QUADRATIC DIFFERENTIALS IN LOW GENUS: EXCEPTIONAL AND NON-VARYING STRATA

DAWEI CHEN AND MARTIN MÖLLER

Abstract. We give an algebraic way of distinguishing the components of the exceptional strata of quadratic differentials in genus three and four. The complete list of these strata is $(9,-1)$, $(6,3,-1)$, $(3,3,3,-1)$ in genus three and (12), $(9,3)$, $(6,6)$, $(6,3,3)$ and $(3,3,3,3)$ in genus four. The upshot of our method is a detailed study regarding the geometry of canonical curves.

This result is part of a more general investigation about the sum of Lyapunov exponents of Teichmüller curves, building on [EKZ11], [Che11] and [CM12]. Using disjointness of Teichmüller curves with divisors of Brill-Noether type on the moduli space of curves, we show that for many strata of quadratic differentials in low genus the sum of Lyapunov exponents for the Teichmüller geodesic flow is the same for all Teichmüller curves in that stratum.

Contents

1. Introduction

The moduli space $\Omega\mathcal{M}_g$ of Abelian differentials, also called the Hodge bundle, parameterize Abelian differentials ω on genus g Riemann surfaces. Let $m_1, \ldots, m_k$ be positive integers such that $\sum_{i=1}^{k} m_i = 2g-2$. Then $\Omega\mathcal{M}_g$ decomposes into

During the preparation of this work the first author is partially supported by NSF grant DMS-1101153 (transferred as DMS-1200329). The second author is partially supported by ERC-StG 257137.

strata $\Omega\mathcal{M}_g(m_1, \ldots, m_k)$ according to the number and multiplicity of the zeros of ω. Since the Teichmüller geodesic flow preserves these strata, many problems in Teichmüller theory can be dealt with stratum by stratum.

Similarly, let $d_1, \ldots, d_n$ be non-zero integers such that $\sum_{j=1}^{n} d_j = 4g - 4$ and $d_j \geq -1$ for all j. The moduli space of quadratic differentials parameterizing pairs (X, q) of a genus g Riemann surface X and a quadratic differential q with at most simple poles is stratified in the same way into $\mathcal{Q}(d_1, \ldots, d_n)$, namely, q has a zero of multiplicity d_i at some point p_i for $d_i > 0$ and has a simple pole at p_j for $d_j = -1$.

Not much is known on the topology of the strata. Kontsevich and Zorich determined in [KZ03] the connected components of $\Omega\mathcal{M}_g(m_1, \ldots, m_k)$. Some strata have hyperelliptic components parameterizing Abelian differentials on hyperelliptic curves that have a single zero or a pair of zeros interchangeable under the hyperelliptic involution, some strata have components distinguished by the spin parity $\dim H^0(X, \mathrm{div}(\omega)/2) \bmod 2$, and the others are connected. The connected components for strata of quadratic differentials were determined by Lanneau in [Lan08]. Some have hyperelliptic components and besides a short list of *exceptional* cases, all the other strata are connected.

To find an algebraic invariant distinguishing the exceptional cases remained an open problem. Our first main result provides a solution to this problem. Let (X, q) be a quadratic differential in $\mathcal{Q}(d_1, \ldots, d_n)$. Suppose q has a zero or pole of order d_i at p_i for $1 \leq i \leq n$. Write $\mathrm{div}(q) = \sum_{i=1}^{n} d_i p_i$ as the total divisor of q and $\mathrm{div}(q)_0 = \sum_{d_i > 0} d_i p_i$ as the zero divisor of q.

Theorem 1.1. *Each of the strata* $(9, -1)$*,* $(6, 3, -1)$ *and* $(3, 3, 3, -1)$ *in genus three has precisely two connected components, distinguished by*

$$\dim H^0(X, \mathrm{div}(q)_0/3) = 1 \quad \textit{resp.} \quad \dim H^0(X, \mathrm{div}(q)_0/3) = 2.$$

We also construct the connected components using techniques from algebraic geometry. This provides a proof of the connectedness (and irreducibility) of the two components that does not rely on any geometry of flat surfaces.

For $g = 4$ we discovered that the list of exceptional strata was incomplete in [Lan08].

Theorem 1.2. *Each of the strata* (12)*,* $(9, 3)$*,* $(6, 6)$*,* $(6, 3, 3)$ *and* $(3, 3, 3, 3)$ *in genus four has precisely two non-hyperelliptic connected components, distinguished by*

$$\dim H^0(X, \mathrm{div}(q)/3) = 1 \quad \textit{resp.} \quad \dim H^0(X, \mathrm{div}(q)/3) = 2.$$

Let us describe the upshots in proving Theorems 1.1 and 1.2, see Sections 6 and 7 for details. Consider the stratum $(9, -1)$ as an example. The canonical model of a non-hyperelliptic, genus three curve X is a plane quartic. If X admits a quadratic differential q with $\mathrm{div}(q) = 9p_1 - p_2$, then there exists a unique plane cubic E such that E and X intersect at p_1 with multiplicity 9. Furthermore, we have $\mathcal{O}_E(9p_1) \sim \mathcal{O}_E(3)$, where $\mathcal{O}(1)$ is the universal line bundle of $\mathbb{P}^2$. Two possibilities can occur, either $\mathcal{O}_E(3p_1) \sim \mathcal{O}_E(1)$ or $\mathcal{O}_E(3p_1) \not\sim \mathcal{O}_E(1)$, which distinguishes the claimed two components. In order to construct these two components, we first fix E and p_1, then consider plane quartics intersecting E at p_1 with multiplicity 9, and finally quotient out the parameter space by the automorphism group of $\mathbb{P}^2$. The same idea applies to the exceptional strata in genus four, using the fact that a canonical curve of genus four is contained in a unique quadric surface in $\mathbb{P}^3$.

In order to use the parity curve E, we need to control its singularities, which boils down to a tedious local analysis. To avoid confusing the reader by technical details, we postpone the argument to Appendix B.

We remark that the criteria related to $\mathrm{div}_0(q)/3$ and $\mathrm{div}(q)/3$ are analogous to that of $\mathrm{div}(\omega)/2$ in distinguishing the odd and even spin components of certain strata of Abelian differentials, see [KZ03] and Sections 6, 7 for more details. It is well-known that the spin parity associated to $\mathrm{div}(\omega)/2$ is a deformation invariant, but the parity associated to $\mathrm{div}_0(q)/3$ and $\mathrm{div}(q)/3$ seems only an isolated example in low genus. Indeed, one can compute $\dim H^0(X, \mathrm{div}(\omega)/2) \bmod 2$ by using the Arf invariant, see [KZ03, Section 3]. But an interpretation of $\dim H^0(X, \mathrm{div}_0(q)/3)$ and $\dim H^0(X, \mathrm{div}(q)/3)$ in terms of flat geometry is not known. We thus leave as interesting *open question:* compute the parity of $\mathrm{div}_0(q)/3$ resp. $\mathrm{div}(q)/3$ using flat geometry only, as for the Arf invariant.

The above results were obtained in parallel with our investigation of sums of Lyapunov exponents for Teichmüller curves. In this sense, the present paper is a continuation to quadratic differentials of our paper [CM12]. A connected component of a stratum was called *non-varying*, if for all Teichmüller curves in this stratum the sum of Lyapunov exponents is the same, and *varying* otherwise. We proved that many strata (components) of Abelian differentials in low genus are non-varying.

Let us recall the basic idea in [CM12]. The Siegel-Veech area constant c, the sum of Lyapunov exponents L and the slope s determine each other for a Teichmüller curve generated by an Abelian differential in $\Omega\mathcal{M}_g(m_1, \ldots, m_k)$:

$$s = \frac{12c}{L} = 12 - \frac{12\kappa}{L},$$

where $\kappa = \frac{1}{12}\sum_{i=1}^{k} \frac{m_i(m_i+2)}{m_1+1}$, see [EKZ11] and [Che11]. Let $\overline{C}$ denote the closure of a Teichmüller curve C in the compactified moduli space of curves $\overline{\mathcal{M}}_g$. We want to construct a divisor D in $\overline{\mathcal{M}}_g$ such that D is disjoint with $\overline{C}$ for all Teichmüller curves C in a given stratum. In the case of Abelian differentials, $\overline{C}$ does not intersect higher boundary divisors δ_i in $\overline{\mathcal{M}}_g$ for $i > 0$. Then we can compute the slope as well as the sum of Lyapunov exponents directly from the equality $\overline{C} \cdot D = 0$.

For quadratic differentials, the hyperelliptic strata were proved to be non-varying by [EKZ11], see Corollary 2.1 for more details. Here as our second main result, we prove that many non-hyperelliptic strata of quadratic differentials in low genus are non-varying.

Theorem 1.3. *Consider the strata of quadratic differentials in low genus.*

(1) In genus one, the strata $\mathcal{Q}(n, -1^n)$ and $\mathcal{Q}(n-1, 1, -1^n)$ are non-varying for $n \geq 2$ (Theorem 8.1).

(2) In genus two, there are 12 non-varying strata among all strata of dimension up to seven (Theorem 9.1).

(3) In genus three, there are 19 non-varying strata among all non-exceptional strata of dimension up to eight (Theorem 10.1) and 6 non-varying strata among all exceptional strata (Theorem 6.2).

(4) In genus four, there are 8 non-varying strata among all non-exceptional strata of dimension up to nine (Theorem 11.1) and 7 non-varying strata among all exceptional strata (Theorem 7.2).

Let us explain the upshot in proving Theorem 1.3 as well as the difference from the case of Abelian differentials. For Teichmüller curves generated by quadratic differentials in $\mathcal{Q}(d_1, \ldots, d_n)$, we have a similar relation between the Siegel-Veech area constant c, the sum of (involution-invariant) Lyapunov exponents L^+ and the slope s:

$$s = \frac{12c}{L^+} = 12 - \frac{12\kappa}{L^+},$$

where $\kappa = \frac{1}{24}\sum_{j=1}^{n} \frac{d_j(d_j+4)}{d_j+2}$, see Propositions 4.1 and 4.2. By using a divisor disjoint from a Teichmüller curve $\overline{C}$, one would naturally expect to read off the value of $L^+(C)$ from the divisor class. However, in the case of quadratic differentials, Teichmüller curves may intersect higher boundary divisors, because a core curve of a cylinder may disconnect the associated flat surface for quadratic differentials, whereas this is impossible for Abelian differentials, see Remark 4.7. Thus, for a divisor D in $\overline{\mathcal{M}}_g$ with class

$$D = a\lambda + b\delta_0 + \sum c_i\delta_i,$$

even if $\overline{C} \cdot D = 0$, we cannot directly deduce the slope $s = (\overline{C} \cdot \delta)/(\overline{C} \cdot \lambda)$, where $\delta = \sum \delta_i$ is the total boundary. Therefore, for a claimed non-varying stratum of quadratic differentials, it requires a considerable amount of work using both algebraic geometry and flat geometry to study the intersection of $\overline{C}$ with higher boundary divisors δ_i occurring in the divisor class of D.

Moreover, for a number of non-varying strata we are only able to construct a disjoint divisor in the moduli space of pointed curves $\overline{\mathcal{M}}_{g,n}$, hence we lift a Teichmüller curve C to $\overline{\mathcal{M}}_{g,n}$ by marking n zeros or poles of its generating differential. Besides λ and the boundary classes, a divisor class in $\overline{\mathcal{M}}_{g,n}$ may also contain the first Chern class ω_i of the relative dualizing line bundle associated to the ith marked point. Consequently we have to understand the intersection $\overline{C} \cdot \omega_i$. This calculation is carried out in Proposition 4.2.

Among the non-varying strata in Theorem 1.3, there are three of them for which our standard method does not work. In other words, we are not able to find divisors disjoint with all Teichmüller curves in these three strata. Instead, we adapt the idea of [YZ12] by using certain filtration of the Hodge bundle, which is treated in Appendix A.

Finally in genus five, we show that even the stratum with a unique zero is varying (Appendix C). Therefore, it seems quite plausible that our list of non-varying strata (including the known hyperelliptic strata by [EKZ11]) is complete. Nevertheless, for a varying stratum it would still be interesting to figure out the value distribution for the sums of Lyapunov exponents for all Teichmüller curves contained in the stratum.

This paper is organized as follows. In Section 2 we provide the background on strata of Abelian and quadratic differentials. A result of independent interest shows that near certain boundary strata of the moduli space the period and plumbing parameters are coordinates of strata of quadratic differentials.

In Section 3 we recall the Picard group of moduli spaces and various divisor classes. Section 4 discusses properties of Teichmüller curves generated by quadratic differentials near the boundary of the moduli space.

In order to prove disjointness of Teichmüller curves with various divisors in genus three and four along the hyperelliptic locus and the Gieseker-Petri locus, the use of the canonical model of a curve is not sufficient. Instead, we need to use the bicanonical model. The necessary background is provided in Section 5. Finally, Sections 6 to 11 contain the discussion of irreducible components and non-varying strata summarized in our main results.

Acknowledgments: We thank Vincent Delecroix and Anton Zorich for supplying us with an ample amount of computer data on which our investigation of non-varying strata and a proof for varying strata in the appendix were based. We also want to thank Izzet Coskun and Joe Harris for helpful suggestions on the geometry of low genus curves. Moreover, we thank Fei Yu for the conversation that led to the results in Appendix A. Finally we want to thank the anonymous referees for many useful suggestions which helped improve the exposition of the paper.

Part of the work was done during the Oberwolfach Meeting 'Billiards, Flat Surfaces and Dynamics on Moduli Spaces', May 2011 and the Park City Summer Program 'Moduli Spaces of Riemann Surfaces', July 2011. We would like to thank the organizers for their invitation and hospitality.

2. Background on moduli spaces

2.1. **Strata and hyperelliptic loci.** For $d_i \geq -1$, $d_i \neq 0$ and $\sum_{i=1}^n d_i = 4g-4$, let $\mathcal{Q}(d_1, \ldots, d_n)$ denote the *moduli space of quadratic differentials*. It parameterizes pairs (X, q) of a genus g Riemann surface X and a quadratic differential q on X that have n distinct zeros or poles of order $d_1, \ldots, d_n$. Here we focus on the case that q is not a global square of an Abelian differential. Otherwise it reduces to the study of strata of Abelian differentials as in [CM12]. The condition $d_i \geq -1$ ensures that the quadratic differentials in $\mathcal{Q}(d_1, \ldots, d_n)$ have at most simple poles and that their total flat volume is thus finite. The pairs (X, q) are called *half-translation surfaces*. We denote by $\mathbb{P}\mathcal{Q}(d_1, \ldots, d_n) = \mathcal{Q}(d_1, \ldots, d_n)/\mathbb{C}^*$ the associated projectivized space.

Let $\Omega\mathcal{M}_g$ denote the *Hodge bundle* of holomorphic one-forms over the moduli space $\mathcal{M}_g$ of genus g curves and let $\mathbb{P}\Omega\mathcal{M}_g$ denote the associated projective bundle. The spaces $\Omega\mathcal{M}_g$ and $\mathbb{P}\Omega\mathcal{M}_g$ are stratified according to the multiplicities of the zeros of one-forms. For $m_i \geq 1$ and $\sum_{i=1}^k m_i = 2g-2$, let $\Omega\mathcal{M}_g(m_1, \ldots, m_k)$ denote the stratum parameterizing one-forms that have k distinct zeros of order $m_1, \ldots, m_k$.

Denote by $\overline{\mathcal{M}}_g$ the Deligne-Mumford compactification of $\mathcal{M}_g$. The Hodge bundle extends to the boundary of $\overline{\mathcal{M}}_g$, parameterizing *stable one-forms* or equivalently sections of the dualizing sheaf. We denote the total space of this extension by $\Omega\overline{\mathcal{M}}_g$.

Points in $\Omega\mathcal{M}_g$, called *flat surfaces*, are usually written as (X, ω) for a one-form ω on X. For a stable curve X, denote the dualizing sheaf by ω_X. We will stick to the notation that points in $\Omega\overline{\mathcal{M}}_g$ are given by a pair (X, ω) with $\omega \in H^0(X, \omega_X)$.

If the quadratic differential is not a global square of a one-form, there is a *canonical double covering* $\pi : Y \to X$ such that $\pi^* q = \omega^2$. This covering is ramified precisely at the zeros of odd order of q and at the poles. It gives a map

$$\phi : \mathcal{Q}(d_1, \ldots, d_n) \to \Omega\mathcal{M}_g(m_1, \ldots, m_k),$$

where the signature $(m_1, \ldots, m_k)$ is determined by the ramification type (see [KZ03] for more details).

If the domain and the range of the map ϕ have the same dimension for some signature, we call the image a *component of hyperelliptic flat surfaces* of the corresponding stratum of Abelian differentials. This can only happen if the domain of ϕ parameterizes genus zero curves. More generally, if the domain of ϕ parameterizes genus zero curves, we call the image a *locus of hyperelliptic flat surfaces* in the corresponding stratum. These loci are often called hyperelliptic loci, e.g. in [KZ03] and [EKZ11]. We prefer to reserve *hyperelliptic locus* for the subset of $\mathcal{M}_g$ (or its closure in $\overline{\mathcal{M}}_g$) parameterizing hyperelliptic curves and thus specify with 'flat surfaces' if we speak of subsets of $\Omega\mathcal{M}_g$.

