains in this way an identification of the tangent space

$$T_{(X^0,\omega^0)}\mathbb{P} \cong H^1(X_0, T_{X^0} \to \Omega^1_{X_0}).$$

What remains to prove the proposition is to show that this identification fits together in the bundle. In order to do so we define φ as the composition map

$$(\Omega^1_{X_\mathbb{P}/\mathbb{P}})^\vee \xrightarrow{\text{can}} f^*\mathcal{O}_\mathbb{P}(1) \xrightarrow{\quad d \quad} f^*\mathcal{O}_\mathbb{P}(1) \otimes \Omega^1_{X_\mathbb{P}}(\log f^{-1}(\infty)) \longrightarrow f^*\mathcal{O}_\mathbb{P}(1) \otimes \Omega^1_{X_\mathbb{P}/\mathbb{P}}$$

and consider it as a complex $\mathfrak{C}^\bullet$ in degrees 0 and 1. Here "can" is the contraction against the universal section, i.e, the dual of

$$f^*\mathcal{O}_{\mathbb{P}}(-1) \to f^*(p^*(f_*\Omega^1_{X/M_g} \oplus \mathcal{O}_{M_g})) \to f^*(p^*f_*\Omega^1_{X/M_g}) = f^*f_*\Omega^1_{X_{\mathbb{P}}/\mathbb{P}} \to \Omega^1_{X_{\mathbb{P}}/\mathbb{P}}$$

and d is the exterior derivative.

We will define a map $T_{\mathbb{P}} \to R^1f_*(\mathfrak{C}^\bullet)$ that boils down to the one described by deformation theory for any thick point $\Delta \to \mathbb{P}$. All we need is to construct a map $\widetilde{\varphi}$, that fits into the following exact sequence of complexes

$$\begin{tikzcd}
0 \arrow[r] & T_{X_{\mathbb{P}}/\mathbb{P}} \arrow[r] \arrow[d, "\varphi"] & T_{X_{\mathbb{P}},f}(-\log f^{-1}(\infty)) \arrow[r] \arrow[d, "\widetilde{\varphi}"] & f^{-1}T_{\mathbb{P}}(-\log\infty) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & f^*\mathcal{O}_{\mathbb{P}}(1) \otimes \Omega^1_{X_{\mathbb{P}}/\mathbb{P}} \arrow[r, equal] & f^*\mathcal{O}_{\mathbb{P}}(1) \otimes \Omega^1_{X_{\mathbb{P}}/\mathbb{P}} \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

where the upper line is the restriction of the standard exact sequence from $f^*T_{\mathbb{P}}(-\log\infty)$ to $f^{-1}T_{\mathbb{P}}(-\log\infty)$. Once we have established this diagram the coboundary map associated to $Rf_*(\cdot)$ will provide the desired map. Observe that after applying $Rf_*(\cdot)$ there is no difference between $f^*T_{\mathbb{P}}(-\log\infty)$ and $f^{-1}T_{\mathbb{P}}(-\log\infty)$, but to construct $\widetilde{\varphi}$ the difference is essential.

For any small enough open set U elements of $T_{X_{\mathbb{P}},f}(-\log f^{-1}(\infty))$ are of the form $\lambda t + \tau$ for $\tau \in f^{-1}T_{\mathbb{P}}(-\log\infty)(U)$, $t \in T_{X_{\mathbb{P}}/\mathbb{P}}(U)$ and $\lambda \in \mathcal{O}_{X_{\mathbb{P}}}(U)$. Contraction of $\lambda t + \tau$ against an element of $\Omega^1_{X_{\mathbb{P}}/\mathbb{P}}$ gives a function on U, well-defined only up to f^{-1} of functions on $\mathbb{P}$. Hence $d\circ\mathrm{can}$ is well-defined up to a section of $f^*\mathcal{O}_{\mathbb{P}}(1)\otimes f^{-1}(\Omega^1_{\mathbb{P}}(\log\infty))(U)$. Consequently, the composition with the projection to $f^*\mathcal{O}_{\mathbb{P}}(1) \otimes \Omega^1_{X_{\mathbb{P}}/\mathbb{P}}$ is indeed the well-defined map $\tilde{\varphi}$ we need.

A two-term complex $\mathcal{E}_0 \to \mathcal{E}_1$ in degrees zero and one admits the so-called stupid filtration by subcomplexes

$$\{0\} \subset [0 \to \mathcal{E}_1] \subset [\mathcal{E}_0 \to \mathcal{E}_1].$$

This filtration naturally defines a short exact sequence of complexes. Applied to $\mathfrak{C}^\bullet$ we obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & T_{\mathbb{P}/M_g}(-\log\infty) \arrow[r] \arrow[d] & T_{\mathbb{P}} \arrow[r] \arrow[d] & p^*T_{M_g} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & f_*\Omega^1_{X_{\mathbb{P}}/\mathbb{P}}(1) \arrow[r] & R^1f_*(\mathfrak{C}^\bullet) \arrow[r] & R^1f_*(\Omega^1_{X_{\mathbb{P}}/\mathbb{P}})^\vee \arrow[r] & 0.
\end{tikzcd}$$

By deformation theory, as above simply forgetting ω_ε, the right vertical arrow is an isomorphism too. This proves the remaining statements claimed in the theorem. $\square$

2.2. **Tangent bundle to a stratum.** Let now $S := \Omega M_g(k_1,\dots,k_r) \hookrightarrow \Omega M_g \hookrightarrow \mathbb{P}$ be a connected component of a stratum. The dimension of S is $2g+r-1$. It is obvious that the map $S \to M_g$ is not a bundle projection, since e.g. for $r = 1$ and $g \geq 4$ the dimension of M_g is larger than $\dim(S)$. There is another point that deserves caution.

Remark 2.2. *In general, the map $p : S \to M_g$ is not equidimensional over its image.*

For $g \leq 3$ the map $p : S \to M_g$ is equidimensional over its image. This is obvious for $g = 2$ and the case by case discussion for $g = 3$ is done in the proof of Theorem 4.3.

Proof of Remark 2.2. An example is the stratum $S = \Omega M_5(2,2,2,2)$. This stratum has dimension 13 while $\dim M_5 = 12$. There is a canonical choice of a theta characteristic κ on S: If

$$\mathrm{div}(\omega) = \sum_{i=1}^{4} 2P_i, \quad \text{let} \quad \kappa = \sum_{i=1}^{4} P_i.$$

Over a hyperelliptic curve X_0 with involution i one has $h^0(\kappa) = 3$, since the divisors of the form $\mathcal{O}_{X_0}(P + i(P) + Q + i(Q))$ are linearly equivalent for all (P, Q). Since $h^0(\kappa) \mod 2$ is deformation invariant ([At71]), the fiber dimension of p at the generic point of M_5 is one or three. Since the latter is impossible by the dimension count, the claim follows. □

We let D be the zero divisor of the universal holomorphic one-form on the stratum S. Since a stratum S is by definition disjoint from the hyperplane at infinity and the zero section, D may equivalently be defined as the divisor such that the composition

$$f^*\mathcal{O}_{\mathbb{P}}(-1)|_S \to f^*p^*(f_*\Omega^1_{X/M_g} \oplus \mathcal{O}_{M_g})|_S \to f^*(f_*\Omega^1_{X/\mathbb{P}}|_S) \to \Omega^1_{X/\mathbb{P}}|_S \otimes \mathcal{O}_X(-D) \tag{2}$$

is an isomorphism. By definition of the strata, D is étale over its image under f. We denote by D_{red} the corresponding reduced divisor (of degree r). The bundle $\mathcal{O}_{\mathbb{P}}(1)|_S$ is isomorphic to $\mathcal{O}_S$, but we keep track of it in order to naturally identify some maps as multiplication maps (see Lemma 2.5). In the sequel we define the *relative tangent bundle* $T_{S/p(S)}$ of $p : S \to p(S)$ to be the kernel of $T_S \to p^*T_{p(S)}$ and similarly for restrictions of p to submanifolds $B \subset S$. Given the Remark 2.2, it would be more cautious to define it in the derived category or to work dually with cotangent bundles. We will do the latter below. In the applications $T_{B/p(B)}$ will always be a vector bundle and fits well with geometric intuition.

The following theorem is equivalent to Veech's theorem ([Ve86]) on the existence of period coordinates.

Theorem 2.3. *The tangent bundle to a stratum can be described by the following hypercohomology sheaf:*

$$\begin{aligned} T_S &\cong R^1 f_* \left((\Omega^1_{X/S})^\vee \to f^*\mathcal{O}_{\mathbb{P}}(1)|_S \otimes \Omega^1_{X/S}(D_{\mathrm{red}} - D) \right) \\ &\cong \mathcal{O}_{\mathbb{P}}(1)|_S \otimes R^1 f_* \left(\mathcal{O}_S(-D) \to \Omega^1_{X/S}(D_{\mathrm{red}} - D) \right). \end{aligned} \tag{3}$$

In particular, T_S carries a linear structure, which coincides with the one defined by period coordinates in Section 1.

Moreover the map $p : S \to p(S) \subset M_g$ induces a map $T_S \to p^*T_{p(S)}$ with kernel

$$T_{S/p(S)} = \mathcal{O}_{\mathbb{P}}(1)|_S \otimes f_*\Omega^1_{X/S}(-D + D_{\mathrm{red}}).$$

Proof. Over the stratum S and by definition of D in equation (2) the map φ used in Theorem 2.1 factors through the composition

$$\varphi_S : (\Omega^1_{X/S})^\vee \xrightarrow{\mathrm{can}} f^*\mathcal{O}_{\mathbb{P}}(1)(-D)|_S \xrightarrow{d} f^*\mathcal{O}_{\mathbb{P}}(1)|_S \otimes \Omega^1_{X/S}(-D + D_{\mathrm{red}}).$$

We denote by $\mathfrak{C}^\bullet_S$ the two-term complex in degrees 0 and 1 given by φ_S. As in the preceding theorem one constructs a map $T_S \to R^1 f_*(\mathfrak{C}^\bullet_S)$ as the connecting homomorphism associated to the appropriate exact sequence of complexes. Over each point the stalk of $R^1 f_*(\mathfrak{C}^\bullet_S)$ is the tangent space to S by deformation theory. Together this implies that the two bundles are isomorphic.

The second identification of T_S stems from the isomorphism of complexes

$$\mathfrak{C}^\bullet_S \cong [f^*\mathcal{O}_{\mathbb{P}}(1)|_S \otimes (\ \mathcal{O}_S(-D) \xrightarrow{d} \Omega^1_{X/S}(D_{\mathrm{red}} - D)\ )]$$

induced by (2) and the projection formula.

The first hypercohomology of the de Rham complex carries a flat connection (see [KaOd68] for an algebraic definition). Furthermore the usual derivative defines a connection on $\mathcal{O}_{\mathbb{P}}(1)|_S$ since $S \cap \infty = \emptyset$. The tensor product of these flat connections yields a flat connection on T_S. Denote by $j : X \smallsetminus D \to X$ the inclusion. One checks that

$$\mathrm{Ker}(d : \mathcal{O}_S(-D) \to \Omega^1_{X/S}(D_{\mathrm{red}} - D)) = j_!\mathbb{C},$$

where $j_!$ denotes the extension by zero. Hence

$$R^1 f_* \left(\mathcal{O}_S(-D) \to \Omega^1_{X/S}(D_{\mathrm{red}} - D)\right) \cong R^1 f_* j_! \mathbb{C}.$$

Now it suffices to unwind definitions to see that the connection just defined on T_S coincides with the linear structure by period coordinates an in Section 1.

The identification of $T_{S/p(S)}$ follows from the stupid filtration of φ_S plus the usual Kodaira-Spencer theory as at the end of the proof of Theorem 2.1. □

It is easy to see the foliation by constant absolute periods in this setting:

Corollary 2.4. *The inclusion $\mathcal{O}_X(-D) \hookrightarrow \mathcal{O}_X$ induces a surjection*

$$q : T_S \twoheadrightarrow \mathcal{O}_{\mathbb{P}}(1)|_S \otimes R^1 f_* \left(\mathcal{O}_X \to \Omega^1_{X/S}\right).$$

The restriction of q to the relative tangent bundle $T_{S/p(S)}$ is injective.