Instead of taking the canonical double covering one can start with (X, q) in a stratum of quadratic differentials, prescribe the topology of a double covering with branch points contained in the set of zeros and poles of q and consider the locus of branched coverings (Y, q_Y) obtained in that way.

The main result of [Lan04] states that only if $g(X) = 0$ and only for the following three types of non-canonical double coverings the dimensions of the strata containing (X, q) resp. (Y, q_Y) coincide.

(1) $\mathcal{Q}(2(g-k)-3, 2k+1, -1^{2g+2}) \to \mathcal{Q}(2(g-k)-3, 2(g-k)-3, 2k+1, 2k+1)^{\mathrm{hyp}}$.
(2) $\mathcal{Q}(2(g-k)-3, 2k, -1^{2g+1}) \to \mathcal{Q}(2(g-k)-3, 2(g-k)-3, 4k+2)^{\mathrm{hyp}}$.
(3) $\mathcal{Q}(2(g-k)-4, 2k, -1^{2g}) \to \mathcal{Q}(4(g-k)-6, 4k+2)^{\mathrm{hyp}}$.

Consequently the images of these maps are connected components of the corresponding strata of quadratic differentials. They will be called *components of hyperelliptic half-translation surfaces*.

2.2. **Sum of Lyapunov exponents and Siegel-Veech constant.** *Lyapunov exponents* measure the Hodge norm growth of cohomology classes under parallel transport along the Teichmüller geodesic flow. Here we consider the flow acting on a Teichmüller curve and the corresponding Lyapunov exponents. The individual exponents are hard to calculate, but their sum is a rational number that can be evaluated, one Teichmüller curve at a time. The same holds for the partial sum over all Lyapunov exponents that belong to a local subsystem, in case the local system with fiber $H^1(X, \mathbb{R})$ over the Teichmüller curve splits into several subsystems. See [Möl11] for a survey on these results and related references.

For a Teichmüller curve C generated by (X, q) in $\mathcal{Q}(d_1, \ldots, d_n)$, let (Y, η) be the canonical double covering. The curve Y comes with an involution τ. Its cohomology splits into the τ-invariant and τ-anti-invariant part. Adapting the notation of [EKZ11] we let $g = g(X)$ and $g_{\mathrm{eff}} = g(Y) - g$. Let λ_i^+ be the Lyapunov exponents of the τ-invariant part of $H^1(Y, \mathbb{R})$ and let λ_i^- be the Lyapunov exponents of the τ-anti-invariant part. The τ-invariant part descends to X and hence the λ_i^+ are the Lyapunov exponents of (X, q) we are primarily interested in. Define

$$\begin{aligned} L^+ &= \lambda_1^+ + \cdots + \lambda_g^+, \\ L^- &= \lambda_1^- + \cdots + \lambda_{g_{\mathrm{eff}}}^-. \end{aligned} \tag{1}$$

The role of L^+ is analogous to the ordinary sum of Lyapunov exponents in the case of Abelian differentials.

The main result of [EKZ11] expresses the sum of Lyapunov exponents as

$$L^+ = c + \kappa, \quad \text{where} \quad \kappa = \frac{1}{24}\Big(\sum_{j=1}^{n} \frac{d_j(d_j+4)}{d_j+2}\Big) \tag{2}$$

and where c is the (area) Siegel-Veech constant of (X, q). We will not give the definition of Siegel-Veech constants here but rather note that a similar formula holds for Y and the Siegel-Veech constants of X and Y are closely related. As a result, [EKZ11] obtain the key formula

$$L^- - L^+ = \frac{1}{4} \sum_{odd\ d_j} \frac{1}{d_j + 2}. \tag{3}$$

Applying this formula to the various double coverings associated to hyperelliptic half-translation surfaces shows that the following components are non-varying.

Corollary 2.1 ([EKZ11]). *Let C be a Teichmüller curve in one of the components of hyperelliptic half-translation surfaces. Then:*

For type (1), we have

$$L^+ = \frac{g+1}{2} - \frac{g+1}{2(2g-2k-1)(2k+3)}.$$

For type (2), we have

$$L^+ = \frac{2g+1}{4} - \frac{1}{4(2g-2k-1)}.$$

For type (3), we have

$$L^+ = \frac{g}{2}.$$

2.3. **Compactification of $\mathcal{Q}(d_1, \ldots, d_n)$.** We now describe the moduli spaces of quadratic differentials algebraically over a compactification of the moduli space of curves. This construction has the main feature that the boundary objects are the stable curves that appear as limit objects of Teichmüller curves as we will see in Section 4.2. Since we allow simple poles, i.e. $d_i = -1$ for some i, the spaces $\mathcal{Q}(d_1, \ldots, d_n)$ are not strata of a single vector bundle, but of several, according to the number of poles.

Given a signature $(d_1, \ldots, d_n)$, let k denote the number of poles, i.e. the number of indices i with $d_i = -1$. We can assume $d_j \geq 0$ for $1 \leq j \leq n-k$ and $d_j = -1$ for $n-k < j \leq n$. From now on we work over the moduli space $\overline{\mathcal{M}}_{g,k}$, the Deligne-Mumford compactification of the moduli space of genus g curves with k marked points. Over $\overline{\mathcal{M}}_{g,k}$ there is a vector bundle $\overline{\mathcal{Q}}_k \to \overline{\mathcal{M}}_{g,k}$, whose fiber over a stable pointed curve $(X, p_1, \ldots, p_k)$ parameterizes the sections

$$q \in H^0(X, \omega_X^{\otimes 2}(p_1 + \cdots + p_k)).$$

Let $\overline{\mathcal{Q}}(d_1, \ldots, d_n)$ be the closure of the subspace of $\overline{\mathcal{Q}}_k$ where the associated divisor of q has zeros (different from the p_i) of order $d_1, \ldots, d_{n-k}$. Thus, a point (X, q) in the interior of $\overline{\mathcal{Q}}(d_1, \ldots, d_n)$ corresponds to a quadratic differential of type $(d_1, \ldots, d_n)$ with simple poles at $p_1, \ldots, p_k$, which are smooth points of X.

2.4. **Period coordinates and plumbing coordinates.** Both $\Omega\mathcal{M}_g(m_1, \ldots, m_k)$ and $\mathcal{Q}(d_1, \ldots, d_n)$ are known to be smooth and they possess a convenient coordinate system given by period coordinates ([Mas82], [Vee90]). To obtain local coordinates on a neighborhood U of (X, ω) in the first case $\Omega\mathcal{M}_g(m_1, \ldots, m_k)$, we fix a basis of $H_1(Y, Z(\omega), \mathbb{Z})$, where $Z(\omega)$ denotes the locus of zeros of ω. For $(X', \omega') \in U$, integration of ω' along this basis provides a coordinate system. In the second case $\mathcal{Q}(d_1, \ldots, d_n)$, near (X, q) we start with the canonical double cover $\pi : Y \to X$

so that $\pi^* q = \omega^2$ for some one-form ω on Y. Let τ be the involution of Y with quotient X and fix a basis of $H_1(Y, Z(\omega), \mathbb{Z})^-$, the τ-anti-invariant part of the relative homology of Y with respect to $Z(\omega)$. For (X', q') in a neighborhood of (X, q) there exists $\pi' : Y' \to X'$ and τ' with the corresponding properties, since the canonical double cover can be constructed in families. Consequently, we can parallel transport the chosen basis of $H_1(Y, Z(\omega), \mathbb{Z})^-$ to $H_1(Y', Z(\omega'), \mathbb{Z})^-$. Integration of a square root of q' along this fixed basis provides maps from a neighborhood of (X, q) to $\mathbb{C}^{\dim H_1(Y,Z(\omega),\mathbb{Z})^-}$, and this is the desired coordinate system.

For stable curves we need one more type of coordinates to deform them into smooth curves, coming from the construction of plumbing in a cylinder (see e.g. [Wol89] or [Ber74] and the proof below). These functions will not be a coordinate system for all stable curves, since ω might be identically zero on one component of a reducible stable curve or a flat surface may have a separating node where ω is holomorphic and thus the location of the node cannot be detected by periods. However, for certain classes of translation and half-translation structures on stable curves the combination of the above functions, that we call *period plumbing coordinates* still forms a coordinate system.

We call a stable curve together with a stable one-form (X, ω) of *polar type* if there does not exist an irreducible component of X on which ω vanishes identically and if ω has a pole at each of the nodes of X. Similarly, we call a pair (X, q) of *polar type* if there does not exist an irreducible component of X on which q vanishes identically and if q has a double pole at each of the nodes of X. Let $\Omega\widetilde{\mathcal{M}}_g(m_1, \ldots, m_k)$ be the partial compactification of $\Omega\mathcal{M}_g(m_1, \ldots, m_k)$ by adding stable flat surfaces of polar type and let $\widetilde{\mathcal{Q}}(d_1, \ldots, d_n)$ be the partial compactification of $\mathcal{Q}(d_1, \ldots, d_n)$ by adding stable half-translation surfaces of polar type.

Note that in a stratum of the moduli space of stable one-forms, the stable curves of polar type cannot possess a separating node, by the residue theorem. On the other hand, for quadratic differentials, a stable half-translation surface of polar type may have separating nodes. In the case of one-forms the argument of the following proposition is due to [Bai07].

Proposition 2.2. *The partial compactifications of the strata $\Omega\widetilde{\mathcal{M}}_g(m_1, \ldots, m_k)$ and $\widetilde{\mathcal{Q}}(d_1, \ldots, d_n)$ are smooth. Local coordinates are given by period plumbing coordinates.*

Proof. We start with the case of stable one-forms. If (X, ω) is of polar type, then the normalization of X is a possibly disconnected smooth curve and the pullback of ω is a one-form, non-zero on each of the components, with at most simple poles at the pre-images of the nodes, say r of them. Let $\Sigma_{\underline{g},r}$ be the topological type of this punctured disconnected surface, where $\underline{g}$ is the tuple of genera of the irreducible components of Σ. Period coordinates are local coordinates on the boundary stratum of (X, ω), or equivalently, coordinates on the Teichmüller space $\Omega\mathcal{T}_{\underline{g},r}(m_1, \ldots, m_k)$ by an easy generalization of the argument of [Mas82] or [Vee90].

We denote the loops around the r punctures of $\Sigma_{\underline{g},r}$ by $\alpha_1, \ldots, \alpha_r$. For (X, ω) as above, let the Dehn space $\Omega\mathcal{D}_{\underline{g},r}(m_1, \ldots, m_k)$ for $\Sigma_{\underline{g},r}$ be the union of the quotient of the $\Omega\mathcal{T}_g(m_1, \ldots, m_k)$ by the group $\mathbb{Z}^r$ generated by the Dehn twists around $\alpha_1, \ldots, \alpha_r$ and the boundary Teichmüller space $\Omega\mathcal{T}_{\underline{g},r}(m_1, \ldots, m_k)$. This space is given the topology and complex structures such that the quotient map to $\Omega\overline{\mathcal{M}}_g$ is a holomorphic covering map onto its image.

We first define an *umplumbing* map

$$\Psi = \psi \times (z_1, \ldots, z_r) : \Omega\mathcal{D}'_{\underline{g},r}(m_1, \ldots, m_k) \to \Omega\mathcal{T}_{\underline{g},r}(m_1, \ldots, m_k) \times \mathcal{C}^r$$

as follows, where the prime denotes the restriction to a sufficiently small neighborhood of the locus of stable curves of polar type in $\Omega\mathcal{T}_{\underline{g},r}(m_1, \ldots, m_k)$. Since each of the α_j corresponds to a loop around a pole, in such a neighborhood each of the curves α_j for $j = 1, \ldots, r$ is homotopic to the core curve of a maximal flat cylinder C_j. For each of them we fix a curve in $H_1(X, Z(\omega), \mathbb{Z})$ crossing C_j once but not crossing the other cylinders. We define ψ as unplumbing on $\Omega\mathcal{D}_{\underline{g},r}(m_1, \ldots, m_k)$, i.e. replacing each of the C_j by a pair of half-infinite cylinders with residue equal to $\int_{\alpha_j} \omega$. On the boundary ψ is the identity and we define $z_j = \exp(2\pi i(\int_{\beta_j} \omega / \int_{\alpha_j} \omega))$, which is obviously well-defined up to the Dehn twists.

We claim that Ψ is biholomorphic onto its image. The converse is given by *plumbing*, i.e. for any surface of polar type in $\Omega\mathcal{T}_{\underline{g},r}(m_1, \ldots, m_k)$ we replace the pair of half-infinite cylinders with residue equal to r_j by a cylinder with core curve α_j and $\int_{\alpha_j} \omega = r_j$ such that $\int_{\beta_j} \omega$ satisfies $z_j = \exp(2\pi i(\int_{\beta_j} \omega / r_j))$. The function Ψ is obviously holomorphic outside the boundary and continuous on all of $\Omega\mathcal{D}'_{\underline{g},r}(m_1, \ldots, m_k)$, hence holomorphic there. Moreover, plumbing is obviously inverse to unplumbing, thus proving the claim.

Together with generalized period coordinates, this claim on Ψ shows that period and plumbing functions are indeed coordinates on $\Omega\widetilde{\mathcal{M}}_g(m_1, \ldots, m_k)$.

The proof for the case of half-translation surfaces is the same. Again, the anti-invariant periods on the canonical double cover give, by the arguments of [Mas82] or [Vee90], coordinates along the boundary of $\widetilde{\mathcal{Q}}(d_1, \ldots, d_n)$, since q is non-zero on each irreducible component. Note that the holonomy around each curve α_j around a puncture of $\Sigma_{\underline{g},r}$ is among these anti-invariant period functions since the double covering map is unramified near the punctures, because by hypothesis q has precisely a double pole there. We now can define the unplumbing map Ψ and its inverse given by plumbing as above. □

3. Divisor classes

In this section we recall the Picard group of the moduli space of curves with marked points and collect the expression of several geometrically defined divisors on the moduli space of curves in low genus with few marked points in terms of the standard generators of the Picard group. The results are basically contained in the literature ([Log03], [Far09]), but in several cases not all boundary terms were calculated in full detail. We will thus perform the calculation for the cases we need.

Use $\mathrm{Pic}(\cdot)$ to denote the rational Picard group $\mathrm{Pic}_{\mathrm{fun}}(\cdot)_{\mathbb{Q}}$ of a moduli stack (see [HM98, Chapter 3.D] for more details). Since the quantities we are interested in, the sum of Lyapunov exponents and slope, are invariant under finite base change, this is the group we want to use, not the Picard group of the coarse moduli scheme.

Recall the standard notation for elements in the Picard group. Let λ denote the first Chern class of the Hodge bundle. Let δ_i, $i = 1, \ldots, \lfloor g/2 \rfloor$ be the boundary divisor of $\overline{\mathcal{M}}_g$ whose generic element is a smooth curve of genus i joined at a node to a smooth curve of genus $g - i$. The generic element of the boundary divisor δ_0 is an irreducible nodal curve of geometric genus $g - 1$. In the literature sometimes δ_0 is denoted by δ_{irr}. We write δ for the total boundary class. All the divisor classes we consider here are stacky. In particular, the divisor class δ_1 equals one-half of the pullback of the corresponding divisor Δ_1 from the coarse moduli scheme, due to the elliptic involution of order 2.

For moduli spaces of curves with marked points we denote by ω the first Chern class of the relative dualizing sheaf of $\overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$ and ω_i its pullback to $\overline{\mathcal{M}}_{g,n}$ via the map forgetting all but the i-th marked point. For a subset $S \subset \{1, \ldots, n\}$ let $\delta_{i;S}$ denote the boundary divisor whose generic element is a smooth curve of genus i joined at a node to a smooth curve of genus $g - i$ such that the component of genus i contains exactly the marked points labeled by S.

Theorem 3.1 ([AC87]). *The rational Picard group of $\overline{\mathcal{M}}_g$ for $g \geq 3$ is freely generated by λ and δ_i, $i = 0, \ldots, \lfloor g/2 \rfloor$.*

More generally, the rational Picard group of $\overline{\mathcal{M}}_{g,n}$ for $g \geq 3$ is freely generated by λ, ω_i, $i = 1, \ldots, n$, by δ_0 and by $\delta_{i;S}$, $i = 0, \ldots, \lfloor g/2 \rfloor$, where $|S| > 1$ if $i = 0$ and $1 \in S$ if $i = g/2$.

Alternatively, we define $\psi_i \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$ to be the class with value $-\pi_*(\sigma_i^2)$ on the universal family $\pi : \mathcal{X} \to C$ with section σ_i corresponding to the ith marked point. We have the relation

$$\omega_i = \psi_i - \sum_{i \in S} \delta_{0;S},$$

see e.g. [Log03, p. 107-108] for details. Consequently, a basis of $\mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$ can be formed by λ, the ψ_i and the boundary classes as well.

Given the structure of the Picard group of $\overline{\mathcal{M}}_g$, it is natural to define the *slope* of a Teichmüller curve C as

$$s(C) = \frac{\overline{C} \cdot \delta}{\overline{C} \cdot \lambda}.$$

In general, slope can be defined for any one-parameter family of stable genus g curves, which measures how the complex structures vary with respect to the number of singular fibers in the family.

For Abelian differentials, the slope of a Teichmüller curve in $\Omega\mathcal{M}_g(\mu)$ carries as much information as the Siegel-Veech constant or the sum of Lyapunov exponents (see [EKZ11, Theorem 1] and [Che11, Theorem 1.8]):

$$s(C) = \frac{12c(C)}{L(C)} = 12 - \frac{12\kappa_\mu}{L(C)}. \tag{4}$$

Not surprisingly, a similar relation holds for Teichmüller curves generated by quadratic differentials after replacing $L(C)$ by $L^+(C)$, see (15) in Proposition 4.2.

Occasionally we need to mark some zeros or poles of a quadratic differential and lift the corresponding Teichmüller curve to $\overline{\mathcal{M}}_{g,n}$. Therefore, we introduce several divisor classes on $\overline{\mathcal{M}}_{g,n}$ for later use.

Let W be the divisor of Weierstrass points in $\overline{\mathcal{M}}_{g,1}$. It has divisor class (see e.g. [Cuk89])

$$W = -\lambda + \frac{g(g+1)}{2}\omega_1 - \sum_{i=1}^{g-1} \frac{(g-i)(g-i+1)}{2}\delta_i. \tag{5}$$

Let $BN^1_{g,(s_1,\ldots,s_r)}$ be the pointed Brill-Noether divisor parameterizing pointed curves $(X, z_1, \ldots, z_r)$ in $\overline{\mathcal{M}}_{g,r}$ where $\sum_{i=1}^r s_i = g$ such that $h^0(X, \sum_{i=1}^r s_i z_i) = 2$. In particular, $BN^1_{g,(g)}$ is just the divisor W of Weierstrass points.

The divisor class of $BN^1_{g,(1,\dots,1)}$ was fully worked out in [Log03, Section 5]. The divisor class of $BN^1_{g,(s_1,\dots,s_r)}$ was also implicitly calculated there. Below we give explicitly the divisor classes for the cases we need.

3.1. **Genus 3.** Let H be the divisor of hyperelliptic curves in $\overline{\mathcal{M}}_3$. It has divisor class

$$H = 9\lambda - \delta_0 - 3\delta_1, \tag{6}$$

see e.g. [HM98, Chapter 3.H].

We also have pointed Brill-Noether divisor classes as follows.

$$\begin{aligned} BN^1_{3,(1,1,1)} = & -\lambda + \omega_1 + \omega_2 + \omega_3 - \sum_{i,j} \delta_{0;\{i,j\}} \\ & - 3\delta_{0;\{1,2,3\}} - \sum_{i,j} \delta_{1,\{i,j\}} - \delta_{1;\emptyset} - 3\delta_{1;\{1,2,3\}}. \end{aligned} \tag{7}$$

$$BN^1_{3,(2,1)} = -\lambda + 3\omega_1 + \omega_2 - 2\delta_{0;\{1,2\}} - \delta_{1;\emptyset} - \delta_{1;\{1\}} - 3\delta_{1;\{1,2\}}. \tag{8}$$

As noted above, the class of $BN^1_{3,(1,1,1)}$ was calculated in [Log03, Section 5]. The class of $BN^1_{3,(2,1)}$ essentially follows from $BN^1_{3,(1,1,1)}$. We skip this calculation and instead, we will prove a completely analogous but harder case in genus four.

3.2. **Genus 4.** In genus four we need the following pointed Brill-Noether divisors.

Lemma 3.2. *The pointed Brill-Noether divisors in genus four have divisor classes as follows.*

$$W = BN^1_{4,(4)} = -\lambda + 10\omega - 6\delta_1 - 3\delta_2 - \delta_3.$$

$$\begin{aligned} BN^1_{4,(1,1,1,1)} = & -\lambda + \omega_1 + \omega_2 + \omega_3 + \omega_4 - \sum_{|S|\geq 2} \frac{|S|(|S|-1)}{2}\delta_{0;S} \\ & - \sum_{|S|\neq 1} \frac{(||S|-1|)(||S|-1|+1)}{2}\delta_{1;S} \\ & - \sum_{|S|\neq 2} \frac{(||S|-2|)(||S|-2|+1)}{2}\delta_{2;S}. \end{aligned} \tag{9}$$

$$\begin{aligned} BN^1_{4,(2,1,1)} = & -\lambda + 3\omega_1 + \omega_2 + \omega_3 - 2\delta_{0;\{1,2\}} - 2\delta_{0;\{1,3\}} \\ & - \delta_{0;\{2,3\}} - 5\delta_{0;\{1,2,3\}} - \delta_{1;\emptyset} - \delta_{1;\{1\}} - \delta_{1;\{2,3\}} - 3\delta_{1;\{1,3\}} \\ & - 3\delta_{1;\{1,2\}} - 6\delta_{1;\{1,2,3\}} - 6\delta_{2;\emptyset} - 2\delta_{2;\{2\}} - 2\delta_{2;\{3\}}. \end{aligned}$$

$$\begin{aligned} BN^1_{4,(3,1)} = & -\lambda + 6\omega_1 + \omega_2 - 3\delta_{0;\{1,2\}} \\ & - \delta_{1;\emptyset} - 3\delta_{1;\{1\}} - 6\delta_{1;\{1,2\}} - 6\delta_{2;\emptyset} - 2\delta_{2;\{1\}}. \\ BN^1_{4,(2,2)} = & -\lambda + 3\omega_1 + 3\omega_2 - 4\delta_{0;\{1,2\}} \\ & - \delta_{1;\emptyset} - \delta_{1;\{1\}} - \delta_{1;\{2\}} - 6\delta_{1;\{1,2\}} - 6\delta_{2;\emptyset}. \end{aligned} \tag{10}$$

Proof. Let $\pi_i : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n-1}$ be the map forgetting the ith marked point. Then

$$\pi_{n*}(BN^1_{g,(a_1,\dots,a_n)} \cdot \delta_{0;\{n-1,n\}}) = BN^1_{g,(a_1,\dots,a_{n-2},a_{n-1}+a_n)}.$$

Let us first determine the class of $BN^1_{4,(2,1,1)}$. Interchanging the first and the third marked points in the above equality, we have

$$\pi_{4*}(BN^1_{4,(1,1,1,1)} \cdot \delta_{0;\{1,4\}}) = BN^1_{4,(2,1,1)}.$$

Moreover, based on [Log03, Table 1, p. 112] we have

$$\begin{array}{ll}
\pi_{4*}(\omega_1 \cdot \delta_{0;\{1,j\}}) = \omega_j, & \text{for} \quad j = 1,2,3, \\
\pi_{4*}(\omega_4 \cdot \delta_{0;\{1,4\}}) = \omega_1, & \pi_{4*}(\lambda \cdot \delta_{0;\{1,4\}}) = \lambda, \\
\multicolumn{2}{l}{\pi_{4*}(\delta_{0;\{1,4\}} \cdot \delta_{0;\{1,4\}}) = -\psi_1 = -\omega_1 - \sum_{1\in S} \delta_{0;S},} \\
\pi_{4*}(\delta_{i;S} \cdot \delta_{0;\{1,4\}}) = \delta_{i;S},\ \{1,4\} \cap S = \emptyset, & \pi_{4*}(\delta_{i;S} \cdot \delta_{0;\{1,4\}}) = 0,\ 1 \in S,\ 4 \notin S, \\
\pi_{4*}(\delta_{i;S\cup\{4\}} \cdot \delta_{0;\{1,4\}}) = \delta_{i;S},\ 1 \in S, & \pi_{4*}(\delta_{i;S\cup\{4\}} \cdot \delta_{0;\{1,4\}}) = 0,\ 1 \notin S.
\end{array}$$

As a consequence,

$$\begin{aligned}
BN^1_{4,(2,1,1)} = {} & -\lambda + 2\omega_1 + \omega_2 + \omega_3 + \omega_1 + \sum_{1\in S} \delta_{0;S} - \delta_{0;\{2,3\}} - 3\delta_{0;\{1,2\}} - 3\delta_{0;\{1,3\}} \\
& -6\delta_{0;\{1,2,3\}} - \delta_{1;\emptyset} - \delta_{1;\{1\}} - \delta_{1;\{2,3\}} - 3\delta_{1;\{1,3\}} - 3\delta_{1;\{1,2\}} - 6\delta_{1;\{1,2,3\}} \\
& -3\delta_{2;\emptyset} - \delta_{2;\{2\}} - \delta_{2;\{3\}} - \delta_{2;\{1,2\}} - \delta_{2;\{1,3\}} - 3\delta_{2;\{1,2,3\}}.
\end{aligned}$$

Using $\pi_{3*}(BN^1_{4,(2,1,1)} \cdot \delta_{0;\{1,3\}}) = BN^1_{4,(3,1)}$, we deduce that

$$\begin{aligned}
BN^1_{4,(3,1)} = {} & -\lambda + 3\omega_1 + \omega_2 + \omega_1 + 2\omega_1 + 2\delta_{0;\{1,2\}} - 5\delta_{0;\{1,2\}} \\
& -\delta_{1;\emptyset} - 3\delta_{1;\{1\}} - 6\delta_{1;\{1,2\}} - 6\delta_{2;\emptyset} - 2\delta_{2;\{2\}}.
\end{aligned}$$

Similarly, using $\pi_{3*}(BN^1_{4,(2,1,1)} \cdot \delta_{0;\{2,3\}}) = BN^1_{4,(2,2)}$, we conclude that

$$\begin{aligned}
BN^1_{4,(2,2)} = {} & -\lambda + 3\omega_1 + \omega_2 + \omega_2 + \omega_2 + \delta_{0;\{1,2\}} - 5\delta_{0;\{1,2\}} \\
& -\delta_{1;\emptyset} - \delta_{1;\{1\}} - \delta_{1;\{2\}} - 6\delta_{1;\{1,2\}} - 6\delta_{2;\emptyset}.
\end{aligned}$$

In each case, after simplifying we get the result stated above. □

3.3. **Genus** 1**.** Recall the divisor theory of $\overline{\mathcal{M}}_{1,n}$. Its rational Picard group is freely generated by λ and $\delta_{0;S}$ for $2 \leq |S| \leq n$. Moreover, we have

$$\delta_0 = 12,$$

$$\psi_i = \lambda + \sum_{i\in S} \delta_{0;S}.$$

3.4. **Genus** 2**.** On the moduli space $\overline{\mathcal{M}}_2$, the rational Picard group is generated by λ, δ_0 and δ_1 with the relation (see [AC98, Theorem 2.2])

$$\lambda = \frac{\delta_0}{10} + \frac{\delta_1}{5}. \tag{11}$$

The following result is quite useful to prove non-varying strata in genus two.

Lemma 3.3. *If a one-dimensional family of stable curves of genus two does not intersect* δ_1*, then its slope is* 10.

Proof. This follows from the definition of slope and the relation (11). □

The rational Picard group of $\overline{\mathcal{M}}_{2,n}$ is generated by the divisor classes λ, ω_i, $i = 1, \ldots, n$, by δ_0, by $\delta_{0;S}$ with $|S| > 1$ and by $\delta_{0,S}$ with $1 \in S$. By [AC98, Theorem 2.2] the only relation among them is

$$5\lambda = 5\psi + \delta_0 - \sum_{|S|>1} \delta_{0,S} + 7 \sum_{1 \in S} \delta_{1,S},$$

where $\psi = \sum_{i=1}^{n} \psi_i$ is the total ψ class. We will use two divisors in genus two. As a special case of (5), the divisor of Weierstrass points in $\overline{\mathcal{M}}_{2,1}$ has class

$$W = -\lambda + 3\omega_1 - \delta_1. \tag{12}$$

The pointed Brill-Noether divisor $BN^1_{2,(1,1)}$ in $\overline{\mathcal{M}}_{2,2}$ has class

$$BN^1_{2,(1,1)} = -\lambda + \omega_1 + \omega_2 - \delta_{0;\{1,2\}} - \delta_{1;\emptyset}. \tag{13}$$

4. Properties of Teichmüller curves

4.1. **Computing intersection numbers.** Let C be a Teichmüller curve generated by a half-translation surface in $\mathcal{Q}(d_1, \ldots, d_n)$. We always work with an appropriate unramified cover of C whose uniformizing group is torsion free and we denote this cover still by C. In particular we will assume that over C there is a universal family $\pi : \mathcal{X} \to C$. This also implies that $\chi = 2g(C) - 2 + |\Delta|$, where Δ is the set of cusps in C and χ is the orbifold Euler characteristic of C. Denote by S_j the section of $\mathcal{X}$ corresponding to the zero or pole of order d_j. Use ω_π to denote the relative dualizing sheaf of π.

Proposition 4.1 ([Kon97]; [BM10])**.** *For a Teichmüller curve C generated by a half-translation surface in $\mathcal{Q}(d_1, \ldots, d_n)$, we have*

$$L^+(C) = \frac{2 \deg \lambda}{\chi}.$$

Proposition 4.2. *Intersection numbers of the Teichmüller curves with various divisor classes and the sum of Lyapunov exponents are related as follows:*

$$\begin{aligned} S_j^2 &= -\frac{\chi}{d_j + 2}, \qquad & S_j \cdot \omega_\pi &= \frac{\chi}{d_j + 2}, \\ \overline{C} \cdot \delta &= 6\chi \cdot c, \qquad & \overline{C} \cdot \lambda &= \frac{\chi}{2} \cdot (c + \kappa), \end{aligned} \tag{14}$$

where $c = c(C)$ is the Siegel-Veech constant of C related to L^+ by (2). In particular, we have

$$s(C) = \frac{12c(C)}{L^+(C)} = 12 - \frac{12\kappa}{L^+(C)}. \tag{15}$$

Proof. Let $\mathcal{F}$ be the universal line bundle on C parameterizing the quadratic differentials that generate C. Denote by S the union of the sections S_j for $j = 1, \cdots, n$. By the exact sequence

$$0 \to \pi^* \mathcal{F} \to \omega_\pi^{\otimes 2} \to \mathcal{O}_S \Big(\sum_{j=1}^{n} d_j S_j \Big) \to 0$$

and the fact that $\deg \mathcal{F} = \chi$ (see [Möl06]), one calculates that

$$S_j^2 = -S_j \cdot \omega_\pi = -\frac{\chi}{d_j + 2},$$

which shows the first two formulas. Moreover, it implies that

$$c_1^2(\omega_\pi) = \frac{\chi}{4} \cdot \Big(\sum_{j=1}^{n} \frac{d_j(d_j+4)}{d_j+2} \Big) = 6\chi \cdot \kappa.$$

By Noether's formula, we know that

$$12\lambda = \delta + c_1^2(\omega_\pi).$$

Dividing both sides by 6χ, the left hand side equals $L^+(C)$ by the preceding proposition and the right hand side equals $\frac{\delta}{6\chi} + \kappa$. By (2), we reads off $c = \frac{\delta}{6\chi}$. Hence the intersection numbers with δ and λ follow immediately.

Finally, by (2) and the definition of slope we obtain that

$$s(C) = \frac{12c}{c+\kappa} = \frac{12(L^+(C) - \kappa)}{L^+(C)} = 12 - \frac{12\kappa}{L^+(C)}.$$

□

Remark 4.3. We can also deduce the above formulas by passing to the canonical double cover. Note that $\mathcal{Q}(d_1, \ldots, d_n) \to \Omega\mathcal{M}(\cdots, d_i/2, d_i/2, \cdots, d_j+1, \cdots)$ for d_i even and for d_j odd, since the double cover is branched at the singularities of odd order. Restrict this to a Teichmüller curve C in $\mathcal{Q}(d_1, \ldots, d_n)$. Then it gives rise to a Teichmüller curve isomorphic to C in the corresponding stratum of Abelian differentials. We have the following commutative diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[rr, "f"] \arrow[dr] & & \mathcal{X} \arrow[dl] \\ & C & \end{tikzcd}$$

and let S'_j be the section of $\mathcal{X}' \to C$ over S_j in case d_j is odd and $S_{j,1}, S_{j,2}$ be the sections over S_j in case d_j is even. Then we have

$$\begin{aligned} f_* S'_j &= S_j, \qquad & f^* S_j &= 2S'_j, \\ f_*(S_{j,1} + S_{j,2}) &= 2S_j, \qquad & f^* S_j &= S_{j,1} + S_{j,2}. \end{aligned}$$

In the case when d_j is odd, we have

$$S_j^2 = (f_* S'_j) \cdot S_j = 2(S'_j)^2 = -\frac{\chi}{d_j+2}.$$

In the case when d_j is even, we have

$$S_j^2 = \frac{1}{2}(f_*(S_{j,1} + S_{j,2})) \cdot S_j = \frac{1}{2}(S_{j,1}^2 + S_{j,2}^2) = -\frac{\chi}{d_j+2}.$$

Hence we recover the self-intersection formula.

4.2. **Boundary behavior.** The following results are needed later for the proofs of non-varying strata. Roughly speaking, they imply that degenerate half-translation surfaces parameterized in a Teichmüller curve behave similarly to the smooth ones and, as in the case of Abelian differentials, the corresponding stable curves are obtained by squeezing core curves of cylinders (see [Möl11, Proposition 5.9]).