This restriction will be denoted by q_{rel} in the sequel.

Proof. The commutative diagram

$$\begin{tikzcd}
(\Omega^1_{X/S})^\vee \arrow[r, "\mathrm{can}"] \arrow[d, "\varphi_S"] & f^*\mathcal{O}_{\mathbb{P}}(1)(-D)|_S \arrow[r, hook] & f^*\mathcal{O}_{\mathbb{P}}(1) \arrow[d, "d"] \\
f^*\mathcal{O}_{\mathbb{P}}(1)|_S \otimes \Omega^1_{X/S}(-D + D_{\mathrm{red}}) \arrow[rr, hook] & & f^*\mathcal{O}_{\mathbb{P}}(1)|_S \otimes \Omega^1_{X/S}
\end{tikzcd} \tag{4}$$

together with the projection formula defines the map q. The surjectivity of q can be checked pointwise at each $s \in S$. The fiber at s of the complex φ_S is quasiisomorphic to $j_!\mathbb{C}$, where $j : X \smallsetminus D \to X$ is the inclusion. The map of complexes in the diagram (4) comes from the inclusion $j_!\mathbb{C} \hookrightarrow \mathbb{C}$. The spectral sequence

$$E_2^{i,j} = H^i(X_s, H^j(\mathfrak{C}^\bullet_S))$$

has non-zero terms only for $j = 0$, hence degenerates at E_2. Thus, q restricts fiberwise to the map

$$H^1(X_s, j_!\mathbb{C}) \to H^1(X_s, \mathbb{C}),$$

which is obviously a surjection.

With the identifications of Theorem 2.3, the map q_{rel} is the injection on the bottom of diagram (4). □

2.3. **The dual viewpoint.** We now dualize the results obtained in Section 2.2. The following diagram follows immediately from the definitions and Serre duality. The middle and right column describe the identification of the cotangent bundle to a stratum. The map from left column to there is the dual of q defined above. Precisely the concrete description of $q^\vee|_{\mathrm{Ker}(\pi)}$, with π defined below, will play a key role in the sequel. Let $\mathrm{dR} = [\mathcal{O}_X \to \Omega^1_{X/S}]$ denote the relative de Rham complex.

(5)
$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}}(-1)|_S \otimes \mathrm{Ker}(\pi) \arrow[r, "q^\vee|_{\mathrm{Ker}(\pi)}"] \arrow[d, hook] & f_*(\Omega^1_{X/S})^{\otimes 2} \arrow[r, two heads] \arrow[d] & p^*\Omega^1_{p(S)} \arrow[d, hook] \\
\mathcal{O}_{\mathbb{P}}(-1)|_S \otimes R^1 f_* \mathrm{dR} \arrow[r, "q^\vee"] \arrow[d, "\mathrm{id}\otimes\pi"] & R^1 f_*((\mathfrak{C}^\vee)^\bullet_S) \arrow[r, "\cong"] \arrow[d] & \Omega^1_S \arrow[d] \\
\mathcal{O}_{\mathbb{P}}(-1)|_S \otimes R^1 f_* \mathcal{O}_X(D - D_{\mathrm{red}}) \arrow[r, "="] & \mathcal{O}_{\mathbb{P}}(-1)|_S \otimes R^1 f_* \mathcal{O}_X(D - D_{\mathrm{red}}) \arrow[r, "\cong"] & \Omega^1_{S/p(S)}
\end{tikzcd}$$

The map in the complex $(\mathfrak{C}^\vee)^\bullet_S = [f^*\mathcal{O}_{\mathbb{P}}(-1)|_S \otimes \mathcal{O}_X(D - D_{\mathrm{red}}) \to (\Omega^1_{X/S})^{\otimes 2}]$ is the composition of derivation and the identification (2). The vertical arrow on top in the middle is not injective in general, its kernel is the image of $\mathcal{O}_{\mathbb{P}}(-1)|_S \otimes f_*\mathcal{O}_X(D - D_{\mathrm{red}})$. The map $q^\vee$ is defined by applying $R^1 f_*(\cdot)$ to the map of complexes

(6)
$$\begin{tikzcd}
f^*\mathcal{O}_{\mathbb{P}}(-1) \arrow[r, hook] \arrow[d, "d"] & f^*\mathcal{O}_{\mathbb{P}}(-1) \otimes \mathcal{O}_X(D - D_{\mathrm{red}}) \arrow[d] \\
f^*\mathcal{O}_{\mathbb{P}}(-1)|_S \otimes \Omega^1_{X/S} \arrow[r] & (\Omega^1_{X/S})^{\otimes 2}
\end{tikzcd}$$

induced by (2). The map $\pi : R^1 f_* \mathrm{dR} \to R^1 f_* \mathcal{O}_X(D - D_{\mathrm{red}})$, induced by the map of complexes

$$[\mathcal{O}_X \to \Omega^1_{X/S}] \to [\mathcal{O}_X(D - D_{\mathrm{red}}) \to 0],$$

completes the description of the diagram. The proof of the projection formula implies immediately:

Lemma 2.5. *With the identifications $i : \mathcal{O}_{\mathbb{P}}(-1)|_S \to f_*\Omega^1_{X/S}(-D)$ the map $q^\vee|_{\mathrm{Ker}(\pi)}$ restricted to $f_*\Omega^1_{X/S} \subset \mathrm{Ker}(\pi)$ is the multiplication map of one-forms.*

3. Submanifolds of strata with linear structure: Examples

In this section we describe basic properties of submanifolds of strata with linear structure. Recall our convention, that linear structures are always inherited from the one on the stratum.

We let $j : X \smallsetminus D \to X$ the inclusion over the stratum of the complement of the universal divisor into the universal family of curves $f : X \to S$. With the preparation made in the previous section the following criterion is immediate. In the sequel, the subvarieties B are always supposed to be closed inside a stratum, but in general they are not compact.

Theorem 3.1. *If B is a closed submanifold of a stratum S with linear structure, then there are local subsystems*

$$\widetilde{\mathbb{L}}_B \subset R^1(f|_{X_B})_*(j_!\,\mathbb{C}) \quad \textit{and} \quad \mathbb{L}_B \subset R^1(f|_{X_B})_*\mathbb{C} =: \mathbb{V}_B$$

such that $T_B \to T_S|_B$ has the image $\mathcal{O}_{\mathbb{P}}(1)|_B \otimes \widetilde{\mathbb{L}}_B$ and the composition of q and the tangent map of the inclusion

$$\psi : T_B \to T_S|_B \to \mathcal{O}_{\mathbb{P}}(1)|_B \otimes \mathbb{V}_B$$

maps to and onto $\mathcal{O}_{\mathbb{P}}(1)|_B \otimes \mathbb{L}_B$.

Conversely, suppose there is a local subsystem $\widetilde{\mathbb{L}}_B \subset R^1(f|_{X_B})_*(j_! \mathbb{C})$ such that $T_B \to T_S|_B$ has the image $\mathcal{O}_{\mathbb{P}}(1)|_B \otimes \widetilde{\mathbb{L}}_B$. Then B is a linear manifold.

Proof. One direction is immediate from the definition of a linear submanifold. For the converse note that T_B carries a flat connection by hypothesis. This is sufficient by Lemma 1.1. □

Corollary 3.2. *The restriction $\psi_{rel} : T_{B/M} \to (\mathcal{O}_{\mathbb{P}}(1)|_B \otimes \mathbb{L}_B)$ of ψ to the relative tangent bundle is an isomorphism onto $\mathcal{O}_{\mathbb{P}}(1)|_B \otimes (\mathbb{L}_B \cap f_*\omega_{X/B}(D_{\mathrm{red}} - D))$. In particular, for B contained in the generic stratum, the fiber dimension of $B \to M$ is constant.*

Proof. Injectivity was shown in Corollary 2.4 and surjectivity follows from the definition of $\mathbb{L}_B$ in the preceding theorem and the description of the image of the relative tangent bundle in Theorem 2.3. In the generic stratum, $D_{\mathrm{red}} = D$ and the image of ψ_{rel} is $(\mathbb{L}_B)^{(1,0)}$. This is a vector bundle, by Hodge theory. □

To give a flavor of the notion we now present two familiar examples of submanifolds with linear structure in strata of ΩM_g in our language.

Eigenforms over Teichmüller curves ([Mö06a])

A *Teichmüller curve* is an algebraic curve $C \to M_g$ in the moduli space of curves, totally geodesic for the Teichmüller metric. Restricting to the oriented case, they are constructed as $\mathrm{GL}_2^+(\mathbb{R})$-orbit of a pair (X_0, ω_0), called a *Veech surface*. Recall that we abuse M_g instead of replacing it by $M_g^{[n]}$. Let $f : X \to C$ be the universal family over a Teichmüller curve. By construction the bundle $f_*\omega_{X/C}$ comes with a distinguished subbundle $\mathcal{L}$ such that the fibers $(X_0, \mathcal{L}_0)$ over any $0 \in C$ are Veech surfaces. Let $p : B \to C$ be the total space minus the zero section of the bundle $\mathcal{L}$. We hope that no confusion arises by using p both for $\Omega M_g \to M_g$ and for its restrictions.

Proposition 3.3. *The total space B of the distinguished bundle of one-forms over a Teichmüller curve is a closed submanifold of some stratum in ΩM_g with linear structure. It is defined over $\mathbb{R}$.*

This is obvious, if we consider a Teichmüller curve as the image of a $\mathrm{GL}_2^+(\mathbb{R})$-orbit. We reprove it, starting from the characterization of a Teichmüller curve by its variation of Hodge structures (VHS).

Proof. The VHS over Teichmüller curve has a unique rank two summand $\mathbb{L}$, defined over $\mathbb{R}$, whose Higgs field is an isomorphism (see Section 1). The summand $\mathbb{L}$ has the property that its $(1,0)$-part $\mathbb{L}^{(1,0)} \cong \mathcal{L}$ is the distinguished subbundle. We refer to [Mö06a] for details and indicate here only the consequences we need. We use moreover the fact that the lift B of a Teichmüller curve to ΩM_g lies entirely in some stratum S. We are not aware of an algebraic proof of this fact.

We denote by $\mathbb{L}_B$ the pullback of the local system $\mathbb{L}$ on C to B. We need to check that there is a local system underlying the image of $T_B \to T_S|_B$.

By construction of B as total space of the bundle $\mathcal{L}$ we have $\mathcal{O}_{\mathbb{P}}(1)|_B \cong p^*\mathcal{L}^{-1}$. The relative tangent bundle $T_{B/C}$ is trivialized by the section $\partial/\partial z$ where z is a fiber coordinate. Consider the map q defined in Proposition 2.4. The restriction of q to the relative tangent bundle is an inclusion

$$T_{B/C} \cong \mathcal{O}_B \to \text{image} \otimes p^*\mathcal{L}^{-1} \subset f_*\omega_{X/B} \otimes p^*\mathcal{L}^{-1}.$$

We conclude that image $\cong p^*\mathcal{L}$ as subbundles of $f_*\omega_{X/B}$. On the other hand on p^*T_C the map q equals the composition of $p^*T_C \to p^*\mathcal{L}^{-2} \subset R^1f_*\omega_{X/B}^{-1}$ and the restriction of the natural map

$$R^1f_*\omega_{X/B}^{-1} \to R^1f_*\mathcal{O} \otimes \mathcal{O}_{\mathbb{P}}(1)|_B \cong R^1f_*\mathcal{O} \otimes p^*\mathcal{L}^{-1}.$$

Both composition factors are maps that arise as p^* of maps between bundles on C: For the second map this is obvious from the definition and the first map is the Kodaira-Spencer map that depends only on the deformation of the underlying curve, not on the additional one-form. Moreover, all bundles and maps extend over the compactification $\overline{C}$ and yield a composition map

$$T_{\overline{C}} \to R^1f_*\omega_{X/C}^{-1} \to R^1f_*\mathcal{O} \otimes \mathcal{L}^{-1}.$$

The property 'maximal Higgs' implies that this composition is injective onto $\mathcal{L}^{-1} \otimes \mathcal{L}^{-1}$ and splits.