Proposition 4.4. *Suppose C is a Teichmüller curve generated by a quadratic differential in $\mathcal{Q}(d_1, \ldots, d_n)$. The pointed stable curves in $\overline{\mathcal{M}}_{g,k}$ corresponding to the boundary points Δ of C are obtained by choosing a parabolic direction of a generating half-translation surface (X, q) and replacing each cylinder by a pair of half-infinite cylinders whose points at $i\infty$ resp. at $-i\infty$ are identified.*

Proof. The cusps of Teichmüller curves are obtained by applying the Teichmüller geodesic flow $(e^{t/2}, e^{-t/2})$ to a direction in which (X, q) decomposes completely into cylinders. Once we have shown that the object resulting from the above cylinder replacement construction is stable (including the punctures), the rest of the proof is the same as in [Möl11, Propositions 5.9 and 5.10].

We need to show that each rational tail (i.e. a genus zero component of a stable curve joined to the rest of the curve at a separating node and without nodes joining the tail to itself) has at least two punctures. If we cut along the core curve γ that produces the separating node and glue the two halves of γ together, we obtain a closed half-translation surface $(\mathbb{P}^1, q_{\mathbb{P}})$ of genus zero with two simple poles on the glued γ. Since $\deg(\mathrm{div}(q_{\mathbb{P}})) = -4$ and since poles are simple, there exist two more poles somewhere on this $\mathbb{P}^1$, proving our claim. □

The proof cited above provides a geometric way of constructing a quadratic differential q_∞ on the degenerate fibers. Since the zeros and poles of q_∞ lie in the complement of the union of open cylinders for any given direction, they are untouched by the surgery performed while degeneration. We summarize the consequences of this as follows.

Corollary 4.5. *The section q of $\omega_X^{\otimes 2}$ of each smooth fiber X over a Teichmüller curve extends to a section q_∞ of $\omega_{X_\infty}^{\otimes 2}$ for each degenerate fiber X_∞ over the closure of a Teichmüller curve. The signature of zeros and poles of q_∞ is the same as q.*

Corollary 4.6. *Let (X_∞, q_∞) be a degenerate fiber of a Teichmüller curve generated by a half-translation surface (X, q). Then (X_∞, q_∞) is of polar type.*

Moreover, every irreducible component of X_∞ contains at least one singularities of q_∞. In particular, the number of irreducible components of X_∞ is bounded from above by the number of singularities of q.

Remark 4.7. If q is a global square of an Abelian differential, then X_∞ does not have separating nodes, as a consequence of the topological fact that the core curve of a cylinder does not disconnect a flat surface. In other words, Teichmüller curves generated by Abelian differentials do not intersect δ_i for $i > 0$ in $\overline{\mathcal{M}}_g$. On the other hand, if q is not a global square, then the Teichmüller curve generated by q may intersect δ_i for $i > 0$.

Proposition 4.8. *Let C be a Teichmüller curve generated by a half-translation surface in $\mathcal{Q}(d_1, \ldots, d_n)$. Let $\overline{C}$ be the closure of the lift of C to $\overline{\mathcal{M}}_{g,m}$ using the first $m \leq n$ singularities. Then $\overline{C}$ is disjoint with the boundary divisors that have non-zero coefficients in the divisor classes of the Brill-Noether divisors given in Section 3, if the tuple $(g, m, \mathcal{Q}(d_1, \ldots, d_n))$ and the divisor are listed in Table 1.*

Proof. An irreducible component Z of a degenerate half-translation surface X_∞ over a cusp of C contains at least one zero or pole of the degenerate quadratic differential q_∞. Moreover, $\omega_{X_\infty}^{\otimes 2}$ restricted to Z has degree equal to $4g(Z) - 4 + 2m$,

FIGURE 1. Divisors disjoint with Teichmüller curves.

g	m	Stratum	Divisor
2	1	$\mathcal{Q}(3,2,-1)$, $\mathcal{Q}(6,-1,-1)$, $\mathcal{Q}(5,1,-1,-1)$, $\mathcal{Q}(7,-1,-1,-1)$	W
2	2	$\mathcal{Q}(3,1,1,-1)$, $\mathcal{Q}(2,2,1,-1)$, $\mathcal{Q}(4,2,-1,-1)$, $\mathcal{Q}(3,3,-1,-1)$, $\mathcal{Q}(3,2,1,-1,-1)$, $\mathcal{Q}(4,3,-1,-1,-1)$,	$BN^1_{2,(1,1)}$
3	1	$\mathcal{Q}(8)$, $\mathcal{Q}(7,1)$, $\mathcal{Q}(9,-1)$, $\mathcal{Q}(8,1,-1)$, $\mathcal{Q}(10,-1,-1)$, $\mathcal{Q}(9,1,-1,-1)$	W
3	2	$\mathcal{Q}(6,2)$, $\mathcal{Q}(5,3)$, $\mathcal{Q}(4,4)$, $\mathcal{Q}(6,1,1)$, $\mathcal{Q}(7,2,-1)$, $\mathcal{Q}(5,4,-1)$, $\mathcal{Q}(4,3,1)$, $\mathcal{Q}(5,2,1)$, $\mathcal{Q}(6,3,-1)$, $\mathcal{Q}(5,3,1,-1)$, $\mathcal{Q}(7,3,-1^2)$	$BN^1_{3,(2,1)}$
3	3	$\mathcal{Q}(4,2,2)$, $\mathcal{Q}(3,3,2)$, $\mathcal{Q}(3,3,3,-1)$, $\mathcal{Q}(4,3,2,-1)$ $\mathcal{Q}(3,2,2,1)$, $\mathcal{Q}(3,3,1,1)$	$BN^1_{3,(1,1,1)}$
4	1	$\mathcal{Q}(13,-1)$, $\mathcal{Q}(12)$, $\mathcal{Q}(11,1)$	W
4	2	$\mathcal{Q}(10,2)$, $\mathcal{Q}(9,3)$, $\mathcal{Q}(8,4)$, $\mathcal{Q}(8,3,1)$	$BN^1_{4,(3,1)}$
4	2	$\mathcal{Q}(6,6)$	$BN^1_{4,(2,2)}$
4	3	$\mathcal{Q}(7,3,2)$, $\mathcal{Q}(6,3,3)$, $\mathcal{Q}(5,4,3)$	$BN^1_{4,(2,1,1)}$
4	4	$\mathcal{Q}(3,3,3,3)$	$BN^1_{4,(1,1,1,1)}$

where $m = \#(Z \cap \overline{X_\infty \backslash Z})$. Using these facts, the claim follows easily by a case-by-case study. For instance, let us show that a Teichmüller curve C generated by a half-translation surface in $\mathcal{Q}(7,1)$ does not intersect δ_1 in $\overline{\mathcal{M}}_{3,1}$. Otherwise, there exists a degenerate half-translation surface X_∞ consisting of two components Z_1 and Z_2 of genus 1 and 2, respectively, joined at a node such that $\mathrm{div}(q_\infty) = 7p_1 + p_2$ for two distinct points $p_1, p_2 \in X_\infty$. But the degree of $\omega_{X_\infty}^{\otimes 2}$ restricted to Z_1 is 2, in particular, not equal to 7 or 1. Hence Z_1 does not contain any zero of q_∞, contradicting Corollary 4.6. $\square$

Proposition 4.9. *Let C be a Teichmüller curve parameterizing half-translation surfaces (X_t, q_t) such that for generic t the quadratic differential q_t is not a global square of an Abelian differential. Then q_0 is not a global square of a stable one-form on the special fiber X_0.*

Proof. If q_t has a singularity of odd order, the claim is obvious. Assume that all singularities are of even order. Let $\mathcal{X} \to C$ be the universal curve and $\Gamma \subset \mathcal{X}$ the divisor parameterizing the singularities of q_t. Define $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(\Gamma/2)$, which is a well-defined line bundle by the assumption. Denote by $s \in H^0(\mathcal{X}, \mathcal{L}^{\otimes 2})$ the section whose vanishing locus is Γ. Then there exists a canonical double covering $\pi : \mathcal{X}' \to \mathcal{X}$ such that $\pi^*\mathcal{L}$ possesses a section s' satisfying $(s')^2 = \pi^* s$. In other words, along each fiber $X'_t \to X_t$ the pullback of q_t is a square of an Abelian differential. Moreover, q_t is a global square if and only if X'_t is disconnected. Now the result follows from the fact that a family of connected curves cannot specialize to a disconnected one. $\square$

The following proposition says that some hyperelliptic and non-hyperelliptic strata stay disjoint even along the boundary of $\overline{\mathcal{M}}_g$. For a motivation the reader may compare with the corresponding statement for Abelian differentials in [CM12, Proposition 4.4].

Proposition 4.10. *Let C be a Teichmüller curve generated by a half-translation surface (X, q). Suppose that (X_∞, q_∞) is a point corresponding to one of the cusps $\overline{C} \setminus C$. If (X_∞, q_∞) is a hyperelliptic half-translation surface and X_∞ is irreducible, then (X, q) is a hyperelliptic half-translation surface.*

If (X, q) is a half-translation surface in one of the strata

$$\mathcal{Q}(6,2), \mathcal{Q}(6,1,1), \mathcal{Q}(3,3,2), \mathcal{Q}(10,2), \mathcal{Q}(6,-1,-1), \mathcal{Q}(3,3,-1^2) \text{ or } \mathcal{Q}(10,-1^2),$$

then the conclusion holds without the irreducibility assumption.

For the stratum $\mathcal{Q}(3,3,1,1)$ the same conclusion holds for the cusps parameterizing stable curves with a separating node.

Proof. Let (X_0, q_0) denote the 'core' of the half-translation surface (X_∞, q_∞), that is obtained by removing from (X_∞, q_∞) the maximal half-infinite cylinders, i.e. the cylinders corresponding to the nodes of X_∞ that are swept out by closed geodesics in the direction of the residue of q_∞ at a node.

If X_∞ is hyperelliptic, there is an admissible double cover of a semistable model of X_∞ to $\mathbb{P}^1$. Suppose first that X_∞ is irreducible with h nodes. Then the double cover induces an involution ρ of the semistable model of X_∞ that, by our hypothesis, acts as (-1) on q_∞. This involution ρ preserves the central component X_∞ as well as all the non-stable projective lines. We deduce that ρ preserves X_0 and interchanges each pair of half-infinite cylinders that is glued together to form a node. Moreover, ρ has $2(g-h)+2$ fixed points on X_0.

Nearby surfaces (X, q) in C are obtained by replacing the pairs of infinite cylinders with cylinders of finite height. Since all pairs of infinite half-cylinders are preserved by ρ, we can extend ρ to an involution on (X, q) still acting as (-1) on q with 2 fixed points in each cylinder. This gives $2g+2$ fixed points in total and (X, q) is a hyperelliptic half-translation surface, as we claimed.

Boundary points of the Teichmüller curve generated by a half-translation surface in one of the above strata are either irreducible or consist of two components X_1 of genus one and X_2 of genus $g-1>1$ in the first four cases and $g(X_1)=0$, $g(X_2)=g$ in the next three cases. The involution ρ induced by the admissible double covering cannot exchange the two components and it has to fix the unique node joining the two components. Reproducing the preceding argument for nodes joining the irreducible components to itself, if there are, we conclude that ρ fixes all pairs of half-infinite cylinders. We can now complete the proof as in the preceding case.

Thanks to the restrictions on the cusps in question we are just in position to apply the argument again to the last case. □

5. Limit canonical curves in genus three and four

For a smooth, non-hyperelliptic curve X of genus three, its canonical embedding is a smooth plane quartic. In that case, the zeros of a holomorphic quadratic differential correspond to the intersection of a unique plane conic with X. This picture holds more generally if X is nodal, non-hyperelliptic and ω_X is very ample, i.e. the dual graph of X is 3-connected (see e.g. [Has99, Proposition 2.3]). If X is 2-connected or less, or if X is hyperelliptic, its canonical map may no longer be an embedding. We need a replacement for this picture, when the canonical map of a curve of genus three (and also of genus four) fails to be an embedding.

Since quadratic differentials are sections of $\omega_X^{\otimes 2}$, naturally we should consider the bicanonical map of X. The bicanonical linear system on a stable genus three curve X provides an embedding to $\mathbb{P}^5$, unless X possesses an elliptic tail. Now conics in $\mathbb{P}^2$ corresponding to quadratic differentials become hyperplane sections in $\mathbb{P}^5$. Then the ambient $\mathbb{P}^2$ containing X turns out to be a surface in $\mathbb{P}^5$ as the image of the Veronese embedding induced by $|\mathcal{O}_{\mathbb{P}^2}(2)|$. As X degenerates, say to a smooth hyperelliptic curve X_0, we do not have a canonical embedding of X_0 in $\mathbb{P}^2$. Nevertheless, in $\mathbb{P}^5$ we have a bicanonical embedding of X_0 contained in a singular surface which is a degeneration of the Veronese $\mathbb{P}^2$. In summary, the idea is to treat the pair $(\mathbb{P}^2, X)$ as a *log surface*, i.e. a surface plus a divisor with mild singularities, and degenerate $\mathbb{P}^2$ as well. This procedure was carried out by Hassett completely for stable genus three curves [Has99]. We summarize Hassett's results as well as the degenerations we need in genus four. This will be useful when we analyze the boundary behavior of Teichmüller curves in the exceptional strata.

Recall that a *Hirzebruch surface* F_d is a ruled surface over $\mathbb{P}^1$ such that the section e with minimal self-intersection number has $e^2 = -d$. Let f be a ruling of F_d. The rank of the Picard group of F_d is two, hence any curve class of F_d can be written as a linear combination $ae + bf$ with integer coefficients a, b and $f^2 = 0$, $f \cdot e = 1$. Moreover, the canonical line bundle of F_d has class

$$K_{F_d} = -2e - (2 + d)f.$$

5.1. **Genus three.** We start with a description of the generic situation. Consider the Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ by the complete linear system $|\mathcal{O}(2)|$. Denote by S the image surface of degree four. A plane curve of degree d maps to a curve of degree $2d$ in $S \subset \mathbb{P}^5$. Holomorphic quadratic differentials on X modulo scalar are in bijection to conics Q in $\mathbb{P}^2$. They become hyperplane sections of S in $\mathbb{P}^5$.

Proposition 5.1. *In the above setting, suppose a family of log surfaces (S_t, X_t) degenerates such that X_0 is hyperelliptic but still* 3*-connected. Then S_t degenerates to a cone over a rational normal quartic, whose resolution is a Hirzebruch surface F_4. On F_4, the curve X_0 has class $2e + 8f$.*

Two points p_1, p_2 are conjugate in X_0 if and only if they are cut out by a ruling. A point p is a Weierstrass point of X_0 if and only if there is a ruling tangent to X_0 at p.

The degeneration of divisors associated to holomorphic quadratic differentials on X_t has two possibilities, either it consists of four rulings or it has class $e + 4f$, depending on whether the hyperplane section Q passes through the singularity, respectively.

Let S_4 be the degeneration of S_t in the above, i.e. S_4 is a cone over a rational normal quartic curve in $\mathbb{P}^5$. More precisely, take a smooth, degree four, rational curve R that spans a hyperplane in $\mathbb{P}^5$, and a point v not contained in that hyperplane. Then S_4 consists of the union of lines connecting v with each point of R. Those lines are called rulings of S_4. The vertex v is a surface singularity of type A_3. Blowing up S_4 at v, we obtain a smooth Hirzebruch surface of type F_4. More details on this and the following proposition can be found in [Has99].

The second case we need is precisely 2-connected.

Proposition 5.2. *In the above setting, suppose a family of log surfaces (S_t, X_t) degenerates such that X_0 is hyperelliptic and consists of two components X_1, X_2,* *both of genus one, meeting at two nodes* t_1, t_2*. Then* S_t *degenerates to a cone* $S_{2,2}$ *over a one-nodal union of two conics in* $\mathbb{P}^5$ *Let* S_1, S_2 *be the two components of* $S_{2,2}$*, containing* X_1, X_2*, respectively. Then their common ruling contains* t_1, t_2*.*

The zero divisor of the degenerate quadratic differential on X_0 *has two possibilities, either it is cut out by two conics in each of the* S_i *or it consists of four rulings, two in each of the* S_i*, depending on whether the hyperplane section* Q *passes through the singularity of* $S_{2,2}$*, respectively.*

5.2. **Genus four.** We start again with a description of the generic situation. For a general genus four curve X, its canonical embedding lies in a smooth quadric surface Q in $\mathbb{P}^3$. Let f_1, f_2 be the two ruling classes of Q. Then X has class $3f_1 + 3f_2$. For any holomorphic quadratic differential q on X, there exists a unique elliptic curve E (possibly singular) as an element of the linear system $|2f_1 + 2f_2|$ such that $E \cdot X = \operatorname{div}(q)$.

With the hyperelliptic situation in mind, we need to consider the bicanonical embedding. Then Q is embedded into $\mathbb{P}^8$ by the linear system $|2f_1 + 2f_2|$, i.e. by its anti-canonical system $|-K_Q|$. The image of Q is a surface S of degree eight. In particular, S is a conic bundle correspondence between two conics, i.e. the rulings of Q map to conics in $\mathbb{P}^8$.