Together we have shown that q restricts to an isomorphism

$$\psi : T_B \to \mathbb{L}_B \otimes \mathcal{O}_{\mathbb{P}}(1)|_Y \subset \mathbb{V}_B \otimes \mathcal{O}_{\mathbb{P}}(1)|_Y.$$

Taking an unramified cover, we may assume that the zero divisor $D \to B$ consists of sections s_i of f. By [Mö06b] Corollary 3.4 the difference $s_i - s_j$ of any two sections is torsion in $\mathrm{Pic}^0(X/B)$. This implies that the relative periods are rational multiples of the absolute periods. Equivalently, the local system $\mathbb{L}_B$ lifts uniquely to a local system $\widetilde{\mathbb{L}}_B$ such that $T_B \to \mathbb{L}_B \otimes \mathcal{O}_{\mathbb{P}}(1)|_B$ factors through $T_B \to \widetilde{\mathbb{L}}_B \otimes \mathcal{O}_{\mathbb{P}}(1)|_B$. Obviously $\widetilde{\mathbb{L}}_B$ is again defined over $\mathbb{R}$ and satisfies what is needed to apply Theorem 3.1. □

Covering constructions

It is well-known that from a $\mathrm{GL}_2^+(\mathbb{R})$-invariant manifold one can construct a new $\mathrm{GL}_2^+(\mathbb{R})$-invariant manifold in higher genus by a Hurwitz space construction. The dimension will increase by the number of ramification points. We reprove this fact, showing that it does not depend on the linear structure being defined over $\mathbb{R}$.

Proposition 3.4. *Let $B \subset S := \Omega M_g(k_1, \ldots, k_r)$ be a manifold with linear structure and $f : X \to p(B)$ be the universal family over $p(B)$. Let $H \to p(B)$ be the Hurwitz space parameterizing covers of fibers of f ramified over s points and of some fixed type. Then $B' := H \times_{p(B)} B$ is a linear submanifold of some stratum $S' := \Omega M_{g'}(k'_1, \ldots, k'_{r'})$.*

Sketch of proof: Let $f : X' \to B'$ be the pullback of the universal family, let $\pi : Y' \to X'$ be the universal covering of X' and let $g = f \circ \pi$. We let also $j_Y : Y' \smallsetminus D' \to Y'$ be the inclusion of the complement of the universal divisor.

By hypothesis there is a linear subsystem $\widetilde{\mathbb{L}}_B \subset R^1f_*(j_X)_!\,\mathbb{C}$ such that

$$T_B \cong \mathcal{O}_{\mathbb{P}}(1)|_B \otimes \widetilde{\mathbb{L}}_B \subset T_S|_B.$$

We let $\mathbb{L}_B$ be the image of $\widetilde{\mathbb{L}}_B$ in $R^1f_*(j_X)_!\,\mathbb{C}$. We have an identification

$$T_{S'}|_{B'} \cong \mathcal{O}_{\mathbb{P}}(1)|_{B'} \otimes R^1g_*(j_Y)_!\,\mathbb{C} \cong \mathcal{O}_{\mathbb{P}}(1)|_{B'} \otimes R^1f_*((j_X)_!\,\pi_*\mathbb{C}).$$

Via the split inclusion $\mathbb{C} \to \pi_*\mathbb{C}$ we identify $R^1 f_*(\jmath_X)_!\mathbb{C} \subset R^1 f_*((\jmath_X)_!\pi_*\mathbb{C})$ as a direct summand. Moreover, we write $D = \sum_{i=1}^r k_i D_i$ and $D' = \sum_{i=1}^{r'} k'_i D'_i$ for the universal divisor and we use $\mathbb{C}_{k_i D_i}$ to denote the skyscraper sheaf of length k_i along D_i. For the same reason as above, the inclusion $f_*(\oplus_{i=1}^r \mathbb{C}_{k_i D_i}) \to f_*(\oplus_{i=1}^{r'} \pi_*\mathbb{C}_{k'_i D'_i})$ is split and we let $\mathbb{M}$ denote the complement.

It is now easy to check that

$$T_{B'} \cong \mathcal{O}_{\mathbb{P}}(1)|_{B'} \otimes (\widetilde{\mathbb{L}}_{B'} \oplus \mathbb{M}) \subset T_{S'}|_{B'}.$$

This shows that B' has a linear structure. □

4. Linear structures and endomorphisms of the Jacobian

The aim of this section is to show one of the central results:

Theorem 4.1. *Suppose that B is a closed algebraic submanifold of the generic stratum $S = \Omega M_g(1, \ldots, 1)$ with linear structure. Then there are three possibilities:*

(i) *B is a connected component of S.*
(ii) *$g \geq 3$ and B is the preimage in S of the hyperelliptic locus in M_g.*
(iii) *B parameterizes curves with a Jacobian whose endomorphism ring is strictly larger than $\mathbb{Z}$.*

Case iii) also contains the covering constructions, see the end of the previous section, since the Jacobian splits in these cases. The non-generic strata need closer inspection. We do this case by case in genus $g = 3$.

In this section we will rely on Deligne's semisimplicity for the VHS over the manifolds under consideration. That is why we need B to be algebraic or, a priori slightly weaker, that any plurisubharmonic function on B which is bounded above is in fact constant.

We continue with the notation used in the previous section, in particular in Theorem 3.1.

Lemma 4.2. *If $\mathbb{L}_B \subsetneqq \mathbb{V}_B$ then B parameterizes curves with a Jacobian whose endomorphism ring is strictly larger than $\mathbb{Z}$.*

Proof. By semisimplicity ([De87] Proposition 1.13) the $\mathbb{C}$-VHS on the local system $\mathbb{V}_B$ decomposes as

$$\mathbb{V}_B = \bigoplus_{i \in I} (\mathbb{L}_i \otimes W_i) \tag{7}$$

with irreducible local systems $\mathbb{L}_i$ of rank d_i carrying a weight one VHS, whose Higgs field is non-zero, and with W_i vector spaces carrying a weight zero VHS.

We claim that, as in [ViZu04] Lemma 3.2, each of the $\mathbb{L}_i$ may be chosen to be defined over some number field L_i. In fact, let $\mathcal{G}(d_i, \mathbb{V}_B)$ be the set of local systems in $\mathbb{V}_B$ of rank d_i and let $\mathrm{Grass}(d_i, V)$ be the Grassmann variety of subspaces of dimension d_i of a the vector space V, which is defined as the fiber of $\mathbb{V}_B$ at some base point. The set $\mathcal{G}(d_i, \mathbb{V}_B)$ is naturally a subvariety of

$$\mathrm{Grass}(d_i, V) \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\,\mathbb{C}$$

consisting of $\pi_1(B)$-invariant points. The subset in $\mathcal{G}(d_i, \mathbb{V}_B)$ of rank d_i local subsystems, whose projection to $\mathbb{L}_i$ along its complement under the polarization is non-zero, is Zariski-open. It hence contains an element, which is defined over a finite extension L_i of $\mathbb{Q}$. By the irreducibility of $\mathbb{L}_i$ the non-zero map must in fact be an isomorphism.

From the hypothesis $\mathbb{L}_B \subsetneqq \mathbb{V}_B$ we deduce that the decomposition (7) is non-trivial. Let K be the Galois closure of $L_1/\mathbb{Q}$. Then for all $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ the local systems $\mathbb{L}_1^\sigma$ are also among the $\mathbb{L}_i$. Let $L \subset K$ be the field fixed by all automorphisms σ such that $\mathbb{L}_1^\sigma \cong \mathbb{L}_1$ as sub-local systems of $\mathbb{V}$. For $a \in L$ the map

$$\bigoplus_{\sigma \in \mathrm{Gal}(K/\mathbb{Q})/\mathrm{Gal}(K/L)} \sigma(a) \cdot \mathrm{id}_{\mathbb{L}_{i(\sigma)} \otimes W_{i(\sigma)}} \in \mathrm{End}(\mathbb{V}_B), \quad \text{where } \mathbb{L}_{i(\sigma)} := \sigma(\mathbb{L}_1)$$

This endomorphism is of bidegree $(0,0)$ and defined over $\mathbb{Q}$, hence a non-trivial $\mathbb{Q}$-endomorphism of the family of Jacobians over B. If $L = \mathbb{Q}$ and $|I| > 1$ then $\mathbb{Q} \cdot \mathrm{id}_{\mathbb{L}_1 \otimes W_1}$ are non-trivial $\mathbb{Q}$-endomorphisms. Finally, if $L = \mathbb{Q}$ and $|I| = 1$ then W_1 has dimension at least two. Choose a one-dimensional subvectorspace $W_0 \subset W_1$ and take the non-trivial endomorphisms $\mathbb{Q} \cdot \mathrm{id}_{\mathbb{L}_1 \otimes W_0}$. $\square$

Proof of Theorem 4.1. Suppose that neither (i) nor (iii) holds for B. By Corollary 3.2 $T_B \to (p^* T_{M_g})|_B$ is a locally split surjection of vector bundles. Hence $(p^*\Omega^1_{M_g})|_B \to \Omega^1_B$ is a locally split inclusion. We can say more here: Following the notations of Theorem 3.1 and by Lemma 4.2 the map

$$\psi : T_B \to \mathcal{O}_{\mathbb{P}}(1)|_B \otimes \mathbb{V}_B$$

is surjective. A first consequence of this fact is that over each point in $p(B)$ the fibers of $p|_S$ and $p|_B$ coincide. Thus, there is locally over any sufficiently small open set $U \subset M_g$, a splitting $s_U : \Omega^1_{p(B)}|_U \to \Omega^1_{M_g}|_U$ of the surjection $\Omega^1_{M_g}|_{p(B)} \to \Omega^1_{p(B)}$.

Second, the composition

$$p^* T_B \to p^* T_{M_g}|_B = p^* R^1 f_* (\Omega^1_{X/M_g}|_B)^{-1} \to \mathcal{O}_{\mathbb{P}}(1)|_B \otimes R^1 f_* \mathcal{O}_{X_B}$$

has to be surjective, too. Using the notation of Lemma 2.5, we have $f_*\Omega^1_{X/S} = \mathrm{Ker}(\pi)$, since all zeroes are simple on the generic stratum. Dualizing this composition yields that the multiplication map

$$\psi^\vee|_{\mathrm{Ker}(\pi)} : \mathcal{O}_{\mathbb{P}}(-1)|_B \otimes (p^* f_* \Omega^1_{X/M_g})|_B \to (p^* f_* (\Omega^1_{X/M_g})^{\otimes 2})|_B \to p^* \Omega^1_{p(B)}$$

has to be injective and factors, over say $p^{-1}(U)$, through $\mathcal{F}_U := p^*(s_U(\Omega^1_{p(B)}))$. By Lemma 2.5 this implies that for all line bundles $\mathcal{E}_U \subset f_*\omega_{X/M_g}|_U$ generated by a differential with simple zeroes the image of the multiplication map

$$\mathcal{E}_U \otimes f_*\omega_{X/M_g}|_U \to f_*\omega^2_{X/M_g}|_U$$

lies in the subbundle $\mathcal{F} := p^*(s_U(\Omega^1_{p(B)}))$. Since one-forms with simple zeroes span the space of holomorphic one-forms, this implies that the whole image of the multiplication map

$$f_*\omega_{X/M_g}|_U \otimes f_*\omega_{X/M_g}|_U \to f_*\omega^2_{X/M_g}|_U$$

lies in $\mathcal{F}$.

By M.Noether's theorem (e.g. [ACGH85] III.§ 2), for $g = 2$ or at a point outside the hyperelliptic locus, the multiplication map is surjective. Hence if $p(B)$ is not contained in the hyperelliptic locus, $\mathcal{F} = p^*(s_U(\Omega^1_{p(B)}))$ and B coincides with the whole stratum.

If $p(B)$ is contained in the hyperelliptic locus, the image of the multiplication map is a subbundle in $f_*\omega_{X/M_g}^{\otimes 2}$ of rank $2g-1$ as can be checked easily on an explicit basis of one-form. See Theorem 4.3 for more involved arguments in the same style. For dimension reasons, $p(B)$ has to coincide with the hyperelliptic locus.

Finally we remark that this case actually occurs, i.e. that the bundle B over the hyperelliptic locus has a linear structure defined over $\mathbb{R}$. For this purpose we check that B is $\mathrm{GL}_2^+(\mathbb{R})$-invariant. This easily follows from the fact that -1 is in the center of $\mathrm{GL}_2^+(\mathbb{R})$ or, alternatively, that B is the image of stratum of quadratic differentials with $2g+2$ simple poles and $g-1$ double zeros on $\mathbb{P}^1$ under a canonical double covering map, see e.g. [KoZo03] Lemma 1. □

Theorem 4.3. *For a closed submanifold B with linear structure in a stratum $S \neq \Omega M_3(1,\ldots,1)$ of ΩM_3 there are the following possibilities:*

i) *B is a connected component of S.*
ii) *B parameterizes curves whose Jacobian has an endomorphism ring strictly larger than $\mathbb{Z}$.*
iii) *B is the preimage of the hyperelliptic locus in $\Omega M_3(2,1,1)$ or in the unique component $\Omega M_3(2,2)^{\mathrm{odd}}$ of $\Omega M_3(2,2)$ that does not exclusively consist of hyperelliptic curves.*
iv) *B has dimension 5, is disjoint from the hyperelliptic locus and $B \subset \Omega M_3(2,2)^{\mathrm{odd}}$ or $B \subset \Omega M_3(3,1)$.*

We do not claim, that the manifolds in iv) exist. In fact we doubt their existence, but cannot not prove it.

Proof. Case $S = \Omega M_3(4)$: This stratum has two connected components one contained in the hyperelliptic locus, the other one disjoint from the hyperelliptic locus. In both cases the fibers of $p|_S$ are one-dimensional, since zeros of order ≥ 3 on a curve of genus 3 are Weierstraß points, hence there are only finitely many of them on a fixed curve. Consequently, the image of both connected components in M_3 is 5-dimensional. If $B \subset S$ belongs not to case ii) then q is surjective by Lemma 4.2. In particular the induced map on $p^*T_{p(S)}|_B$ surjects onto the whole of $(\mathcal{O}_{\mathbb{P}}(1) \otimes R^1 f_*(\mathcal{O}_B \to \omega^1_{X/B}))/q_{\mathrm{rel}}(T_{S/p(S)})$. This bundle has rank 5. Hence $\dim p(B) = 5$ and B belongs to case i).

Case $S = \Omega M_3(3,1)$: This stratum does not intersect the hyperelliptic locus since the hyperelliptic involution would have to fix the zeros. But on a hyperelliptic curve differentials have zeros at Weierstraß points of even order only. Hence the canonical model of a curve in S is a plane quartic. The fibers of p are one-dimensional: A zero of order three corresponds a Weierstraß point, in fact to an inflexion line of the plane quartic. The simple zero is the 4th point of intersection of this line with the quartic, hence fixed once the Weierstraß point is chosen. Since $\dim(S) = 7$ we deduce that $p(S)$ is of dimension 6, dense in M_3.

Suppose that $B \subset S$ does not belong to case ii). This implies that

$$\overline{\psi} : p^*T_{p(B)} \to (\mathcal{O}_{\mathbb{P}}(1) \otimes R^1 f_*(\mathcal{O}_B \to \omega^1_{X/B}))/q_{\mathrm{rel}}(T_{S/p(S)})$$

is surjective and that B has dimension at least 5. Hence B belongs to case i) or iv).

Case $S = \Omega M_3(2,2)$: By [KoZo03] Theorem 2, there are two components of this stratum, both of dimension $\dim S = 7$: the component $S^{\mathrm{hyp}} := \Omega M_3(2,2)^{\mathrm{hyp}}$ consisting of hyperelliptic curves such that the hyperelliptic involution interchanges the two zeroes, and $S^{\mathrm{odd}} := \Omega M_3(2,2)^{\mathrm{odd}}$, the component with odd spin structure (see [KoZo03] for the definition).

Consider $B \subset S^{\mathrm{hyp}}$ first. Since the hyperelliptic locus has dimension 5 and since the generic differential on any curve is not of type $(2,2)$ all fibers of $S^{\mathrm{hyp}} \to p(S^{\mathrm{hyp}})$ are two-dimensional and the image $p(S^{\mathrm{hyp}})$ is dense in the hyperelliptic locus. If B does not belong to case ii) we use the same argument as in Theorem 4.1 to show that $B = S^{\mathrm{hyp}}$: We have to show that the images of multiplication map

$$\mathcal{E}_U \otimes f_*\omega_{X/M_g}|_U \to f_*\omega^2_{X/M_g}|_U,$$

for all line bundles $\mathcal{E}_U$ generated by a differential with two double zeroes that are interchanged by the hyperelliptic involution, generate the subspace of $f_*\omega^2_{X/S}$ acted on by $+1$ under the hyperelliptic involution. On a hyperelliptic genus 3 curve

$$y^2 = \prod_{i=1}^{7}(x - x_i) \quad \text{with}\, x_i \neq x_j \quad \text{for} \quad i \neq j$$

a basis of holomorphic one-forms is $x^i dx/y$ for $i = 0,1,2$. The $+1$-eigenspace is generated by $x^i(dx/y)^2$ for $i = 0,\ldots,5$. The one-forms $(x - x_0)^2 dx/y$ for $x_0 \neq x_i$, $i = 0,\ldots,7$ all have the prescribed type of zeros. Obviously the vector space they generate consists of all holomorphic one-forms. Hence their products with the $x^i dx/y$ for $i = 0,1,2$ indeed generates the $+1$-eigenspace.

Consider now $B \subset S^{\mathrm{odd}}$. A point in this stratum defines an odd theta characteristic, as in the proof of Remark 2.2. There are only finitely many, in fact 28, such theta characteristics on each curve. Moreover the space of global sections of a fixed theta characteristic cannot be three-dimensional. Hence it is one-dimension for all curves in $p(S^{\mathrm{odd}})$ by [At71]. It follows that the fibers of $p : S^{\mathrm{odd}} \to p(S)$ are all one-dimensional.

As above, if B does not belong to case ii), the map $\overline{\psi}$ induced by ψ on $p^*T_{p(B)}|_B$ surjects onto the whole of $(\mathcal{O}_{\mathbb{P}}(1) \otimes R^1 f_*(\mathcal{O}_B \to \omega^1_{X/B}))/q_{\mathrm{rel}}(T_{S/p(S)})$. This bundle has now again rank 5. Hence B belongs to case i) or iv).

Case $S = \Omega M_3(2,1,1)$: This stratum has dimension 8 and maps surjectively to M_3 with 2-dimensional fibers. In order to argue as in the proof of Theorem 4.1 we have to show that for a genus 3 curve the one-forms of type $(2,1,1)$ generate the space of holomorphic one-forms. For hyperelliptic curves this is follows as in the case $\Omega M_3(2,2)^{\mathrm{odd}}$. For a smooth plane quartic, any tangent line except for finitely many bitangents and inflexion lines corresponds to a one-form of type $(2,1,1)$. The tangent lines to a quartic form a (singular) curve C^* of degree 12 in the dual of $\mathbb{P}^2$. If the generation result we need was false, C^* was contained in, hence equal to, a hyperplane in $(\mathbb{P}^2)^\vee$. This is absurd. □

5. Hilbert modular varieties

Fix a totally real number field K with $[K : \mathbb{Q}] = g$ and an order $\mathfrak{o} \subset K$. We denote by $\mathfrak{o}^\vee$ the inverse different and we let

$$\mathrm{SL}(\mathfrak{o} \oplus \mathfrak{o}^\vee) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(K) : \, a \in \mathfrak{o},\, b \in (\mathfrak{o}^\vee)^{-1},\, c \in \mathfrak{o}^\vee,\, d \in \mathfrak{o} \right\}.$$

The Hilbert modular variety $\mathbb{H}^g/\mathrm{SL}(\mathfrak{o} \oplus \mathfrak{o}^\vee)$ parameterizes principally polarized abelian g-folds A together with the choice of some real multiplication, i.e. with a map $\mathfrak{o} \to \mathrm{End}(A)$.

Forgetting this choice yields a map from $\mathbb{H}^g/\mathrm{SL}(\mathfrak{o}\oplus\mathfrak{o}^\vee)$ to the moduli space of principally polarized abelian g-folds. This map ramifies along diagonals in $\mathbb{H}^g$ unless we keep track of the stack structure of A_g. We avoid this by passing right away to a suitable level structure:

Principally polarized abelian g-folds A together with the choice of real multiplication and a choice of a basis of $\mathfrak{o}\oplus\mathfrak{o}^\vee$, symplectic for the pairing

$$\langle (x,y),(x',y')\rangle := \mathrm{tr}(xy' - x'y), \tag{8}$$

are parameterized by $\mathbb{H}^g$. Forgetting the choice of real multiplication and fixing a $\mathbb{Z}$-basis $a_1,\ldots,a_g$ of $\mathfrak{o}$ yields an embedding

$$j : \mathbb{H}^g \to \mathbb{H}_g, \quad (\tau_1,\ldots,\tau_g) \mapsto M^T \cdot \mathrm{diag}(\tau_1,\ldots,\tau_g)\cdot M,$$

where $\mathbb{H}_g$ denotes the Siegel upper half plane, where $M = (\sigma_j(a_i))$ and $\sigma_j : K \to \mathbb{R}$ are the g real embeddings of K. The map j is equivariant with respect to the action of $\mathrm{SL}(\mathfrak{o}\oplus\mathfrak{o}^\vee)$ on $\mathbb{H}^g$ and the action on $\mathbb{H}_g$ of its image in $\mathrm{Sp}(2g,\mathbb{Z})$, obtained by expressing the entries of $\mathrm{SL}(\mathfrak{o}\oplus\mathfrak{o}^\vee)$ with respect to $a_1,\ldots,a_g$ and its dual basis with respect to the pairing (8).

Let $\mathrm{Sp}^{[n]}(2g,\mathbb{Z}) := \mathrm{Ker}(\mathrm{Sp}(2g,\mathbb{Z}) \to \mathrm{Sp}(2g,\mathbb{Z}/n\mathbb{Z}))$. For $n \geq 3$ the quotient map $\mathbb{H}_g \to \mathbb{H}_g/\mathrm{Sp}^{[n]}(2g,\mathbb{Z})$ is unramified and if we let

$$\mathrm{SL}^{[n]}(\mathfrak{o}\oplus\mathfrak{o}^\vee) = \mathrm{SL}(\mathfrak{o}\oplus\mathfrak{o}^\vee) \cap \mathrm{Sp}^{[n]}(2g,\mathbb{Z})$$

and $A_g^{[n]} := \mathbb{H}_g/\mathrm{Sp}^{[n]}(2g,\mathbb{Z})$ we obtain the two right columns of the following diagram.

$$\begin{tikzcd}
M_g^{[n]} \arrow[d] \arrow[rrdd, "t^{[n]}"] & \mathbb{H}^g \arrow[r, "j"] \arrow[dd] & \mathbb{H}_g \arrow[dd] \\
M_g^{[n]}/\pm 1 \arrow[rrd, "t^{[n]}/\pm 1"] \arrow[dd] & & \\
& Y := \mathbb{H}^g/\mathrm{SL}^{[n]}(\mathfrak{o}\oplus\mathfrak{o}^\vee) \arrow[r, "j^{[n]}"] \arrow[d] & A_g^{[n]} \arrow[d] \\
M_g \arrow[rr, bend right] & \mathbb{H}^g/\mathrm{SL}(\mathfrak{o}\oplus\mathfrak{o}^\vee) \arrow[r] & A_g
\end{tikzcd} \tag{9}$$

In particular the map $j^{[n]}$ is an embedding. We will from now on call, slightly abusively, Y a *Hilbert modular variety.*

Let $\overline{Y}$ be a smooth compactification with $S := \overline{Y} \smallsetminus Y$ a normal crossing divisor. We specify the compactification below. Recall from [ViZu07] the decomposition of the VHS over Y. The logarithmic cotangent bundle decomposes as

$$\Omega^1_{\overline{Y}}(\log S) \cong \bigoplus_{i=1}^{g} \Omega_i$$

defined as follows. The restriction of Ω_i to Y is generated by dz_i where the z_i are coordinates of factors of the universal covering of Y. If the compactification is chosen as in [Mu77], the bundles $\Omega_i|_Y$ extend to $\overline{Y}$ and their sum is the logarithmic cotangent bundle (Theorem 3.1 and Proposition 3.4 in loc. cit.).