The *Gieseker-Petri divisor* is the (closure of) the locus of genus four curves such that the quadric surface Q is singular. We call X contained in the Gieseker-Petri divisor as *Gieseker-Petri special*.

Proposition 5.3. *In the above setting, suppose* X *is Gieseker-Petri special and* 3*-connected, but not hyperelliptic. Then its canonical image lies on a quadric cone* Q_0 *in* $\mathbb{P}^3$*. Blowing up the vertex gives a Hirzebruch surface* F_2*. On* F_2 *the class of* X *is* $6f + 3e$ *and the zeros of a holomorphic quadratic differential correspond to a divisor of class* $4f + 2e$*.*

Proposition 5.4. *In the above setting, suppose* X *is hyperelliptic and* 3*-connected. Then its bicanonical image lies on a surface scroll isomorphic to a Hirzebruch surface* F_5*. On* F_5 *the class of* X *is* $10f + 2e$ *and the zeros of a holomorphic quadratic differential correspond to the union of a line and a divisor of class* $6f + e$*.*

Proof. Let X be a hyperelliptic curve of genus four, embedded in $\mathbb{P}^8$ by its bicanonical system. A pair of conjugate points under the hyperelliptic involution of X span a line in $\mathbb{P}^8$. Take the union of these lines and we obtain a surface scroll of degree seven containing X. It is easy to check that the resulting surface is $S_{1,6}$, i.e. the union of line correspondences between a line L and a rational normal sextic. Namely, $S_{1,6}$ is a component of a degree eight surface S, which is a degeneration of the degree eight anti-canonical embedding of Q, hence the other component of S must have degree one, i.e. a plane $\mathbb{P}^2$. The plane is spanned by L and a ruling F, since S is embedded in $\mathbb{P}^8$ by its anti-canonical system.

Note that $S_{1,6}$ is the image of the Hirzebruch surface F_5 by $|e + 6f|$. Under this embedding, the distinguished section e maps to L and f maps to a ruling. Consequently, the class of X is $2e + 10f$. Moreover, the elliptic curve E cutting out the zeros of a quadratic differential on X has the same class as the anti-canonical class of the log surface $S = \mathbb{P}^2 \cup S_{1,6}$, i.e. E is a hyperplane section of S. Its restriction to $S_{1,6}$ has class $e + 6f$, whose sections are rational normal curves in the hyperplane $\mathbb{P}^7$. Since $(e + 6f) \cdot e = (e + 6f) \cdot f = 1$, such a rational normal curve

R in $\mathbb{P}^7$ intersects L at t_1 and intersects F at t_2. So E is the union of the line $\overline{t_1t_2}$ and R, i.e. two rational curves jointed at two nodes. $\square$

6. Genus three: exceptional strata

By [Lan08] in genus three the strata with an exceptional number of connected components are

$$\mathcal{E}_3 = \{(9,-1),(6,3,-1),(3,3,3-1)\}.$$

For each of the strata $\mathcal{Q}(k_1,k_2,k_3,-1)$ in this list, the following properties are known to hold.

i) The stratum $\mathcal{Q}(k_1,k_2,k_3,-1)$ has exactly two components.
ii) Only one of the two components $\mathcal{Q}(k_1,k_2,k_3,-1)^{\mathrm{reg}}$ is adjacent to $\mathcal{Q}(8)$.
iii) The stratum $\mathcal{Q}(9,-1)^{\mathrm{reg}}$ is obtained from $\mathcal{Q}(5,-1)$ by gluing in a cylinder with angle in $\{\pi,2\pi,4\pi\}$ and $\mathcal{Q}(9,-1)^{\mathrm{irr}}$ is obtained by gluing in a cylinder with angle 3π.

We stick to Zorich's notation ([Zor08]) on the labeling 'reg' and 'irr', corresponding to *regular* and *irregular*, respectively, for a reason that will soon become clear. Originally [Lan08] used the labels *reduced* and *irreducible*, to indicate which of the strata by property ii) could be reduced to the stratum $\mathcal{Q}(8)$. But this mnemonic works in $g=3$ only.

Let us explain the meaning of 'adjacent' in ii). Say, for a quadratic differential in $\mathcal{Q}(9,-1)^{\mathrm{reg}}$, one can merge the simple pole and the zero and obtain a non-degenerate flat surface in $\mathcal{Q}(8)$, while merging the simple pole and the zero for a quadratic differential in $\mathcal{Q}(9,-1)^{\mathrm{irr}}$ would necessarily degenerate the underlying Riemann surface, see [Zor08, Appendix A].

The gluing construction used in iii) gives a topological distinction of the two components. In [Lan08] Lanneau asked for an algebraic distinction. We provide a solution to this question as one of the main results of this section. In fact, we will give an algebraic proof, independent of [Lan08], for the existence and construction of the two components for each stratum in $\mathcal{E}_3$.

For a quadratic differential q, let $\mathrm{div}(q)_0$ (resp. $\mathrm{div}(q)_\infty$) be the divisor of zeros (resp. of poles) of q. Define a divisor

$$\mathcal{L}(X,q) = \mathrm{div}(q)_0/3$$

and also use the same notation to denote its associated line bundle on X.

Theorem 6.1. *Each stratum in $\mathcal{E}_3$ has exactly two components, distinguished by the following parity condition:*

iii') *The surface (X,q) belongs to*

$$\mathcal{Q}(k_1,k_2,k_3,-1)^{\mathrm{irr}} \quad \textit{iff} \quad \dim H^0(X,\mathcal{L}(X,q)) = 2$$

and it belongs to

$$\mathcal{Q}(k_1,k_2,k_3,-1)^{\mathrm{reg}} \quad \textit{iff} \quad \dim H^0(X,\mathcal{L}(X,q)) = 1.$$

Moreover, the component $\mathcal{Q}(k_1,k_2,k_3,-1)^{\mathrm{irr}}$ is not adjacent to $\mathcal{Q}(8)$.

Consequently, we may say that in the irregular components the linear system $|\mathcal{L}(X,q)|$ has an irregularly high dimension.

In spirit, this behavior is very similar to Abelian differentials. There, in a stratum $\Omega\mathcal{M}_g(k_1,\dots,k_r)$ where all the k_i are even, the two components are distinguished by the parity of the spin structure, i.e. by the parity of $H^0(X, \mathrm{div}(\omega)/2)$. Whereas the parity of the spin structure is well-known to be deformation invariant, the invariance of the number of sections of a third root of the zero divisor of a quadratic differential appears rather strange.

To prove the above theorem, the first step is an equivalent interpretation of the parity condition in terms of a torsion order on a secretly underlying elliptic curve. Next, we construct the components of the strata in $\mathcal{E}_3$ from moduli spaces and projective bundles that are known to be irreducible.

As a by-product, we found that almost all the exceptional strata are non-varying.

Theorem 6.2. *All the components of the exceptional strata in genus $g = 3$ with the exception of $\mathcal{Q}(3,3,3,-1)^{\mathrm{irr}}$ are non-varying. The values are collected in the following table:*

	component $\mathcal{Q}^{\mathrm{irr}}$	component $\mathcal{Q}^{\mathrm{reg}}$
$\mathcal{Q}(9,-1)$	$L^+ = 14/11$	$L^+ = 12/11$
$\mathcal{Q}(6,3,-1)$	$L^+ = 7/5$	$L^+ = 23/20$
$\mathcal{Q}(3,3,3,-1)$	varying	$L^+ = 6/5$

6.1. **Parity given by torsion conditions.** Let $\mathcal{Q}(k_1,k_2,k_3,-1)$ be a stratum in $\mathcal{E}_3$. We consider first a half-translation surface (X,q) in $\mathcal{Q}(k_1,k_2,k_3,-1)$ that is not hyperelliptic. Later, in Lemma 6.7 we will see that in all strata this is generically the case. We write $\mathrm{div}(q)_0 = k_1z_1 + k_2z_2 + k_3z_3$ and let p be the pole of q.

Consider the canonical embedding $X \hookrightarrow \mathbb{P}^2$. Take a general line L passing through the pole of q but not through its zeros. Then L intersects X at three points r_1, r_2 and r_3. By the long exact sequence of cohomology associated to the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{O}_{\mathbb{P}^2}(3) \to \mathcal{O}_X(3) \to 0$$

we obtain an isomorphism $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3)) \cong H^0(X, \mathcal{O}_X(3))$. Note that $\mathcal{O}_X(3) \cong \omega_X^{\otimes 3}$. Thus there exists a unique plane cubic E such that

$$E \cdot X = \mathrm{div}(q)_0 + r_1 + r_2 + r_3. \tag{16}$$

Proposition 6.3. *Fix a half-translation surface (X,q) in $\mathcal{Q}(k_1,k_2,k_3,-1)$ and suppose it is not hyperelliptic. For a generic choice of L the plane cubic E is irreducible and $\mathrm{div}(q)_0$ is contained in the smooth locus of E.*

To help the reader quickly grasp our idea, we postpone the proof of the above technical statements to Appendix B.1 and continue with the parity construction. From $r_1 + r_2 + r_3 \sim \mathcal{O}_E(1)$ and (16) we deduce that

$$3\mathcal{L}(X,q) \sim \mathcal{O}_E(3). \tag{17}$$

The key observation is that linear equivalence may or may not hold when dividing both sides by three, thus providing a parity to distinguish the two components.

Proposition 6.4. *In the above setting, the parity $\dim H^0(X, \mathcal{L}(X,q)) = 2$ if and only if $\mathcal{L}(X,q) \sim \mathcal{O}_E(1)$ and the parity $\dim H^0(X, \mathcal{L}(X,q)) = 1$ if and only if $\mathcal{L}(X,q) \not\sim \mathcal{O}_E(1)$ but $3\mathcal{L}(X,q) \sim \mathcal{O}_E(3)$.*

Moreover, in a family of quadratic differentials (X_t, q_t) with X_t a smooth non-hyperelliptic curve for all $t \in \Delta$, the special member (X_0, q_0) for $t = 0$ has the same parity as the generic member.

Proof. Since E is smooth at the z_i by Proposition 6.3, the condition $\mathcal{L}(X,q) \sim \mathcal{O}_E(1)$ says for $(X,q) \in \mathcal{Q}(9,-1)$ that the tangent line to E at z_1 is a flex line, i.e. a tangent line with contact of order three at z_1. Since E and X have contact of order 9 at z_1, either they possess the same flex line or they both have simple tangent lines at z_1. Rephrasing in the language of linear systems, z_1 being a flex of X is equivalent to $h^0(X, 3z_1) = 2$. For $(X,q) \in \mathcal{Q}(6,3,-1)$ the same argument works for the secant line $\overline{z_1 z_2}$ being tangent to X at z_1 and for $(X,q) \in \mathcal{Q}(3,3,3,-1)$ the same argument works for the collinearity of z_1, z_2 and z_3.

If Δ parameterizes a family of half-translation surfaces with generic member satisfying $h^0(X_t, \mathcal{L}(X_t,q_t)) = 2$, then any special member (X_0,q_0) also satisfies $h^0(X_0, \mathcal{L}(X_0,q_0)) = 2$, because the dimension of the linear system is upper semi-continuous. Suppose for a family of half-translation surfaces in this stratum we have $h^0(X, \mathcal{L}(X_t,q_t)) = 1$ for $t \neq 0$. Then we need to prove that they cannot specialize to (X_0,q_0) with $h^0(X, \mathcal{L}(X_0,q_0)) = 2$. Since the support of $\mathcal{L}(X_t,q_t)$ is in the smooth locus of E_t by Proposition 6.3, $\mathcal{L}(X_t,q_t) \otimes \mathcal{O}_E(-1)$ is a well-defined family of Cartier divisors for all $t \in \Delta$. It is well-known that two distinct torsion sections are disjoint in a family of elliptic fiberations (see e.g. [Mir89]). In our situation, it implies that if the torsion order in the Picard group of E_t is three at a generic point t, it cannot drop to one at a special point. □

Note that at this stage we have not yet shown that the dimension of the locus with $\dim H^0(X, \mathcal{L}(X,q)) = 2$ is the same as the corresponding stratum. The possibility of this locus being of smaller dimension will be ruled out in the next section.

6.2. **Construction of components.** Suppose we deal with an exceptional stratum $\mathcal{Q}(k_1,\ldots,k_n,-1)$ with n zeros and one simple pole. Consider the moduli spaces $\overline{\mathcal{M}}_{1,n+1}^{\mathrm{reg}}$ and $\overline{\mathcal{M}}_{1,n+1}^{\mathrm{irr}}$ of stable pointed elliptic curves $(E, z_0, z_1, \ldots, z_n)$ with the additional property that

$$\sum_{i=1}^{n} \frac{k_i}{3} z_i \sim 3z_0$$

in the case 'irr', respectively

$$\sum_{i=1}^{n} \frac{k_i}{3} z_i \not\sim 3z_0 \quad \text{but} \quad \sum_{i=1}^{n} k_i z_i \sim 9z_0$$

in the case 'reg'. Mapping this tuple to $(E, z_1, \ldots, z_n)$ exhibits these moduli spaces as finite connected unramified coverings of $\overline{\mathcal{M}}_{1,n}$. We embed the elliptic curve as a plane cubic in $\mathbb{P}^2$ using the linear system $|3z_0|$, i.e. such that $3z_0 \sim \mathcal{O}_E(1)$. Now choose moreover a line L in $\mathbb{P}^2$, i.e. a section of $|3z_0|$ and let $L \cdot E = r_1 + r_2 + r_3$. Define two parameter spaces $B^{\mathrm{reg}}_{(k_1,\ldots,k_n,-1)}$ and $B^{\mathrm{irr}}_{(k_1,\ldots,k_n,-1)}$ parameterizing tuples $(E, z_0, z_1, \ldots, z_n, L)$. They are fiber bundles over $\overline{\mathcal{M}}_{1,n+1}^{\mathrm{reg}}$ and $\overline{\mathcal{M}}_{1,n+1}^{\mathrm{irr}}$ respectively.

We let

$$f : S^{\mathrm{reg}}_{(k_1,\ldots,k_n,-1)} \to B^{\mathrm{reg}}_{(k_1,\ldots,k_n,-1)} \quad \text{resp.} \quad f : S^{\mathrm{irr}}_{(k_1,\ldots,k_n,-1)} \to B^{\mathrm{irr}}_{(k_1,\ldots,k_n,-1)}$$

be the subspace of plane quartics X whose fiber over $(E, z_0, z_1, \ldots, z_n, L)$ parameterizes those X such that $X \cdot E = \sum k_i z_i + r_1 + r_2 + r_3$.

Proposition 6.5. *For both indices* irr *and* reg *the parameter spaces $B_{(k_1,\ldots,k_n,-1)}$ are irreducible of dimension $n+2$ and the parameter spaces $S_{(k_1,\ldots,k_n,-1)}$ are irreducible of dimension $n+5$, which is the dimension of the incidence correspondence* *consisting of a point in* $\mathbb{P}\mathcal{Q}(k_1, \ldots, k_n, -1)$ *together with a line in* $\mathbb{P}^2$ *passing through the unique pole.*

Moreover, the generic quartic X *parameterized by* $S_{(k_1,\ldots,k_n,-1)}$ *is a smooth curve of genus* 3 *in all the cases.*

Proof. The irreducibility of $\overline{\mathcal{M}}^{\mathrm{reg}}_{1,n+1}$ (resp. $\overline{\mathcal{M}}^{\mathrm{irr}}_{1,n+1}$) for $n = 1$ is a consequence of the irreducibility of the space of elliptic curves with marked points together with the choice of a primitive 9-torsion point (resp. a primitive 3-torsion point). The case $n > 1$ is reduced to the previous case by using the irreducibility of the fiber under the addition map $(r_1, \ldots, r_n) \mapsto \sum_{i=1}^n k_i r_i \in E$. Irreducibility of $B^{\mathrm{reg}}_{(k_1,\ldots,k_n,-1)}$ and $B^{\mathrm{irr}}_{(k_1,\ldots,k_n,-1)}$ follows because they are fiber bundles over the previous parameter spaces. Tensoring the defining sequence of the ideal sheaf of E with $\mathcal{O}_{\mathbb{P}^2}(4)$, we obtain an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(1) \to \mathcal{O}_{\mathbb{P}^2}(4) \to \mathcal{O}_E(4) \to 0.$$

The associated long exact sequence of cohomology shows that any two quadrics in a fiber of f differ by a section of $\mathcal{O}_{\mathbb{P}^2}(1)$. This shows first the irreducibility. Moreover, both for $\bullet = \mathrm{reg}$ and $\bullet = \mathrm{irr}$ we have $\dim \mathcal{M}^{\bullet}_{1,n+1} = n$, hence $\dim B^{\bullet}_{(k_1,\ldots,k_n,-1)} = n + 2$ and since $h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) = 3$ we finally obtain $\dim S^{\bullet}_{(k_1,\ldots,k_n,-1)} = n + 5$.