Let $g : A \to Y$ be the universal family over Y. Then the VHS decomposes as

$$\mathbb{V}_{\mathbb{C}} := R^1 g_* \mathbb{C} = \bigoplus_{i=1}^{g} \mathbb{L}_i, \tag{10}$$

where the $\mathbb{L}_i$ are rank two local systems, in fact defined over K and Galois conjugate. Moreover all the $\mathbb{L}_i$ are maximal Higgs, that is the Higgs field

$$\theta : \mathbb{V}_{\mathbb{C}}^{(1,0)} \to \mathbb{V}_{\mathbb{C}}^{(0,1)} \otimes \Omega^1_Y(\log S)$$

is the direct sum of isomorphisms

$$\theta_i : \mathcal{L}_i := \mathbb{L}_i^{(1,0)} \to \mathbb{L}_i^{(0,1)} \otimes \Omega_i \tag{11}$$

composed with the natural inclusions $\Omega_i \to \Omega^1_Y(\log S)$. Recall also that $\mathbb{L}_i^{(0,1)} = \mathcal{L}_i^{-1}$ and hence $\Omega_i \cong \mathcal{L}_i^{\otimes 2}$.

From this can describe the tangent map to the inclusion $t^{[n]}$. Choose a smooth compactification $\overline{A_g}$ of A_g with boundary S' and choose $\overline{Y}$ to be the closure of Y in $\overline{A_g}$.

Lemma 5.1. *The tangent map to $t^{[n]}$*

$$T_Y(-\log S) \cong \bigoplus_{i=1}^{g} \Omega_i^{\vee} \to T_A(-\log S')|_Y \cong (\mathrm{Sym}^2(\bigoplus_{i=1}^{g} \mathcal{L}_i^{-1}))^{\vee}$$

is given by the injection onto the summands $\oplus_{i=1}^{g} \mathcal{L}_i^{-2}$. More precisely, there is no non-zero map

$$m_{i,j,k} : \mathcal{L}_i \otimes \mathcal{L}_j \to \mathcal{L}_k^2$$

unless $k = i = j$, in which case this map is an isomorphism.

Proof. The first statement follows from the second and the second follows from [ViZu07] Lemma 1.6: The $\mathcal{L}_i$ are line bundles with the same slope with respect to $\omega_Y(\log S)$. Hence maps of the form $m_{i,j,k}$ are either zero or an isomorphism. Comparing $c_1(\mathcal{L}_i)$ we deduce from loc. cit. that the latter cannot happen. $\square$

5.1. **Linearity of Hilbert modular surfaces: Reproving** [McM03]**.** Let Y_M be the preimage of a Hilbert modular surface in $M_2^{[n]}$. Since all curves with $g = 2$ are hyperelliptic, this is an unramified double covering of Y and all statements of Higgs fields or Kodaira-Spencer maps being isomorphisms remain true on Y_M. Let $p : B^{\mathrm{total}} \to Y_M$ be the *eigenform locus* over Y_M, i.e. the subset of ΩM_2 consisting of pairs (X, ω) where $\mathfrak{o}$ acts as $a \cdot \omega = \iota(a)\omega$ for a fixed embedding $\iota : K \to \mathbb{R}$ for all $a \in \mathfrak{o}$.

The aim of this section is to reprove Theorem 7.1 in [McM03] stating that B^{total} is $\mathrm{GL}_2^+(\mathbb{R})$-invariant. In fact we will show:

Proposition 5.2. *The eigenform locus $B := B^{\mathrm{total}} \cap \Omega M_2(1,1)$ of a Hilbert modular surface in the stratum $\Omega M_2(1,1)$ is a closed submanifold with linear structure defined over $\mathbb{R}$.*

The full strength of Theorem 7.1 in loc. cit. follows from Proposition 1.3 and a local consideration around $B^{\mathrm{total}} \cap \Omega M_2(2)$, the Teichmüller curves discovered in [McM03] Theorem 1.3.

Proof. We need to show that there is a rank two local system $\mathbb{L}_B \subset \mathbb{V}_B$ defined over $\mathbb{R}$ such that $\psi(T_B) = \mathcal{O}_{\mathbb{P}}(1)|_B \otimes \mathbb{L}_B$. Then, since the fibers of q are one-dimensional on $\Omega M_2(1,1)$ and since B is three-dimensional, the image of T_B in $T_S|_B$ coincides with the full preimage of $\mathcal{O}_{\mathbb{P}}(1)|_B \otimes \mathbb{L}_B$. It thus carries a flat structure. The conclusion then follows from Theorem 3.1.

Changing the enumeration of the summands if necessary, we may suppose that $a \in \mathfrak{o}$ acts on the VHS via $a \cdot \mathrm{id}_{\mathbb{L}_1} \oplus \sigma(a) \cdot \mathrm{id}_{\mathbb{L}_2}$, where σ generates $\mathrm{Gal}(K/\mathbb{Q})$. Hence by definition $\mathcal{O}_{\mathbb{P}}(1)|_B \cong p^*\mathcal{L}_1^{-1}|_B$ and ψ restricts to

$$\psi_{\mathrm{rel}} : T_{B/Y_M} \cong \mathcal{O}_B \xrightarrow{\sim} \mathcal{L}_1|_B \otimes \mathcal{O}_{\mathbb{P}}(1)|_B \subset f_*\omega_{X/B} \otimes \mathcal{O}_{\mathbb{P}}(1)|_B .$$

It remains to show that the quotient

$$\overline{\psi} : p^*T_{Y_M} \to R^1 f_*\omega_{X/B}^{-1} \to R^1 f_*\mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}}(1)|_B$$

has an image equal to $p^*\mathcal{L}_1^{-1}|_B \otimes \mathcal{O}_{\mathbb{P}}(1)|_B$. All bundles and maps involved in the composition arise as pullback p^* of bundles and maps on $p(B) \subset Y_M$. In fact the first map is the Kodaira-Spencer map and the second is the dual of the multiplication map. Hence we need to show that

$$T_{p(B)} \to R^1 f_*\omega_{X/p(B)}^{-1} \to R^1 f_*\mathcal{O}_X \otimes \mathcal{L}_i^{-1}|_{p(B)}$$

maps onto $\mathcal{L}_i^{-1}|_{p(B)} \otimes \mathcal{L}_i^{-1}|_{p(B)}$. Dualizing yields the map

$$\mathcal{L}_i|_{p(B)} \otimes f_*\omega_{X/p(B)} \to \mathrm{Sym}^2(f_*\omega_{X/p(B)}) = \Omega^1_{A_2^{[n]}}|_{p(B)} \to f_*\omega_{X/p(B)}^{\otimes 2} = \Omega^1_{M_2^{[n]}}|_{p(B)} \to \Omega^1_{p(B)}$$

which is the multiplication of one-forms composed with the dual Kodaira-Spencer map. Ignoring the factorization through $\Omega^1_{M_2^{[n]}}|_{p(B)}$ we know this map from Lemma 5.1: It is the natural inclusion composed with the projection along the direct summand

$$\oplus_{i \neq j} \mathcal{L}_i \otimes \mathcal{L}_j \subset \mathrm{Sym}^2(f_*\omega_{X/p(B)}).$$

Dualizing again we conclude that ψ is as desired. □

5.2. **Obstacles to linearity for Hilbert modular threefolds.** In this section we analyze to which extent the linearity results for Hilbert modular surfaces apply to higher dimensions. For $g \geq 4$ the image of M_g is no longer dense in A_g. It fact, it is shown in [dJZh07] that Hilbert modular g-folds lie generically outside M_g for $g \geq 4$ with possible exceptions for $g = 4$. For $g = 3$ the situation is better:

Lemma 5.3. *A Hilbert modular threefold $Y \subset A_3^{[n]}$ lies generically in $t^{[n]}(M_3^{[n]})$. The complement has codimension at least two.*

Proof. For $g = 3$ the image of $t^{[M_3]}$ is dense in A_3, the complement W of the abelian threefolds, that are reducible as principally polarized abelian varieties. Suppose that $A \cong A_1 \times A_2$ is a generic fiber in a component W_0 of $W \cap Y$. We number the factors such that $\dim A_i = i$. If A_2 was a simple abelian variety or if none of factors of A_2 is isogenous to A_1, then

$$\mathrm{End}_{\mathbb{Q}}(A) = \mathrm{End}_{\mathbb{Q}}(A_1) \times \mathrm{End}_{\mathbb{Q}}(A_2)$$

and $\mathrm{End}_{\mathbb{Q}}(A_2)$ is a $\mathbb{Q}$-algebra of rank 2 or 4. The same argument applies if A_2 is isogenous to $E_1 \times E_2$ and say E_1 is isogenous to A_1 while E_2 is not. Hence we may assume A_2 is isogenous to $E_1 \times E_2$ where E_1, E_2 and A_1 are all isogenous. But this limits the possibilities for A_2 given A_1 to a countable number and we conclude that $\dim W_0 \leq 1$. $\square$

As in the case $g = 2$, let Y_M be the preimage of Y in $M_3^{[n]}$ and let $\Omega M_g^{[n]} \supset B^{\text{total}} \to Y_M$ be the $\mathbb{C}^*$-bundle of eigenfoms with respect to a fixed embedding of $K \to \mathbb{R}$. Let $\Omega M_3(\mathfrak{o})$ be the stratum the generic eigenform maps to and denote by B the intersection of B^{total} and $\Omega M_3(\mathfrak{o})$.

Question 5.4. *What is this stratum $\Omega M_3(\mathfrak{o})$, does it depend on $\mathfrak{o}$?*

For $\mathfrak{o} = \mathbb{Z}[\zeta_7 + \zeta_7^{-1}]$ the stratum $\Omega M_3(\mathfrak{o})$ is the generic stratum $\Omega M_3(1, 1, 1, 1)$. This can be checked using the explicit equation given in [Lo05] of the Teichmüller curve in Veech's series ([Ve89]) generated by $y^2 = x^7 - 1$ and the one-form dx/y.