To prove generic smoothness we may fix a point in $B_{(k_1,\ldots,k_n,-1)}$ corresponding to a smooth elliptic curve E. The base points of the linear system cutting out the X with $X \cdot E = \sum_{i=1}^n k_i z_i + r_1 + r_2 + r_3$ are precisely the points z_i and r_i. Moving L we may move the r_i and by Bertini's theorem for a Zariski open set in each fiber over $\mathcal{M}_{1,n+1}$ the curves X are smooth except possibly at the z_i.

We now argue that at the z_i the generic X is also smooth. In fact, let $D = \sum_{i=1}^n k_i z_i$ and $F = r_1 + r_2 + r_3$, considered as divisors on E. Since $D + F$ is a section of $\mathcal{O}_E(4)$, the line bundle $\mathcal{O}_E(4) \otimes \mathcal{O}_E(-D-F)$ is trivial. It implies that for all quartics in a fiber of f their restrictions to E are unique up to scalars. Lifting to $\mathbb{P}^2$, we conclude that the defining homogeneous polynomial of X can be expressed as $f_0 + le$, where f_0 is the defining equation for a fixed X_0 in S, l is an arbitrary linear form and e is the equation of E. Since E is non-singular at z_i, for a generic choice of l, the vanishing locus of $f_0 + le$ is non-singular at z_i as well. □

Proof of Theorem 6.1. The existence of period coordinates shows that strata of quadratic differentials are smooth. Consequently, disregarding subsets of complex codimension at least one does not change connectivity. By Lemmas 6.6 and 6.7 below we may thus restrict the question on the number of components to the complement of the hyperelliptic locus and to the locus where E is smooth.

On this restricted locus by Proposition 6.4 the parity is deformation invariant and hence there are at least two components. To each point (X, q) in a stratum $\mathcal{Q}(k_1, \ldots, k_n, -1)$ listed in $\mathcal{E}_3$ and the additional choice of L we associated at the beginning of Section 6.1 an elliptic curve E in $\mathbb{P}^2$. By the homogeneity of an elliptic curve we may take $z_0 \in E$ so that the embedding is given by $|3z_0|$. Let $z_1, \ldots, z_n$ be the points in $X \cdot E \setminus L \cdot E$. Finally, (17) implies that $(E, z_0, z_1, \ldots, z_n, L, X)$ defines a point in $S_{k_1,\ldots,k_n}$ with upper index either 'irr' or 'reg'. This space has the expected dimension $2 + \dim(\mathbb{P}\mathcal{Q}(k_1, \ldots, k_n, -1))$ by Proposition 6.5 and the irreducibility statement in that proposition completes the proof. □

Lemma 6.6. *None of the strata in* $\mathcal{E}_3$ *has a component such that for a generic half-translation surface in the component the plane cubic* E *defined by* (16) *is singular.*

Proof. If this was the case, we could reconstruct this component by the argument leading to Proposition 6.5, but replacing $\mathcal{M}^{\bullet}_{1,n+1}$ by a configuration space for $n+1$ points on an irreducible rational nodal or cuspidal curve by Proposition 6.3. If the points $z_1, \ldots, z_n$ lie in the smooth locus of the rational curve, there is still the torsion constraint and this parameter space is of dimension one smaller than $\mathcal{M}^{\bullet}_{1,n+1}$. If one of the z_i lies at a node, the torsion constraint may no longer be well-defined, but z_i being restricted to a node imposes one more condition so that this parameter space is still of dimension at least one smaller than $\mathcal{M}^{\bullet}_{1,n+1}$.

The remaining dimension argument for the fibers of $B^{\bullet}_{(k_1,\ldots,k_n,-1)} \to \mathcal{M}^{\bullet}_{1,n+1}$ and $f : S^{\bullet}_{(k_1,\ldots,k_n,-1)} \to B^{\bullet}_{(k_1,\ldots,k_n,-1)}$ in Proposition 6.5 still holds with E singular but irreducible. In total, the locus with E singular is thus too small to form a component of a stratum in $\mathcal{E}_3$. □

The same type of argument using the description of hyperelliptic curves in Section 5.2 shows the following lemma, whose proof will be given in the appendix.

Lemma 6.7. *None of the strata in $\mathcal{E}_3$ has a component entirely contained in the hyperelliptic locus.*

Finally, let us verify the last sentence in Theorem 6.1. Recall the meaning of 'adjacent' in [Zor08, Appendix A].

Lemma 6.8. *The irregular component of each stratum in $\mathcal{E}_3$ is not adjacent to $\mathcal{Q}(8)$.*

Proof. Suppose there is a family of half-translation surfaces in $\mathcal{Q}(9,-1)^{\mathrm{irr}}$ that degenerates to (X,q) in $\mathcal{Q}(8)$, where $\mathrm{div}(q) = 8z_1$. Since $h^0(X, 3z_1)$ is upper semi-continuous, z_1 is a Weierstrass point on X and $\omega_X \sim 3z_1 + r$ for some $r \in X$. Then from $8z_1 \sim \omega_X^{\otimes 2}$ we obtain that $2z_1 \sim 2r$. If $z_1 = r$, then $\omega_X \sim 4z_1$ and q is a global square, leading to a contradiction. If $z_1 \neq r$, then X is hyperelliptic. Again $\omega_X \sim 4z_1$ and we obtain the same contradiction. The same argument works for the other strata in $\mathcal{E}_3$. □

Remark 6.9. From the above construction it is obvious that $\mathcal{Q}(k_1,k_2,k_3,-1)^{\mathrm{irr}}$ is adjacent to $\mathcal{Q}(k_1+k_2,k_3,-1)^{\mathrm{irr}}$ and that $\mathcal{Q}(k_1,k_2,k_3,-1)^{\mathrm{reg}}$ is adjacent to $\mathcal{Q}(k_1+k_2,k_3,-1)^{\mathrm{reg}}$, while the strata with different upper indices are not adjacent by Proposition 6.4.

Remark 6.10. For the irregular components there is another construction and irreducibility proof. It relies on the following observation. We give details for $\mathcal{Q}(9,-1)^{\mathrm{irr}}$ and the other cases can be dealt with similarly.

Let z_1 be the 9-fold zero of q. Let $B^{\mathrm{irr}}_{(9,-1)}$ parameterize tuples (z_1, L_1, L_2) with z_1 a point and the L_i lines in $\mathbb{P}^2$ such that $z_1 \in L_1$ but $z_1 \notin L_2$. Define $S^{\mathrm{irr}}_{(9,-1)}$ to be the parameter space of plane quartics X such that i) L_1 is a flex line to X at z_1 and ii) L_2 is a flex line to X at the intersection point r of L_1 and L_2. As above one checks that the parameter space $S^{\mathrm{irr}}_{(9,-1)}$ is irreducible of dimension 13 and hence its quotient by the action of $\mathrm{PGL}(3)$ is of dimension 5. One easily checks that an open subset of this quotient is indeed $\mathbb{P}\mathcal{Q}(9,-1)^{\mathrm{irr}}$.

6.3. **The non-varying property.** Throughout this section let C be a Teichmüller curve generated by (X,q) in one of the strata in $\mathcal{E}_3$ and let $\overline{C}$ be its closure over the compactified moduli space of curves. Note that using the language of linear series the components $\mathcal{Q}(9,-1)^{\mathrm{irr}}$, $\mathcal{Q}(6,3,-1)^{\mathrm{irr}}$ and $\mathcal{Q}(3,3,3,-1)^{\mathrm{irr}}$ lie in the divisors $BN^1_{3,(3)}$, $BN^1_{3,(2,1)}$ and $BN^1_{3,(1,1,1)}$, respectively, after a suitable lift to the moduli space of curves with marked points. Consequently these divisors are the natural candidates to prove non-varying for the regular components of the strata in $\mathcal{E}_3$. But we are still left with ruling out possible intersections at the boundary of the components. This is the content of Proposition 2.2 together with Corollary 4.6.

Lemma 6.11. *For each stratum $\mathcal{Q}(k_1,\ldots,k_n,-1)$ in $\mathcal{E}_3$, the closures of its two components do not intersect at a boundary point of a Teichmüller curve in that stratum.*

Proof. Since the two components are of the same dimension, if they intersect, $\widetilde{\mathcal{Q}}(k_1,\ldots,k_n,-1)$ would be singular at the common locus, contradicting Proposition 2.2. □

Proposition 6.12. *If C is generated by (X,q) in $\mathcal{Q}(9,-1)^{\mathrm{irr}}$, then $\overline{C}$ is disjoint with the hyperelliptic locus.*

Proof. Suppose $\mathrm{div}(q) = 9z_1 - p_1$ with z_1 a Weierstrass point. If X is hyperelliptic, we have $\omega_X \sim 4z_1$. By $\omega_X^{\otimes 2} \sim 9z_1 - p_1$, we conclude that $z_1 \sim p_1$, contradicting that $z_1 \neq p_1$. □

Proof of Theorem 6.2. For the components of $\mathcal{Q}(9,-1)$ this is straightforward from the divisor classes given in Section 3, from Lemma 6.11 and Proposition 6.12, respectively, since the stable curves parameterized by the boundary of $\overline{C}$ are irreducible.

In the case of $\mathcal{Q}(6,3,-1)^{\mathrm{reg}}$ we know from Lemma 6.11 that $\overline{C}$ and the divisor $BN^1_{3,(2,1)}$ are disjoint in $\overline{\mathcal{M}}_{3,2}$. For a curve X parameterized in the boundary divisor δ_1, the degree of $\omega_X^{\otimes 2}$ restricted to the genus one component is equal to 2, hence the only marked δ_1-boundary divisor $\overline{C}$ could intersect is the divisor $\delta_{1;\{2\}}$, where the zero of multiplicity 3 and the simple pole are contained in the genus one component. Note that $\delta_{1;\{2\}}$ does not appear in (8). Hence we obtain that

$$\overline{C}\cdot(-\lambda + 3\omega_1 + \omega_2) = 0.$$

By Proposition 4.2, we have

$$\frac{\overline{C}\cdot\omega_1}{\overline{C}\cdot\lambda} = \frac{2}{(6+2)L^+(C)} = \frac{1}{4L^+(C)},$$

$$\frac{\overline{C}\cdot\omega_2}{\overline{C}\cdot\lambda} = \frac{2}{(3+2)L^+(C)} = \frac{2}{5L^+(C)}.$$

Therefore, we obtain that $L^+(C) = 23/20$.

For $\mathcal{Q}(6,3,-1)^{\mathrm{irr}}$, the lift of C is contained in $BN^1_{3,(2,1)}$, so this divisor does not work for the disjointness property. Instead, one can find a divisor inside $BN^1_{3,(2,1)}$ disjoint with $\overline{C}$ and perform the intersection calculation. The trade-off is a tedious study of the Picard group of $BN^1_{3,(2,1)}$ as well as its singularities. Here we take an alternative approach, adapting the idea of [YZ12] and using a filtration of the Hodge bundle on $\overline{C}$. Since it has a different flavor and requires a technical setup, we will treat this approach independently in Appendix A.

Finally, we consider $\mathcal{Q}(3,3,3,-1)$. The boundary terms appearing in the divisor class of $BN^1_{3,(1,1,1)}$ in (7) are either irreducible or, if over δ_1 have any number but one of the three marked points on the elliptic tail X_1. Since $\omega_X^{\otimes 2}$ has degree two restricted to the subcurve X_1, a boundary point of $\overline{C}$ in δ_1 has precisely one of the marked zeros in X_1 along with the pole. Consequently, all the boundary terms are irrelevant for the intersection number and from this we can calculate the sum of Lyapunov exponents. □

7. Genus four: exceptional strata

Among all the strata of quadratic differentials in genus four, only $\mathcal{Q}(12)$ was claimed in [Lan08] to have two components not arising from any hyperelliptic component. This claim is incomplete. In addition, even for $\mathcal{Q}(12)$ in [Lan08] there is no terminology distinguishing the two components.

Consider the following strata

$$\mathcal{E}_4 = \{(12), (9,3), (6,6), (6,3,3), (3,3,3,3)\}.$$

Some of these strata obviously possess a hyperelliptic component according to Section 2. In what follows we will show that each of the strata in $\mathcal{E}_4$ possesses two non-hyperelliptic components. As in the case of genus three, we also provide algebraically a parity to distinguish the two (non-hyperelliptic) components. We thus keep the labels 'irr' and 'reg' as to make the following result look consistent with Theorem 6.1.

Theorem 7.1. *Each of the strata $\mathcal{Q}(6,6)$, $\mathcal{Q}(6,3,3)$ and $\mathcal{Q}(3,3,3,3)$ has a hyperelliptic component, denoted by $\mathcal{Q}(6,6)^{\mathrm{hyp}}$ etc. Besides the hyperelliptic components each of the strata in $\mathcal{E}_4$ has exactly two additional components distinguished as follows. Let $\mathcal{L}(X,q) = \mathrm{div}(q)/3$, then the surface (X,q) belongs to*

$$\mathcal{Q}^{\mathrm{irr}} \quad \textit{iff} \quad \dim H^0(X, \mathcal{L}(X,q)) = 2$$

and it belongs to

$$\mathcal{Q}^{\mathrm{reg}} \quad \textit{iff} \quad \dim H^0(X, \mathcal{L}(X,q)) = 1.$$

Many of the exceptional strata are non-varying.

Theorem 7.2. *The exceptional strata in genus four with the exception of $\mathcal{Q}(6,6)^{\mathrm{irr}}$, $\mathcal{Q}(6,3,3)^{\mathrm{irr}}$ and $Q(3,3,3,3)^{\mathrm{irr}}$ are non-varying. The values are collected as follows:*

	hyperell. comp.	component $\mathcal{Q}^{\mathrm{irr}}$	component $\mathcal{Q}^{\mathrm{reg}}$
$\mathcal{Q}(12)$	$---$	$L^+ = 11/7$	$L^+ = 10/7$
$\mathcal{Q}(9,3)$	$---$	$L^+ = 92/55$	$L^+ = 82/55$
$\mathcal{Q}(6,6)$	$L^+ = 2$	varying	$L^+ = 3/2$
$\mathcal{Q}(6,3,3)$	$L^+ = 11/5$	varying	$L^+ = 31/20$
$\mathcal{Q}(3,3,3,3)$	$L^+ = 12/5$	varying	$L^+ = 8/5$

7.1. **Parity given by torsion conditions.** Let $\mathcal{Q}(k_1, \ldots, k_n)$ be a stratum in $\mathcal{E}_4$ and consider a half-translation surface (X,q) in this stratum that is not hyperelliptic and not in the Gieseker-Petri locus. Later, in Lemma 7.7 we will see that this holds for a generic surface in all components but the hyperelliptic ones. Let $X \hookrightarrow \mathbb{P}^3$ be the canonical embedding. Since X is not in the Gieseker-Petri locus, the image of the canonical embedding lies in a smooth quadric surface $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Let $\mathcal{O}_Q(1)$ denote the hyperplane class in $\mathbb{P}^3$ restricted to Q. It has divisor class $(1,1)$ in the Picard group of Q. A general section E of $|\mathcal{O}_Q(2)|$ is an elliptic curve of degree four with divisor class $(2,2)$. We have the following exact sequence

$$0 \to \mathcal{O}_Q(-1) \to \mathcal{O}_Q(2) \to \mathcal{O}_X(2) \to 0,$$

hence $H^0(Q, \mathcal{O}_Q(2)) \to H^0(X, \mathcal{O}_X(2))$ is an isomorphism. Note that $\mathcal{O}_X(2) \cong \omega_X^{\otimes 2}$. Therefore, a bicanonical divisor of X is uniquely cut out by a section E of $|\mathcal{O}_Q(2)|$. Therefore, we can choose E uniquely corresponding to q modulo scalars, that is

$$E \cdot X = \operatorname{div}(q). \tag{18}$$

We say that $(E, \operatorname{div}(q))$ is *sufficiently smooth* if E is reduced (possibly reducible) and $\operatorname{div}(q)$ is supported on the smooth locus of E. In that case, we also say that X has a *sufficiently smooth parity curve*. This condition allows us to study torsion in the Picard group of divisors supported away from the singular locus of E. Since X is a section of $|\mathcal{O}_Q(3)|$, we have

$$3\mathcal{L}(X, q) \sim \mathcal{O}_E(3). \tag{19}$$

in the Picard group of E. This linear equivalence may or may not hold when dividing both sides by three. Contrary to the case of genus three we cannot just focus on irreducible E. Our substitute is the following lemma. As before, we postpone the proof of all the technical results to the appendix.

Lemma 7.3. *In the above setting, suppose X is neither hyperelliptic nor Gieseker-Petri special. Then either E is sufficiently smooth or* $\dim H^0(X, \mathcal{L}(X, q)) = 1$.

Proposition 7.4. *For X neither hyperelliptic nor Gieseker-Petri special and with sufficiently smooth parity curve, the parity*

$$\dim H^0(X, \mathcal{L}(X, q)) = 2 \quad \text{if and only if} \quad \mathcal{L}(X, q) \sim \mathcal{O}_E(1)$$

and

$$\dim H^0(X, \mathcal{L}(X, q)) = 1 \quad \text{if and only if} \quad \mathcal{L}(X, q) \not\sim \mathcal{O}_E(1) \quad \text{but} \quad 3\mathcal{L}(X, q) \sim \mathcal{O}_E(3).$$

Moreover, in a family of quadratic differentials (X_t, q_t) with X_t a smooth non-hyperelliptic curve, not Gieseker-Petri special and with sufficiently smooth parity curve for all $t \in \Delta$, the special member (X_0, q_0) has the same parity as the generic member.