Theorem 5.5. *Let $B := B_{\mathfrak{o}}$ denote the locus of eigenforms $B := B_{\mathfrak{o}}$ over a Hilbert modular threefold and let $\Omega M_3(\mathfrak{o})$ be the stratum of ΩM_3 the generic eigenform maps to. The manifold B inherits from $\Omega M_3(\mathfrak{o})$ a linear structure if and only if*

i) *there is* no *point $b \in B$ corresponding to a hyperelliptic curve, such that the normal bundle to the hyperelliptic locus at b coincides (as subbundle of the cotangent bundle) with one of the distinguished subbundles Ω_i, the cotangent directions to the natural foliations of a Hilbert modular threefold* and
ii) *either $\Omega M_3(\mathfrak{o}) = \Omega M_3(2, 1, 1)$ or $\Omega M_3(\mathfrak{o}) = \Omega M_3(1, 1, 1, 1) =: S$ and the local system defined by the zeros of the eigenform extends the pullback of $\mathbb{L}_1$ to B to a local subsystem of $R^1 f_*(j_!\mathbb{C})|_{Y_M}$.*

Before proving this, several remarks are necessary. First, concerning the first condition we have:

Proposition 5.6. *The intersection of Y_M with the hyperelliptic locus H in $M_3^{[n]}$ is non-empty, Y_M is not contained in H and some components of $H \cap Y_M$ are codimension one in Y_M.*

Proof. Take the Baily-Borel-Satake compactification of Y. This gives a projective embedding such that the boundary consists of codimension ≥ 2 components. Hence an intersection with general hyperplanes produces a curve $C_Y \subset Y$ that avoids both the boundary and, by Lemma 5.3, the reducible locus. Consequently, $C := (t^{[n]})^{-1}(C_Y)$ is a compact curve lying entirely in Y_M.

Suppose that the intersection $H \cap Y_M$ is empty. The class of $H \in \mathrm{Pic}(\overline{M_3})$ is a positive multiple of the Hodge bundle ([HaMo98] Formula 3.165) and a combination of the boundary divisors. Since C avoids both H and the boundary, this implies that for $f : X \to C$, the pullback of universal family of curves to C, we have $\deg(f_* \omega_{X/C}) = 0$. Consequently, the family of curves is isotrivial, contradicting the construction of C.

Suppose that $Y_M \subset H$. Then $t^{[n]}$ is unramified over Y_M. Since Y_M differs from Y only by the codimension $\geq$ 2-locus $W \cap Y$, we have (choosing some basepoint x)

$$\pi_1(Y_M, x) \cong \pi_1(Y, t^{[n]}(x)) \cong \Gamma \subset \mathrm{SL}(\mathfrak{o} \oplus \mathfrak{o}^\vee).$$

Hence $\pi_1(Y_M, x)$ is a lattice in a Lie group of rank ≥ 2. This contradicts a result of Farb and Masur ([FaMa98]): The image of such a lattice in the mapping class group has to be finite i.e. after base change to a finite unramified cover of $M_3^{[n]}$ the Hilbert modular threefold Y becomes simply connected. This is absurd.

The same argument works if the codimension of $H \cap Y_M$ in Y_M is ≥ 2. Consequently, there is a codimension one component of $H \cap Y_M$. $\square$

The second remark concerns the counterexample in [McM03]. There McMullen found a curve X_{ζ_7} of genus 3 with real multiplication, in fact $X_{\zeta_7} : y^2 = x^7 - 1$, and the following crucial relation between the eigendifferentials $\omega_i = x^{i-1}dx/y$: We have

$$\omega_2^2 = \omega_1 \omega_3.$$

Such a relation was ruled out by Lemma 5.1 over a Hilbert modular variety. It thus can hold on Y_M only on the hyperelliptic locus. The condition in i) is a geometric reformulation of this fact.

Third, observe that even for $\mathfrak{o} = \mathbb{Z}[\zeta_7 + \zeta_7^{-1}]$ we cannot disprove linearity using Theorem 5.5 for the part B of the corresponding Hilbert modular threefold lying in the generic stratum. The problem is that (X_{ζ_7}, ω_2) lies in $\Omega M_3(2,2)$ rather than in the generic stratum. Of course, we can disprove linearity indirectly: If B was linear, it would be $\mathrm{GL}_2^+(\mathbb{R})$-invariant by Proposition 1.3, hence its closure would be $\mathrm{GL}_2^+(\mathbb{R})$-invariant, too, contradicting Theorem 7.5 in [McM03].

Proof of Theorem 5.5. Suppose that B is linear. Then by Theorem 3.1 there is a linear subsystem $\mathbb{L}_B$ of the VHS over B such that ψ maps onto $\mathcal{O}_{\mathbb{P}}(1)|_B \otimes \mathbb{L}_B$. We write t as shorthand for the Torelli map $t^{[n]}$. By construction the fibers of $p : B \to Y_M \subset M_3$ are one-dimensional. Since moreover, via ψ, the relative tangent space maps into the $(1,0)$-part of $\mathbb{L}_B$, we conclude that $\mathbb{L}_B$ has rank two and equals the pullback to B of, say, $t^*\mathbb{L}_1$ on Y. Consequently the fibers of q have dimension 2 when restricted to B. Equivalently, the intersections of B with the leaves of the foliation by relative periods is two-dimensional. This excludes $B \subset \Omega M_3(4), \Omega M_3(2,2)$ and $\Omega M_3(3,1)$ and, using the full statement of Theorem 3.1, we conclude that ii) holds.

The second consequence of ψ mapping surjectively to $\mathcal{O}_{\mathbb{P}}(1)|_B \otimes \mathbb{L}_B$ is, using $\mathcal{O}_{\mathbb{P}}(1)|_B \cong p^*\mathcal{L}_1^{-1}|_B$, the surjectivity of

$$p^*T_{p(B)} \to R^1 f_* \omega_{X/B} \to p^*\mathcal{L}_1^{-1}|_B \otimes p^*\mathcal{L}_1^{-1}|_B \subset p^*\mathcal{L}_1^{-1}|_B \otimes R^1 f_* \mathcal{O}_X.$$

This map is a pullback of a map between bundles pulling back from Y_M. Dualizing we obtain that the restriction to $p(B)$ of the map $\psi^\vee_{Y_M}$ in the diagram

$$\begin{tikzcd}
t^*\mathcal{L}_1 \otimes t^*\mathcal{L}_1 \arrow[r] \arrow[drr, "\psi^\vee|_{Y_m}"] & t^*\mathcal{L}_1 \otimes f_*\omega_{X/Y_M} \arrow[r] \arrow[d] & t^*\Omega^1_Y(\log S) \arrow[d] \\
 & f_*\omega^{\otimes 2}_{X/Y_M} \arrow[r] & \Omega^1_{Y_M}(\log t^{-1}(S)) \arrow[d] \\
 & & \mathcal{O}_H
\end{tikzcd}$$

is injective with locally free cokernel. The right column is the short exact sequence coming from the ramification along the hyperelliptic locus. The maps are the multiplication maps and the Kodaira-Spencer maps by Lemma 2.5 and Lemma 5.1. Since the composition map in the first row is injective and the image splits off by Lemma 5.1, the linearity condition implies i).

The converse implication follows the same lines: Condition i) ensures surjectivity of ψ onto $\mathcal{O}_{\mathbb{P}}(1)|_B \otimes p^*\mathbb{L}_1|_B$ and condition ii) ensures that the image of $T_B \to T_S|_B$ is a local subsystem mapping to $\mathcal{O}_{\mathbb{P}}(1)|_B \otimes p^*\mathbb{L}_1|_B$. $\square$

Remark 5.7. Suppose that $\Omega M_3(\mathfrak{o}) = \Omega M_3(1,1,1,1)$ for some order $\mathfrak{o}$. Then the condition ii) in Theorem 5.5 may be rephrased in terms of variations of mixed Hodge structures. Let $Y_{\text{gen}} \subset Y_M$ be the open subset of the Hilbert modular threefold in the image of the generic stratum, i.e.

$$Y_{\text{gen}} = p(B \cap \Omega M_3(1,1,1,1)),$$

where $B = B(\mathfrak{o})$ is the eigenform locus as above. Let $f : X \to Y_{\text{gen}}$ be the universal family and $D \subset X$ the universal divisor with the inclusion $j : (X \smallsetminus D) \to X$. The direct image $R^1 f_*(j_!\mathbb{C})|_{Y_{\text{gen}}}$ carries a variation of mixed Hodge structures. While the weight one quotient $R^1 f_*\mathbb{C}$ is known to split as in (10), it is an interesting question to the determine direct summands of $R^1 f_*(j_!\mathbb{C})|_{Y_{\text{gen}}}$.
In order to do so, one would like to know the fundamental group of Y_{gen} as well as the representation underlying $R^1 f_*(j_!\mathbb{C})|_{Y_{\text{gen}}}$.

Although it does not seem directly relevant for linear manifolds, the same question arises for $g = 2$: What is the fundamental group of a Hilbert modular surface minus the locus of abelian surfaces that decompose with polarization and minus the image of the locus of eigenforms with a double zero. How can one describe its representation underlying the mixed Hodge structure $R^1 f_*(j_!\mathbb{C})|_{Y_M}$?

6. Linear manifolds not necessarily defined over $\mathbb{R}$: searching for a good definition

In the previous sections we have tested manifolds to have a linear structure defined over $\mathbb{R}$ in the sense of Section 1. In the cases analyzed so far, such a manifold is all of a stratum, the hyperelliptic locus or the linear structure is related to uniformization. The purpose of this section is to make the last sentence more precise in a way that linear structures not necessarily defined over $\mathbb{R}$ also fit into this context.

Eigenforms over Ball quotients ([DeMo86])

We consider the family of cyclic coverings of $\mathbb{P}^1$ of degree d ramified over the $N \geq 4$ points $\{\infty, 0, 1, x_1, \ldots, x_{N-3}\}$. Hence let

$$M := \{(x_i)_{i=1}^{N-3} \,|\, x_i \neq x_j, x_i \notin \{0,1,\infty\}\} \subset (\mathbb{P}^1)^{(N-3)}$$

and let $f : X \to M$ be the universal family of such cyclic coverings for a fixed type of covering. Here, the type contains some more information besides the ramification orders. We fix a type contained in the list on p. 86 of [DeMo86] for $N \geq 5$ and recall the key properties of these families.

In that situation M, or a completion of M adding a finite number of boundary divisors, is shown by Deligne and Mostow to be a quotient of the $(N-3)$-ball. In fact, the VHS over M has a $\mathbb{C}$-summand $\mathbb{L}$ of type $(1, N-3)$ with the following Higgs field Θ. We let $\mathcal{L} = (\mathbb{L} \otimes \mathcal{O}_M)^{(1,0)}_{\text{ext}}$. Then

$$\Theta : \mathcal{L} \cong \omega_X^{1/N} \to (T_Y \otimes \omega_X^{1/N}) \otimes \Omega^1_X \cong (\mathbb{L})^{(0,1)} \otimes \Omega^1_X$$

the canonical diagonal embedding. The analog of the eigenform locus in the Hilbert modular case is given by letting $\Omega M_g \supset B \to M$ be the total space of $\mathcal{L}$. Thus $\dim B = N-2$.

Proposition 6.1. *The total space B is contained in some stratum $S \subset \Omega M_g$. It is a linear submanifold of S and* not *defined over $\mathbb{R}$.*

Proof. As in the case of Hilbert modular surfaces we have $\mathcal{O}_{\mathbb{P}}(1)|_B \cong p^*\mathcal{L}^{-1}$ and $\psi_{\mathrm{rel}} : T_{B/M} \to \mathcal{O}_{\mathbb{P}}(1)|_B \otimes \mathcal{L}$ is the obvious isomorphism. We claim that ψ descends modulo $T_{B/M}$ to

$$\overline{\psi} : p^*T_M \to \mathcal{O}_{\mathbb{P}}(1)|_B \otimes p^*(\mathbb{L})^{(0,1)}.$$

Given the claim, $\psi : T_B \to \mathcal{O}_{\mathbb{P}}(1)|_B \otimes \mathbb{L}$ is an isomorphism for dimension reasons. The cokernel of $f^*\mathcal{L} \to \Omega^1_X$ is étale over B, in fact it coincides with the preimage of the ramification points of the cyclic cover by construction. Thus there is a unique local subsystem $\widetilde{\mathbb{L}} \subset R^1 f_* j_! \mathbb{C}$ that lifts $\mathbb{L}$ to and through which ψ factors. The linearity of B follows now from Theorem 3.1.

That $\mathbb{L}$, hence $\widetilde{\mathbb{L}}$, is not defined over $\mathbb{R}$ follows immediately the fact that its $(1,0)$-part is of rank one, while its complex conjugate is the $(0,1)$-part which is of rank $N-3$.

It remains to check the claim. The quotient of ψ on p^*T_M is the Kodaira-Spencer map with the dual of the multiplication map by Lemma 2.5 pulled back via p^*. It now suffices to apply the description of the Higgs field Θ plus the relation between Θ and the Kodaira-Spencer map (see end of Section 1). $\square$

Example 6.2. *The bundle of one-forms generated by $\omega = ydx/(x(x-1)(x-\lambda))(x-\mu)$ over the 2-dimensional family of curves*

$$y^3 = x^2(x-1)(x-\lambda)(x-\mu), \quad (\lambda,\mu) \in (\mathbb{P}^1 \smallsetminus \{0,1,\infty\})^2, \;\; \lambda \neq \mu$$

is a closed submanifold of $\Omega M_3(4)$ with linear structure. This structure is not defined over $\mathbb{R}$, but only over $\mathbb{Q}(\zeta_3)$.

Remark 6.3. We now list phenomena and pathologies we do not want a 'linear manifold' to have:

i) For any flat surface, i.e. any pair (X_0,ω_0) of a smooth curve and a one-form, the set $\mathbb{C}^* \cdot (X_0,\omega_0)$ has a linear structure. But it is never defined over $\mathbb{R}$ by Riemann's period relations and its image in M_g is a point.

ii) The construction of [DeMo86] also works for several types of $N = 4$ and yields the following, see also [BoMo05]. For example, let $d = 2mn$ for m,n odd. Denote by $g : Y \to C$ the universal family of cyclic coverings $C \to M$ pulled back from $M \cong \mathbb{P}^1 \smallsetminus \{0,1,\infty\}$ to a finite unramified covering in order to kill non-unipotent monodromies. From the construction as cyclic covering the VHS decomposes into eigenspaces

$$R^1 g_* \mathbb{C} = \bigoplus_{(i,m)=1,(j,n)=1} \mathbb{L}(i,j)$$

The $\mathbb{L}(i,j)$ are rank two local systems, not defined over $\mathbb{R}$ as local subsystems of $R^1 g_*\mathbb{C}$. In fact the $\mathbb{L}(i,j)$ are defined over $\mathbb{R}$ as abstract local system. A local subsystem isomorphic to $\mathbb{L}(i,j)$ and defined over $\mathbb{R}$ sits diagonally in a sum of 4 copies of $\mathbb{L}(i,j)$'s. This is a key observation in [BoMo05], but not important in this remark.

As a consequence, the $(1,0)$-part of $\mathbb{L}(i,j)$ is one-dimensional and we let $B(i,j)$ denote its total space. From the construction as cyclic coverings it is immediate that $B(i,j)$ lies completely in a stratum $S(i,j)$, which one depends on (i,j). For all pairs (i,j) the total space is indeed a closed submanifold of $S(i,j)$, since the fibers over $\{0,1,\infty\}$ are always singular curves. Moreover, $S(i,j)$ carries a linear structure as we now check.

The restriction ψ_{rel} of ψ to $T_{B/p(B)}$ is an isomorphism, since B is constructed as total space of a vector bundle on $p(B) = C$. Moreover it is well-known (see [BoMo05] for a proof in this language) that

$$\overline{\psi} : p^*T_{p(B)} \to \mathcal{O}_{\mathbb{P}}(1)|_{B(i,j)} \otimes R^1 f_* \mathcal{O}_B$$

maps to, but not onto, $\mathcal{O}_{\mathbb{P}}(1)|_{B(i,j)} \otimes \mathbb{L}^{(0,1)}$. For all (i,j) the zeros of the differentials generating $\mathbb{L}(i,j)^{(1,0)}$ are the preimages of $\{0,1,\infty,x_1\}$ via the cyclic covering. In particular the difference of any two zeros is torsion in $\mathrm{Pic}^0(Y/C)$. Hence $\mathbb{L}(i,j)$ lifts to a local subsystem of the local system of relative periods, as needed to apply Theorem 3.1.

But, fixing d, only for 4 pairs (i,j) the linear structure controls the uniformization in the sense that the monodromy Γ of $\mathbb{L}(i,j)$ is discrete and $\mathbb{H}/\Gamma$ is a partial compactification C_0 of C. The crux is that in all the other cases, the Kodaira-Spencer map $\overline{\psi}$ naturally extends to C_0, but it is no longer onto $\mathcal{O}_{\mathbb{P}}(1)|_{B(i,j)} \otimes \mathbb{L}^{(0,1)}$ at $C_0 \smallsetminus C$.

iii) It seems natural to remedy the above problem by imposing the existence of a closure of B with good properties. Neither for B (due to the $\mathbb{C}^*$-action) nor for $p(B)$ (since all information about the one-forms is lost) we expect to have such a compactification in general. On the other hand even in the intermediate case, just dividing by $\mathbb{C}^*$, the quotient carries no longer a linear structure. But the tangent mappings still exist on $B/\mathbb{C}^*$. This motivates the following definition, justified by Theorem 6.5.

Fix some stratum S Let $\pi : S \to S/\mathbb{C}^* \subset \mathbb{P}\Omega M_g$ be the quotient map by $\mathbb{C}^*$ and let $\tau : S/\mathbb{C}^* \to M_g$ be the forgetful map. We thus have a factorization $p = \tau \circ \pi$. Recall that the local system $\mathbb{L}_B$ depends only on the Hodge structure of the fibers, not on the one-form chosen. It is thus a pullback $\mathbb{L}_B = p^*\mathbb{L}$ of a local system $\mathbb{L}$ on $p(B)$.

Definition 6.4. *A submanifold B in some stratum $S \subset \Omega M_g$ is called a* linear manifold*, if B is closed in S, if it inherits from S a linear structure, if its image in M_g is not reduced to a point and if there exists a compactification Y of $\pi(B)$ with boundary Δ a normal crossing divisor with the following property: Write $\Delta = T + U$ as disjoint union, such that the monodromy of $\mathbb{L}$ is trivial around components of T and unipotent around components of U. There is a surjective map*

$$\psi_Y : T_Y(-\log U) \to \mathcal{O}_{\mathbb{P}\Omega M_g}(1)|_Y \otimes \tau^*(\mathbb{L}),$$

whose restriction to $\pi(B)$ and pullback via π yields the quotient map of ψ

$$T_B/T_{B/\pi(B)} \to \mathcal{O}_{\mathbb{P}}(1)|_B \otimes \mathbb{L}_B/\psi(T_{B/\pi(B)}).$$

Theorem 6.5. *The following manifolds match the conditions of Definition 6.4*

i) *The connected components of strata and the hyperelliptic locus are $\mathbb{R}$-linear manifolds.*

ii) *The eigenform locus over a Hilbert modular surface is a linear manifold defined over* $\mathbb{R}$.
iii) *The canonical lift of a Teichmüller curve is a linear manifold defined over* $\mathbb{R}$.

Sketch of proof: In case i) for a stratum S, take the closure of $\pi(S)$ in the projectivized total space of the relative dualizing sheaf $\overline{\mathbb{P}\Omega M_g}$ over the Deligne-Mumford compactification of M_g and take a blowup of the lower-dimensional strata to make Δ a normal crossing divisor. In that case, $\mathbb{L}$ coincides with the full de Rham cohomology. Hence $U = \tau^{-1}(\partial M_g) \cap Y$. Around U the Hodge metric has logarithmic growth. This implies that ψ_Y is surjective near U.

For the hyperelliptic locus the same argument works using the compactification by admissible double coverings ([HaMo98]).

For the Hilbert modular surfaces B in ii) take Y as in diagram (9) as compactification of $p(B) = Y_M$. Equation (11) is precisely what we need.

For iii) note that a Teichmüller curve $C \cong \pi(B) \cong p(B)$ does not intersect the locus of curves with separating nodes. Consequently, $U = \overline{C} \smallsetminus C = \Delta$ and the desired surjection follows again from the growth of the Hodge metric. □

To justify this complicated definition, recall that a Teichmüller curve C with compactification $\overline{C}$ and $\overline{C} \smallsetminus C = \Delta$ is characterized by having a rank two local system $\mathbb{L}$ that is maximal Higgs, i.e. such that the Higgs field

$$\theta : \mathbb{L}^{(1,0)} \to \mathbb{L}^{(0,1)} \otimes \Omega^1_{\overline{C}}(\log \Delta)$$

is an isomorphism. A 2-dimensional linear manifold must be *almost Teichmüller* in the following sense: i) By the second condition of Theorem 3.1 and Definition 6.4 it has a rank two local system $\mathbb{L}$ that is maximal Higgs with U, i.e. such that

$$\mathbb{L}^{(1,0)} \to \mathbb{L}^{(0,1)} \otimes \Omega^1_{\overline{C}}(\log U)$$

is an isomorphism. ii) It satisfies the first condition of Theorem 3.1 involving relative periods.

We need to allow this modification of the usual definition, otherwise the locus of reducible Jacobians would cause a Hilbert modular surface not to be a linear manifold. From the construction as closed $\mathrm{GL}_2^+(\mathbb{R})$-orbits we deduce that an $\mathbb{R}$-linear almost Teichmüller curve is in fact a Teichmüller curve. If we drop the condition linearity over $\mathbb{R}$ this is no longer the case, see Section 7.

7. Linear manifolds in low dimension: $g = 2$

We show that the preparation made above yields a short proof of the classification of manifolds with linear structure in ΩM_2 (originally due to McMullen, [McM07]) under the additional hypothesis that the manifold is algebraic and with the slight generalization of not restricting to linear manifolds defined over $\mathbb{R}$.

Referring to the notation of Theorem 3.1, we let for a linear manifold B denote $\tilde{d} := \mathrm{rank}(\widetilde{\mathbb{L}_B})$ and $d := \mathrm{rank}(\mathbb{L}_B)$. If the linear structure is defined over $\mathbb{R}$, then d must be even and $d/2$ is the dimension of the fibers of $p : B \to p(B) \subset M_g$.

Theorem 7.1. *A linear submanifold B in a stratum of ΩM_2 is one of the following possibilities:*

i) *One of the strata $\Omega M_2(2)$ or $\Omega M_2(1,1)$.*
ii) *The locus of eigenforms over a Hilbert modular surface or over a surface parameterizing curves with a reducible Jacobian.*
iii) *The canonical lift of a Teichmüller curve to ΩM_2*
iv) *A family of differentials over a Shimura curve parameterizing Jacobians whose endomorphism ring is a quaternion algebra. This quaternion algebra is indefinite or a matrix algebra.*

Precisely in the cases i), ii) and iii) the linear structure is defined over $\mathbb{R}$.

An example of a quaternionic Shimura curve as in iv) is the family $y^3 = x(x-1)(x-\lambda)^2$. The total spaces of both eigenform bundles for $\mathbb{Q}(\zeta_3)$-multiplication generated by

$$\omega_1 = ydx/x(x-1)(x-\lambda) \quad \text{and by} \quad \omega_2 = y^2dx/x(x-1)(x-x_1)^2$$

are $\mathbb{Q}(\zeta_3)$-linear manifolds. The zeros of ω_1 are the preimages of $x = 0$ and $x = 1$, while the zeros of ω_2 are the preimages of $x = \lambda$ and $x = \infty$. The quotient by the involution

$$(x, y) \mapsto \left(\frac{\lambda}{x}, \frac{(\lambda - x)(x-1)\lambda}{xy}\right)$$

maps the family to a family of elliptic curves. Hence the endomorphism ring is a matrix algebra. The proof of linearity is exactly the same as that of Proposition 6.1.