Proof. By assumption, the canonical embedding of X is contained in a smooth quadric surface Q in $\mathbb{P}^3$. By the Riemann-Roch formula, $\dim H^0(X, \mathcal{L}(X, q)) = 2$ is equivalent to the existence of a unique plane section C of Q such that $C \cdot X = \mathcal{L}(X, q) + r + s$ for some points $r, s \in X$. Since $E \cdot X = 3\mathcal{L}(X, q)$, this is equivalent to $C \cdot E = \mathcal{L}(X, q)$. Since by definition $C \cdot E \sim \mathcal{O}_E(1)$, this equivalence proves the first statement.

For a sufficiently smooth parity curves E, the line bundle $\mathcal{L}(X, q) \otimes \mathcal{O}_E(-1)$ is well-defined and is torsion of order either one or three that cannot jump in a family. As in genus three, using upper semicontinuity of cohomology, we only need to show that in a family of half-translation surfaces whose special fiber is no longer sufficiently smooth, the dimension of $H^0(X, \mathcal{L}(X, q))$ does not increase. This is precisely the content of Lemma 7.3. ∎

Note that at this stage we have not yet shown that the dimension of the locus with $\dim H^0(X, \mathcal{L}(X, q)) = 2$ is the same as the corresponding stratum in $\mathcal{E}_4$. The case of smaller dimension (thus necessarily contained in the hyperelliptic locus or in the Gieseker-Petri divisor) will be ruled out in the next section.

7.2. **Construction of components.** Let $B^{\mathrm{irr}}_{(k_1,\dots,k_n)}$ resp. $B^{\mathrm{reg}}_{(k_1,\dots,k_n)}$ be subset of the moduli space of stable elliptic curves $(E, q_1, r_1, \dots, r_n, D)$ with $n+1$ marked points and additionally with the linear equivalence class D of a divisor of degree two satisfying the condition

$$\sum_{i=1}^{n} \frac{k_i}{3} r_i \sim 2q_1 + D \tag{20}$$

resp. satisfying

$$\sum_{i=1}^{n} \frac{k_i}{3} r_i \not\sim 2q_1 + D \quad \text{but} \quad \sum_{i=1}^{n} k_i r_i \sim 6q_1 + 3D$$

and $2q_1 \not\sim D$. The linear systems $|2q_1|$ and $|D|$ define maps $E \to \mathbb{P}^1$ and by definition of the parameter spaces the product $E \to Q := \mathbb{P}^1 \times \mathbb{P}^1$ is an embedding. Instead of D we could choose a point $q_2 \in E$ up to a two-torsion translation and let D be the divisor class associated to $\mathcal{O}_E(2q_2)$. We will consider $Q \hookrightarrow \mathbb{P}^3$ given by the Veronese embedding $|\mathcal{O}(1,1)|$ and, as above, let $\mathcal{O}_Q(k)$ be the restriction of $\mathcal{O}_{\mathbb{P}^3}(k)$ to Q. Thus, E can be viewed as a section of $|\mathcal{O}_Q(2)|$.

Let $f : S^{\mathrm{irr}}_{(k_1,\dots,k_n)} \to B^{\mathrm{irr}}_{(k_1,\dots,k_n)}$ and $f : S^{\mathrm{reg}}_{(k_1,\dots,k_n)} \to B^{\mathrm{reg}}_{(k_1,\dots,k_n)}$ be the fibrations whose fiber over $(E, q_1, r_1, \dots, r_n, D)$ consists of all sections X of $|\mathcal{O}_Q(3)|$ together with the quadratic differential q obtained from restricting E to X.

Proposition 7.5. *The parameter spaces $B^{\bullet}_{(k_1,\dots,k_n)}$ are irreducible of dimension $n+1$ for both upper indices $\bullet = \mathrm{irr}$ and $\bullet = \mathrm{reg}$ and the parameter spaces $S^{\bullet}_{(k_1,\dots,k_n)}$ are irreducible of dimension $n+5$, which equals $\dim \mathbb{P}\mathcal{Q}(k_1,\dots,k_n)$.*

Moreover, a generic section X parameterized by $S_{(k_1,\dots,k_n)}$ is a smooth curve of genus four in all the cases.

Proof. Forgetting the last marked point and using the interpretation of D exhibit a finite, dominant map to a quotient of the moduli space of elliptic curves with $n+1$ marked points by a finite group action. This proves the first dimension count. Tensoring the defining sequence for E with $\mathcal{O}_Q(3)$, we obtain an exact sequence

$$0 \to \mathcal{O}_Q(1) \to \mathcal{O}_Q(3) \to \mathcal{O}_E(3) \to 0,$$

and read off $h^0(Q, \mathcal{O}_Q(1)) = 4$, which implies that the dimension of $S_{(k_1,\dots,k_n)}$ is four larger than $B_{(k_1,\dots,k_n)}$. The irreducibility of $B_{(k_1,\dots,k_n)}$ for $n = 1$ is a consequence of the irreducibility of the space of elliptic curves with one marked point together with a choice of primitive 4-torsion point (respectively a primitive 12-torsion point). The case $n > 1$ is reduced to the previous case by using the irreducibility of the fiber under the addition map $(r_1, \dots, r_n) \mapsto \sum_{i=1}^n k_i r_i \in E$.

The proof that a generic curve X is smooth is completely parallel to the case of genus three. □

Proof of Theorem 7.1. The existence of period coordinates shows that strata are smooth. Consequently, disregarding subsets of complex codimension at least one does not change connectivity. By Lemmas 7.6 and 7.7 below we may thus restrict the question on the number of components to the complement of the hyperelliptic locus and the Gieseker-Petri locus and to half-translation surfaces with sufficiently smooth parity curves.

On this complement by Proposition 7.4 the parity is deformation invariant and hence there are at least two components for each stratum listed in $\mathcal{E}_4$. Recall that to a point (X,q) in a stratum $\mathcal{Q}(k_1,\ldots,k_n)$ in $\mathcal{E}_4$ we associated at the beginning of Section 7.1 an elliptic curve E and a map $E \to Q \cong \mathbb{P}^1 \times \mathbb{P}^1$. By the homogeneity of an elliptic curve we may define $q_1 \in E$ so that the first projection is given by $|2q_1|$ and let D be the pullback of $\mathcal{O}_{\mathbb{P}^1}(1)$ via the second projection. Since $X \cdot E = \operatorname{div}(q)$, we can associate X canonically n points on Q. Finally, equation (19) implies that $(X, E, q_1, D, \operatorname{div}(q))$ defines a point in $S_{k_1,\ldots,k_n}$ with upper index either irr or reg. The irreducibility statement in Proposition 7.5 completes the proof. □

Lemma 7.6. *None of the strata in $\mathcal{E}_4$ has a component such that for a generic half-translation surface in that component the parity curve defined by* (18) *is singular.*

Proof. The proof is identical to Lemma 6.6. If E is singular but sufficiently smooth, the torsion constraint is still valid, so dimension of the base space drops by one, since the parameter for the j-invariant of E has been lost. If E is singular and at least one of the z_i is at a node, the parity condition (20) no longer makes sense, but this cannot compensate the loss of at least two parameters for the j-invariant of E and for the location of the z_i at a node. □

Lemma 7.7. *Except the hyperelliptic components defined in Section 2.1 no component of a stratum in $\mathcal{E}_4$ is contained in the hyperelliptic locus or contained in the Gieseker-Petri locus.*

The proof of Lemma 7.7 will be given in the appendix.

Remark 7.8. For the irregular components there is another construction and irreducibility proof. It relies on the following observation. We give details for $\mathcal{Q}(12)^{\mathrm{irr}}$ and the other cases can be dealt with similarly.

Let z_1 be the 12-fold zero of q. Since z_1 is a Weierstrass point, $\omega_X \sim 4z_1 + r + s$ for some points r and s. Since $\omega_X^{\otimes 2} \sim 12z_1$, we conclude that $4z_1 \sim 2r + 2s$, hence $\omega_X \sim 3r + 3s$. Conversely, if we can find $r, s \in X$ such that $\omega_X \sim 3r + 3s$ and $\omega_X \sim 4p + r + s$, then one easily checks that $\omega_X^{\otimes 2} \sim 12z_1$. Now, consider (X,ω) in $\Omega\mathcal{M}_4(3,3)^{\text{non-hyp}}$ such that $\operatorname{div}(\omega) = 3r + 3s$. The dimension of this locus modulo scalars is 8. Let z_1 be a Weierstrass point of X and choose a plane section through z_1 whose intersection with the canonical embedding of X is $4z_1+x+y$. If we require $(x,y) = (r,s)$, this imposes two conditions, hence the dimension of the locus where $\omega_X \sim 3r + 3s \sim 4z_1 + r + s$ is equal to $8 - 2 = 6$, which equals $\dim \mathbb{P}\mathcal{Q}(12)$.

7.3. **The non-varying properties.** Throughout this section let C be a Teichmüller curve generated by (X,q) in one of the strata in $\mathcal{E}_3$ and let $\overline{C}$ be its closure in the compactified moduli space. The non-varying property of the hyperelliptic components is an immediate consequence of Corollary 2.1. Note that using the language of linear series the components $\mathcal{Q}(12)^{\mathrm{irr}}$, $\mathcal{Q}(9,3)^{\mathrm{irr}}$, $\mathcal{Q}(6,6)^{\mathrm{irr}}$, $\mathcal{Q}(6,3,3)^{\mathrm{irr}}$ and $\mathcal{Q}(3,3,3,3)^{\mathrm{irr}}$ lie in the divisors $BN^1_{4,(4)}$, $BN^1_{4,(3,1)}$, $BN^1_{4,(2,2)}$, $BN^1_{4,(2,1,1)}$ and $BN^1_{4,(1,1,1,1)}$ respectively, after a suitable lift the moduli space of curves with marked points. Consequently these divisors are the natural candidates to prove non-varying for the regular components. But we are still left with ruling out intersections at the boundary. As for $g = 3$, this is the content of Proposition 2.2 together with Corollary 4.6.

Lemma 7.9. *For any stratum $\mathcal{Q}(k_1,\ldots,k_n)$ in $\mathcal{E}_4$, the closures of its two components $\widetilde{\mathcal{Q}}(k_1,\ldots,k_n)^{\mathrm{irr}}$ and $\widetilde{\mathcal{Q}}(k_1,\ldots,k_n)^{\mathrm{reg}}$ do not intersect at a boundary point of a Teichmüller curve in the stratum lifted to $\overline{\mathcal{M}}_{4,n}$ by marking the zeros of q.*

Proof of Theorem 7.2. By Lemma 7.9 and the disjointness of the boundary of the Teichmüller curve with the boundary terms appearing in Lemma 3.2 as established in Proposition 4.8, the non-varying property of the regular components in $\mathcal{E}_4$ is a direct consequence of Proposition 4.2.

For $\mathcal{Q}(12)^{\mathrm{irr}}$ and $\mathcal{Q}(9,3)^{\mathrm{irr}}$, due to the same issue as $\mathcal{Q}(6,3,-1)^{\mathrm{irr}}$ in genus three, we will take an alternative approach to prove their non-varying property. The details will be given in Appendix A. □

8. Genus one

Theorem 8.1. *In genus one, the following strata are non-varying:*

Stratum	L^+	Stratum	L^+
$\mathcal{Q}(n,-1^n)$	$L^+ = 2/(n+2)$	$\mathcal{Q}(n,1,-1^{n+1})$	$L^+ = 2/(n+2)$

Moreover, the stratum $\mathcal{Q}(4,2,-1^6)$ is varying.

An example justifying that $\mathcal{Q}(4,2,-1^6)$ is varying will be given in the appendix. In general, to show a stratum is varying one only needs to find two Teichmüller curves in the stratum such that they have different values for the sum of Lyapunov exponents. Such examples of Teichmüller curves come from a special branched cover construction, called 'square-tiled surfaces', see Appendix C for more details.

Proof. Let C be a Teichmüller curve generated by a half-translation surface in $\mathcal{Q}(n,-1^n)$ and lift C to $\overline{\mathcal{M}}_{1,n+1}$ by marking all the zeros and poles of q. For any degenerate fiber over $\overline{C}$ the zero z_1 does not lie in a component of genus zero, because $\omega_X^{\otimes 2}$ restricted to this rational tail has degree -2. Hence we conclude that

$$\overline{C} \cdot \sum_{1\in S} \delta_{0;S} = 0,$$

i.e. $C \cdot (\psi_1 - \lambda) = 0$. Using Proposition 4.2 we read off $L^+(C) = \frac{2}{n+2}$.

For the stratum $\mathcal{Q}(n,1,-1^{n+1})$ the same argument works without any change. □

We suspect that all the strata $\mathcal{Q}(a,b,-1^{a+b})$ for $a \geq b > 1$ are varying. Genus one strata are also the first testing ground for finer asymptotic questions on the behavior of Lyapunov exponents as the number of poles grows. This is beyond the scope of this paper. Nevertheless, the above method already provides an upper bound of L^+ for all strata in genus one.

Theorem 8.2. *Let C be a Teichmüller curve in the stratum $\mathcal{Q}(d_1,\dots,d_r)$ in genus one. Suppose d_1 is the largest order of zeros of q. Then $L^+(C) \leq \frac{2}{d_1+2}$.*

Proof. As in the preceding proof, we now have an inequality

$$\overline{C} \cdot \sum_{1\in S} \delta_{0;S} \geq 0,$$

since C is not entirely contained in the boundary of $\overline{\mathcal{M}}_{1,r}$. Using the relation of divisor classes on $\overline{\mathcal{M}}_{1,r}$, we conclude that $C \cdot (\psi_1 - \lambda) \geq 0$. Then by Proposition 4.2 the desired upper bound follows right away. □

9. Genus two

Recall the definition of hyperelliptic half-translation surfaces in Section 2.1. In particular, any Riemann surface of genus two is hyperelliptic, but being a hyperelliptic half-translation surface requires more, i.e. the quadratic differential has to be invariant under the hyperelliptic involution. Throughout we denote by a' the conjugate of a in a hyperelliptic curve under the hyperelliptic involution.

Several strata of (non-hyperelliptic) quadratic differentials with a small number of zeros are indeed empty, e.g. $\mathcal{Q}(4)$, $\mathcal{Q}(3,1)$, $\mathcal{Q}(2,2)^{\text{non-hyp}}$, $\mathcal{Q}(2,1,1)^{\text{non-hyp}}$ and $\mathcal{Q}(1,1,1,1)^{\text{non-hyp}}$. This was shown by [Lan04] and can also be quickly retrieved in our language. For example, for $(X,q) \in \mathcal{Q}(1,1,1,1)$, let $\operatorname{div}(q) = z_1+z_2+z_3+z_4 \sim z_1+z_2+z_1'+z_2'$. Hence $z_3+z_4 \sim z_1'+z_2'$. Up to relabeling, we conclude that z_1 and z_2 are conjugate and so are p_3 and p_4. Consequently (X,q) is a hyperelliptic half-translation surface, hence $\mathcal{Q}(1,1,1,1)^{\text{non-hyp}}$ is empty.

We will verify the following strata in genus two are non-varying. The upshot of the proof is that for Teichmüller curves C in such a non-varying stratum, we exhibit a divisor D disjoint from $\overline{C}$ in the moduli space of curves. Then using $\overline{C}\cdot D = 0$ and the divisor class of D, we can calculate $L^+(C)$ based on Proposition 4.2. As a result, $L^+(C)$ only depends on the signature (and possibly the parity) of the stratum and is independent of C. We thus conclude the desired non-varying property.

Theorem 9.1. *In genus two, besides the components of hyperelliptic flat surfaces and the empty strata mentioned above, the following strata are non-varying:*

Stratum	L^+	Stratum	L^+
$\mathcal{Q}(5,-1)$	$L^+ = 6/7$	$\mathcal{Q}(4,2,-1,-1)$	$L^+ = 5/6$
$\mathcal{Q}(6,-1,-1)^{\text{non-hyp}}$	$L^+ = 3/4$	$\mathcal{Q}(3,3,-1,-1)^{\text{non-hyp}}$	$L^+ = 4/5$
$\mathcal{Q}(4,1,-1)$	$L^+ = 1$	$\mathcal{Q}(3,1,1,-1)$	$L^+ = 16/15$
$\mathcal{Q}(3,2,-1)$	$L^+ = 9/10$	$\mathcal{Q}(2,2,1,-1)$	$L^+ = 1$
$\mathcal{Q}(7,-1,-1,-1)$	$L^+ = 2/3$	$\mathcal{Q}(4,3,-1,-1,-1)$	$L^+ = 11/15$
$\mathcal{Q}(5,1,-1,-1)$	$L^+ = 6/7$	$\mathcal{Q}(3,2,1,-1,-1)$	$L^+ = 9/10$

All the other strata in genus two of dimension less than or equal to seven are varying.

Examples of square-tiled surfaces certifying the varying strata are listed in the appendix.

The proof of Theorem 9.1 uses three types of divisors, grouped in the following sections.