Question 7.2. *Is this the only example of such a curve in genus 2? If not, how to classify them? Are there examples where the quaternion algebra is indefinite?*

Proof of Theorem 7.1. We list the cases for $(\tilde{d}, d)$: Case $d = 1$ is excluded by the definition of a linear manifold, compare Remark 6.3. If $d = 3$, B is not defined over $\mathbb{R}$. The local system $\mathbb{L}_B$ intersects its complex conjugate non-trivially. Hence B parameterizes a family of curves whose Jacobian has a one-dimensional fixed part. For dimension reasons, this is impossible both in $\Omega M_2(1,1)$ and $\Omega M_2(2)$.

In the stratum $\Omega M_2(2)$ the only remaining possibilities are $(2,2)$ and $(4,4)$. Since $\Omega M_2(2)$ is irreducible, $(4,4)$ corresponds to a stratum. The pair $(2,2)$ gives the a $\mathbb{C}^*$-bundle over a curve. We have to show that the manifold is a Teichmüller curve or a quaternionic Shimura curve. This is the content of the Lemma below.

In the stratum $\Omega M_2(1,1)$ the possibilities are first $(5,4)$, which is the whole stratum since it is irreducible, second $(2,2)$, which is again a Teichmüller curve by the same arguments, and finally $(3,2)$. The case $(4,4)$ is impossible by the argument used in the proof of Theorem 4.1: The fiber dimension is 2 in this case and the multiplication map implies that $p(B)$ is at least 3-dimensional.

In the case $(3,2)$ the local system $\mathbb{L}$ is irreducible for dimension reasons and since the linear manifold B does not map to a point in M_2. In particular, the fibers of $B \to p(B)$ are generically one-dimensional. Suppose that $\mathbb{L}$ is not defined over $\mathbb{R}$. Then $p(B)$ parameterizes curves whose Jacobian has endomorphisms by a complex field. By [Sh63] Proposition 19 this endomorphism ring is in fact even larger. Such curves are parameterized by Shimura curves. We conclude that $p(B)$ is one-dimensional and obtain a contradiction since $\dim(B) = 3$.

Hence the family of Jacobians either splits or has real multiplication. In the second case, again for dimension reasons, $p(B)$ has to be dense in the corresponding Hilbert modular surface. Recall that over a Hilbert modular surface the VHS splits into $\mathbb{L}_1 \oplus \mathbb{L}_2$. Hence for $i = 1$ or $i = 2$ the map

$$\psi_{\mathrm{rel}} : T_{B/p(B)} \to \mathcal{O}_{\mathbb{P}}(1) \otimes p^*\mathbb{L}_i^{(0,1)}$$

is an isomorphism. We deduce that B is the total space of the eigenform bundle over the Hilbert modular surface. The case of split Jacobians is similar.

The linearity of the manifolds in the list has been shown in Theorem 6.5. □

Lemma 7.3. *A linear manifold of dimension 2, not necessarily $\mathbb{R}$-linear, in a stratum of ΩM_2 is the canonical lift of a Teichmüller curve or a quaternionic Shimura curve parameterizing Jacobians whose endomorphism ring is a quaternion algebra, either indefinite or a matrix algebra.*

Proof. Since multiplication by $\mathbb{C}^*$ does not change the Hodge structure, the local system $\mathbb{L}_B$ (as in Theorem 3.1) is a pullback from $p(B) =: C$. Being isomorphic to T_B and of rank 2, the local system $\mathbb{L}_B$ must be irreducible. Hence either the VHS over C splits over $\mathbb{Q}$ and the Jacobians of the universal family $X \to C$ split up to isogeny. Or the family of Jacobians of f has multiplication by a field K of degree 2 over $\mathbb{Q}$.

In the first case, though $\mathbb{L}_B$ is defined over $\mathbb{Q}$, we cannot a priori not conclude that $\widetilde{\mathbb{L}_B}$ is defined over $\mathbb{R}$. But we know that the universal family $f : X \to C$ admits a map $h : X \to E$ to a family of elliptic curves $E \to C$. Moreover, since

$$\psi_{\mathrm{rel}} : T_{B/p(B)} \to \mathcal{O}_{\mathbb{P}}(1) \otimes p^*\mathbb{L}_B^{(0,1)}$$

is an isomorphism, we know moreover that the differentials parameterized by B pull back from E. On fibers of E there are only two absolute periods, $\mathbb{R}$-linearly independent. Hence if the relative period on X depends linearly on the absolute ones, it is possible to change the defining equation in order to obtain linear dependence defined over $\mathbb{R}$.

The second case is K is real, in particular $\mathbb{L}_B$ is defined over $\mathbb{R}$. Again we can not yet conclude that $\widetilde{\mathbb{L}_B}$ is defined over $\mathbb{R}$, too. But since

$$\psi_{\mathrm{rel}} : T_{B/p(B)} \to \mathcal{O}_{\mathbb{P}}(1) \otimes p^*\mathbb{L}_B^{(0,1)}$$

is an isomorphism, we know moreover that B lies in the eigenform locus of real multiplication. By Theorem 5.2 (or [McM07]) the eigenform locus is linear and defined over $\mathbb{R}$.

The last case is K not real. In this case the endomorphism ring of the family of Jacobians is larger than K ([Sh63] Proposition 19). It is an indefinite quaternion algebra, either a division algebra and the generic Jacobian is simple or a matrix algebra and the family of Jacobians splits. In both cases, abelian varieties with such an endomorphism ring are parameterized by a countable union Shimura curves. Hence for dimension reasons, C coincides with such a Shimura curve. □

8. Linear manifolds in low dimension: The hyperelliptic locus $\mathcal{H}$ in the non-hyperelliptic component of $\Omega M_3(2,2)$

Combining the analysis of the VHS as in Section 7 and the use of the multiplication map as in Theorem 4.1, one can write down for a given stratum a list of possibilities of closed algebraic submanifolds. We will do this for a special case, the hyperelliptic locus $\mathcal{H}$ in the non-hyperelliptic component of $\Omega M_3(2,2)$, i.e. the one with odd spin structure

$\Omega M_3(2,2)^{\text{odd}}$. Recall that $\mathcal{H}$ is of dimension 5 and up to $\mathbb{C}^*$ it is finite over the hyperelliptic locus. This choice is motivated by [HLM06]: On one hand, this locus is, besides $g = 2$ probably the simplest one to classify $\mathrm{GL}_2^+(\mathbb{R})$-orbit closures using connected sum constructions and Ratner's theorem for products of tori. On the other hand, $\mathrm{GL}_2^+(\mathbb{R})$-invariant submanifolds of ΩM_2 are classified, but $\mathrm{GL}_2^+(\mathbb{R})$-orbit closures in the whole cotangent bundle to M_2, i.e. consisting of a Riemann surface plus a quadratic differential, are not yet classified. By a well-known covering construction (recalled e.g. in [HLM06]), they correspond bijectively to $\mathrm{GL}_2^+(\mathbb{R})$-orbit closures in $\mathcal{H}$.

It is convenient to introduce some more definitions. Given an algebraic manifold B with linear structure, consider the monodromy group Γ of the local system $\mathbb{L}_B$ as in Theorem 3.1. Recall from the proof of Lemma 4.2 that $\mathbb{L}_B$ is isomorphic to a local system defined over a number field. In particular $K := \mathbb{Q}(\mathrm{tr}(\gamma), \gamma \in \Gamma)$ is a number field, called the *trace field* of B. Here the more important notion is the field F fixed by all elements σ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\sigma(\mathbb{L}_B) \cong \mathbb{L}_B$. We call F the *fixed field* of B. Obviously $K \subset F$. For Teichmüller curves ([Mö06a] Lemma 2.3) and Hilbert modular surfaces both fields coincide.

Theorem 8.1. *A closed submanifold B in $\mathcal{H}$ with linear structure defined over $\mathbb{R}$ is one of the following possibilities:*

i) *the whole locus $\mathcal{H}$, or*
ii) *a Teichmüller curve, or*
iii) *a manifold of dimension 4 with fixed field of degree 1 or 2 over $\mathbb{Q}$.*

Proof. We first claim that the foliation by absolute periods and the hyperelliptic locus intersect transversely in $\Omega M_3(2,2)^{\text{odd}}$. Consider the diagram (5) restricted to a point $(X_0 : y^2 = \prod(x - x_i), \omega_0)$ in $\Omega M_3(2,2)^{\text{odd}}$. The hyperelliptic involution h acts on all bundles involved. Since $D - D_{\text{red}}$ is fixed by h, the involution h fixes $\mathrm{Ker}(\pi)$, too. On the whole de Rham cohomology h acts by (-1), since $x^{i-1}dx/y$ for $i = 1,2,3$ is a basis of the one-forms and $[y/x^i] \in \Gamma(X_0 \smallsetminus \{0,\infty\}, \mathcal{O}_{X_0})$ for $i = 1,2,3$ is a basis, using Czech cohomology, of $H^1(X_0, \mathcal{O}_{X_0})$. Hence the image of $q^\vee|_{\mathrm{Ker}(\pi)}$ is in the $+1$-eigenspace of $H^0(X_0, (\Omega^1_{X_0})^{\otimes 2})$. This eigenspace is precisely the cotangent space to the hyperelliptic locus (e.g. [OS80]). Since, dually, the kernel of q defines the tangent space to the foliation by absolute periods, the claim follows.

From the claim we conclude that the possible dimensions of $\widetilde{\mathbb{L}}_B$ and $\mathbb{L}_B$ are $\tilde{d} = d \in \{6,2,4\}$. These give the rough classification.

It remains to check the claim on the degree of the fixed field F in the case $d = 4$. If $[F : \mathbb{Q}] = 3$, then curves parameterized by B have real multiplication by F. For dimension reasons $p(B)$ equals a dense subset of a Hilbert modular threefold. This contradicts Proposition 5.6. $\square$

An example for case iii) with fixed field $\mathbb{Q}$ is to take for $p(B)$ a component of the Hurwitz space of unramified double coverings of a genus 2 surface and for B the pullback of elements in $\Omega M_2(1,1)$ under this covering map. For a suitable choice of the double covering, the surfaces in $\Omega M_3(2,2)$ thus obtained are indeed in $\Omega M_2(2,2)^{\text{odd}}$ and hyperelliptic.

Corollary 8.2. *Let Δ be a Teichmüller disc generated by a pair $(X_0, \omega_0) \in \mathcal{H}$ which is stabilized by a pseudo-Anosov diffeomorphism with trace field of degree 3 over $\mathbb{Q}$. If* *the closure of the orbit* $\mathrm{GL}_2^+(\mathbb{R}) \cdot (X_0, \omega_0)$ *is an algebraic manifold, then it is either the canonical lift of a Teichmüller curve to* $\mathcal{H}$ *or as big as possible, i.e.* $\overline{\mathrm{GL}_2^+(\mathbb{R}) \cdot (X_0, \omega_0)} = \mathcal{H}$.

Proof. Suppose the claim is wrong. Then by Theorem 8.1 we deduce that the closure $B := \overline{\mathrm{GL}_2^+(\mathbb{R}) \cdot (X_0, \omega_0)}$ has dimension 4 and $\dim p(B) = 3$. The pseudo-Anosov diffeomorphism corresponds to a closed geodesic γ in the image of Δ of Δ. The monodromy of the subspace $V := \langle \mathrm{Re}\omega_0, \mathrm{Im}\omega_0 \rangle$ along γ has a trace $\mathrm{tr}(\gamma)$ that generates a field with $[\mathbb{Q}(\mathrm{tr}(\gamma)) : \mathbb{Q}] = 3$. Suppose that for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ we had $\sigma(\mathbb{L}_B) \cong \mathbb{L}_B$. Then this holds in particular when restricting the local system to γ. Moreover it holds for the invariant subspace V of $\mathbb{L}_B|_\gamma$. Hence σ fixes $\mathbb{Q}(\mathrm{tr}(\gamma))$ and statement iii) in Theorem 8.1 yields the contradiction. $\square$

In [HLM06] this closure statement was shown for a particular Teichmüller disc, that does not descend to a Teichmüller curve, without any a priori manifold hypothesis on the closure.

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