9.1. Irreducible degenerations.

Lemma 9.2. *For a Teichmüller curve C generated by (X,q) in one of the strata $\mathcal{Q}(5,-1)$ or $\mathcal{Q}(4,1,-1)$ all the stable curves parameterized by points in the boundary of C are irreducible. Consequently, Theorem 9.1 follows in these cases from Lemmas 3.3 and* (4)*.*

Proof. This follows immediately from Corollary 4.6. □

9.2. The Weierstrass divisor.

Lemma 9.3. *For a Teichmüller curve C generated by (X,q) in one of the strata $\mathcal{Q}(3,2,-1)$, $\mathcal{Q}(7,-1,-1,-1)$, $\mathcal{Q}(5,1,-1,-1)$ or $\mathcal{Q}(6,-1,-1)^{\text{non-hyp}}$, lift C to $\overline{\mathcal{M}}_{2,1}$*

using the first zero. Then $\overline{C}$ *does not intersect the divisor of Weierstrass points, i.e.* $W \cdot \overline{C} = 0$.

Using the divisor class (12) and Proposition 4.2, we can readily calculate $L^+(C)$ from $W \cdot \overline{C} = 0$, hence this completes the theorem in these cases.

In the proof of the lemma and in several other places we will use the notion of *admissible covers*, which parameterize certain branched covers of nodal curves. Just as stable nodal curves provide a nice compactification for the moduli space of smooth curves, admissible covers provide a nice compactification for the Hurwitz space of branched covers of smooth curves, see [HM98, Chapter 3.G] for an excellent introduction to this topic.

Proof. Suppose to the contrary X is in the intersection. In the first case $\mathrm{div}(q) = 3z_1 + 2z_2 - p_1$ together with $2z_1 \sim \omega_X$ this implies $z_1 + p_1 \sim 2z_2$ for X irreducible, impossible for $p_1 \neq p_3$. This reasoning is also valid if X is reducible, consisting of two elliptic components X_1, X_2, where X_1 contains z_1, p_1, X_2 contains p_2 and they intersect at a node t. Analyzing the admissible double cover, we see that Figure 2 is the only possibility and we obtain $2z_1 \sim 2t$ in X_1. Restricting q to X_1, we also

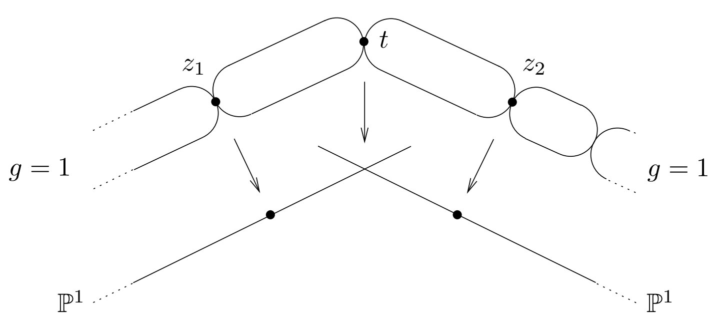

FIGURE 2. Admissible double cover for a reducible degeneration with z_i a Weierstrass point

have $3z_1 - p_1 \sim 2t$. Hence we conclude that $z_1 \sim z_3$, leading to a contradiction.

In the second case, $\mathrm{div}(q) = 7z_1 - p_1 - p_2 - p_3$ and $2z_1 \sim \omega_X$, hence $4z_1 \sim \mathrm{div}(q)$. Consequently $3z_1 \sim p_1 + p_2 + p_3$. By Riemann-Roch, we know $h^0(X, 3z_1) = h^0(X, 2z_1) = 2$. This shows that z_1 is a base point of the linear system $|3z_1|$. But $p_1 + p_2 + p_3$ is a section of this linear system, contradicting that $z_1 \neq p_1, p_2, p_3$.

In the third case, $\mathrm{div}(q) = 5z_1 + z_2 - p_1 - p_2$ and $4z_1 \sim \omega_X^{\otimes 2}$ imply $p_1 + p_2 \sim z_1 + z_2$, hence $z_2 \sim z_1' = z_1$, impossible for $z_1 \neq z_2$.

In the last case, $\mathrm{div}(q) = 6z_1 - p_1 - p_2$ and $4z_1 \sim \omega_X^{\otimes 2}$ imply $p_1 + p_2 \sim 2z_1$, hence (X, q) is a hyperelliptic half-translation surface, contradicting Proposition 4.10. □

9.3. The Brill-Noether divisor $BN^1_{2,(1,1)}$.

Lemma 9.4. *For a Teichmüller curve C generated by (X, q) in one of the strata $\mathcal{Q}(3,1,1,-1)$, $\mathcal{Q}(2,2,1,-1)$, $\mathcal{Q}(4,3,-1,-1,-1)$, $\mathcal{Q}(3,2,1,-1,-1)$, $\mathcal{Q}(4,2,-1,-1)$ or $\mathcal{Q}(3,3,-1,-1)^{\text{non-hyp}}$, lift C to $\overline{\mathcal{M}}_{2,2}$ using the first two zeros. Then $\overline{C}$ does not intersect the Brill-Noether divisor $BN^1_{2,(1,1)}$, i.e. $BN^1_{2,(1,1)} \cdot \overline{C} = 0$.*

Next, one checks that in each of the cases, $\overline{C}$ does not intersect the boundary terms appearing in (13). Thus one can readily calculate $L^+(C)$ based on $BN^1_{2,(1,1)} \cdot \overline{C} = 0$.

Proof. In the first case $\operatorname{div}(q) = 3z_1 + z_2 + z_3 - p_1$. We claim that z_1 and z_2 are not conjugate. Otherwise $z_1 + z_2 \sim \omega_X$ and consequently $z_2 + p_1 \sim z_1 + z_3$. Hence p_1 and z_1 are both conjugate to z_2, impossible for $p_1 \neq z_1$. This reasoning is also valid if X is reducible, consisting of two elliptic components X_1, X_2, where X_1 contains z_1, p_1, X_2 contains z_2, z_3 and they intersect at a node. Analyzing the double cover, we see that it is impossible for z_1 and z_2 to have the same image.

The strata $\mathcal{Q}(2,2,1,-1)$, $\mathcal{Q}(4,2,-1,-1)$ and $\mathcal{Q}(3,2,1,-1,-1)$ can be solved by the same argument.

In the case $\mathcal{Q}(4,3,-1,-1,-1)$ let $\operatorname{div}(q) = 4z_1 + 3z_2 - p_1 - p_2 - p_3$. If z_1, z_2 are conjugate, then $z_1 + z_2 \sim \omega_X$ and $2z_1 + 2z_2 \sim \operatorname{div}(q)$. Consequently we have $2z_1 + z_2 \sim p_1 + p_2 + p_3$. By Riemann-Roch, we know $h^0(X, 2z_1 + z_2) = h^0(X, z_1 + z_2) = 2$, hence z_1 is a base point of the linear system $|2z_1 + z_2|$. But $p_1 + p_2 + p_3$ is also a section of this linear system, contradicting that $z_1 \neq p_1, p_2, p_3$.

Finally in the case $\mathcal{Q}(3,3,-1,-1)^{\text{non-hyp}}$, let $\operatorname{div}(q) = 3z_1 + 3z_2 - p_1 - p_2$. If z_1 and z_2 are conjugate, then $2z_1 + 2z_2 \sim \operatorname{div}(q)$ and $p_1 + p_2 \sim z_1 + z_2$, contradicting that (X, q) is a non-hyperelliptic half-translation surface by Proposition 4.10. $\square$

10. Genus three: non-exceptional strata

Theorem 10.1. *In genus three, besides the components of hyperelliptic flat surfaces and those non-varying exceptional components, the following strata are non-varying:*

Stratum	L^+	Stratum	L^+
$\mathcal{Q}(8)$	$L^+ = 6/5$	$\mathcal{Q}(8,1,-1)$	$L^+ = 6/5$
$\mathcal{Q}(7,1)$	$L^+ = 4/3$	$\mathcal{Q}(7,2,-1)$	$L^+ = 7/6$
$\mathcal{Q}(6,2)^{\text{non-hyp}}$	$L^+ = 5/4$	$\mathcal{Q}(5,4,-1)$	$L^+ = 25/21$
$\mathcal{Q}(5,3)$	$L^+ = 44/35$	$\mathcal{Q}(10,-1,-1)^{\text{non-hyp}}$	$L^+ = 1$
$\mathcal{Q}(4,4)$	$L^+ = 4/3$	$\mathcal{Q}(5,3,1,-1)$	$L^+ = 44/35$
$\mathcal{Q}(6,1,1)^{\text{non-hyp}}$	$L^+ = 17/12$	$\mathcal{Q}(4,3,2,-1)$	$L^+ = 37/30$
$\mathcal{Q}(5,2,1)$	$L^+ = 19/14$	$\mathcal{Q}(3,3,1,1)$	$L^+ = 22/15$
$\mathcal{Q}(4,3,1)$	$L^+ = 7/5$	$\mathcal{Q}(3,2,2,1)$	$L^+ = 7/5$
$\mathcal{Q}(4,2,2)$	$L^+ = 4/3$	$\mathcal{Q}(7,3,-1,-1)$	$L^+ = 16/15$
$\mathcal{Q}(3,3,2)^{\text{non-hyp}}$	$L^+ = 13/10$		

All the other strata in genus three of dimension less than or equal to eight are varying.

Examples of square-tiled surfaces certifying the varying strata are listed in the appendix.

Again, we prove Theorem 10.1 by the disjointness of Teichmüller curves with various divisors on moduli spaces of genus three curves with marked points.

10.1. The Weierstrass divisor.

Lemma 10.2. *For a Teichmüller curve C generated by (X, q) in $\mathcal{Q}(8)$, $\mathcal{Q}(7,1)$, $\mathcal{Q}(8,1,-1)$ or $\mathcal{Q}(10,-1,-1)^{\text{non-hyp}}$, lift C by the first zero to $\overline{\mathcal{M}}_{3,1}$. Then $\overline{C}$ does not intersect the Weierstrass divisor W.*

Proof. Suppose (X,q) lies in the intersection of $\overline{C}$ with W. For the first stratum, $\operatorname{div}(q) = 8z_1$ but by definition q is not a global square of an Abelian differential, namely $4z_1 \not\sim \omega_X$. This implies that $h^0(X, 4z_1) = 2$. If $h^0(X, 3z_1) = 2$, then $h^0(X, \omega_X(-3z_1)) = 1$, hence $\omega_X \sim 3z_1 + r$ for some $r \neq z_1$. This holds even if X is stable, since by Corollary 4.6 it is irreducible and consequently has a canonical morphism to $\mathbb{P}^2$ with z_1 as a flex point. Then $6z_1 + 2r \sim 8z_1$, $2r \sim 2z_1$, hence X is hyperelliptic. Since z_1 is a Weierstrass point, we conclude that $4z_1 \sim \omega_X$, contradicting that q is not a global square.

For the second stratum, we have $\operatorname{div}(q) = 7z_1 + z_2 \sim \omega_X^{\otimes 2}$ for $z_1 \neq z_2$. Note that $3z_1 + z_2 \not\sim \omega_X$, since otherwise $z_1 \sim z_2$, impossible. The hypothesis implies $h^0(X, 4z_1) = 2$. If $h^0(X, 3z_1) = 2$, then $\omega_X \sim 3z_1 + r$ for some $r \neq z_1, z_2$. This holds even if X is degenerate, since by Corollaries 4.5 and 4.6 X is irreducible and consequently has a canonical embedding in $\mathbb{P}^2$ with z_1 as a flex point. Consequently, $6z_1 + 2r \sim 7z_1 + z_2$ implies that X is hyperelliptic and z_1 and z_2 are conjugate. But then $2z_1 + 2z_2 \sim \omega_X \sim 3z_1 + r$ and z_2 has to be a Weierstrass point, contradicting that it is conjugate to z_1.

For the third stratum, we have $\operatorname{div}(q) = 8z_1 + z_2 - p_1 \sim \omega_X^{\otimes 2}$. First consider the case when X is irreducible. Note that z_1 cannot be a Weierstrass point. Otherwise we would have $\omega_X \sim 3z_1 + r$, hence $2r + p_1 \sim 2z_1 + z_2$. Then $\omega_X \sim 2z_1 + z_2 + s$ and $z_1 + r \sim z_2 + s$. If X is hyperelliptic, then $\omega_X \sim 4z_1$ and $z_2 \sim p_1$, impossible. Otherwise we have $r = z_2$ and $s = z_1$. Then $z_2 + p_1 \sim 2z_1$, which still implies that X is hyperelliptic, leading to the same contradiction.

If X is reducible, by Corollaries 4.5 and 4.6, it consists of two irreducible components X_0 and X_2 of genus 0 and 2, respectively, meeting at two nodes t_1 and t_2, where X_0 contains z_2, p_1 and X_2 contains z_1. If $h^0(X, 3z_1) = 2$, by analyzing the associated triple admissible cover, we obtain that $3z_1 \sim 2t_1 + t_2$ in X_2 (up to relabeling t_1, t_2). Since $8z_1 + z_2 - p_1 \sim \omega_X^{\otimes 2}$, we have $8z_1 \sim \omega_{X_2}^{\otimes 2}(2t_1 + 2t_2) \sim 2z_1 + 2z_1' + 3z_1 + t_2$, hence $2t_1 + t_2 \sim 3z_1 \sim t_2 + 2z_1'$ and t_1, z_1' are both Weierstrass points of X_2. Then z_1 is also a Weierstrass point. Since $2z_1 \sim 2t_1$, then we get $z_1 \sim t_2$, impossible for z_1 being contained in the smooth locus of X.

For the last stratum, we have $\operatorname{div}(q) = 10z_1 - p_1 - p_2 \sim \omega_X^{\otimes 2}$. First consider the case when X is irreducible. We can write $\omega_X \sim p_1 + p_2 + r + s$ for some $r, s \in X$. If z_1 is a Weierstrass point, then $\omega_X \sim 3z_1 + t$. Hence $3\omega_X \sim 9z_1 + 3t \sim 10z_1 + r + s$ and consequently $3t \sim z_1 + r + s$. Then $\omega_X \sim z_1 + r + s + u \sim p_1 + p_2 + r + s$ and $z_1 + u \sim p_1 + p_2$. We conclude that X is hyperelliptic, contradicting that this is a non-hyperelliptic stratum and Proposition 4.10.

If X is reducible, by Corollaries 4.5 and 4.6, it consists of two irreducible components X_0 and X_3 of genus 0 and 3, respectively, meeting at a node t such that X_0 contains p_1, p_2 and X_3 contains z_1. If $h^0(X, 3z_1) = 2$, by analyzing the associated triple admissible cover, we see that $h^0(X_3, 3z_1) = 2$, hence $\omega_{X_3} \sim 3z_1 + s$ for some $s \in X_3$. Since $10z_1 \sim \omega_{X_3}^{\otimes 2}(2t) \sim 6z_1 + 2s + 2t$, we conclude that $4z_1 \sim 2s + 2t$. We may write $\omega_{X_3} \sim s + t + u + v$. Then $t + u + v \sim 3z_1$ and $u + v + z_1 \sim 2s + t$. Note that $z_1 \neq t$. If $z_1 = s$, then $2z_1 \sim 2t$ and X_3 is hyperelliptic, contradicting the non-hyperelliptic assumption. If $z_1 \neq s$, we have $\omega_{X_3} \sim u + v + z_1 + w$ and $s + t \sim z_1 + w$. Since $z_1 \neq t, s$, the curve X_3 is hyperelliptic and we deduce the same contradiction. □

For a Teichmüller curve C in the first two strata, since the stable curves parameterized by $\overline{C}$ are irreducible, the theorem follows in these cases from the above lemma together with (5) and Proposition 4.2. For the last two strata of the lemma, by Proposition 4.8 $\overline{C}$ does not intersect any boundary terms in the divisor class (5) of W. Using $\overline{C} \cdot W = 0$ we can thus calculate the value of $L^{+}(C)$.

10.2. **The Brill-Noether divisor** $BN^1_{3,(2,1)}$.

Lemma 10.3. *For a Teichmüller curve C generated by (X, q) in one of the strata $\mathcal{Q}(6,2)^{\text{non-hyp}}$, $\mathcal{Q}(6,1,1)^{\text{non-hyp}}$, $\mathcal{Q}(5,3)$, $\mathcal{Q}(5,2,1)$, $\mathcal{Q}(4,4)$, $\mathcal{Q}(4,3,1)$, $\mathcal{Q}(5,4,-1)$, $\mathcal{Q}(5,3,1,-1)$, $\mathcal{Q}(7,2,-1)$ or $\mathcal{Q}(7,3,-1,-1)$, lift C to $\overline{\mathcal{M}}_{3,2}$ using the first two zeros of q. Then $\overline{C}$ does not intersect the Brill-Noether divisor $BN^1_{3,(2,1)}$.*

Proof. The proofs for $\mathcal{Q}(6,2)^{\text{non-hyp}}$ and $\mathcal{Q}(6,1,1)^{\text{non-hyp}}$ are completely analogous, so we deal with $\mathcal{Q}(6,2)^{\text{non-hyp}}$ only. Suppose that (X, q) is in the intersection with the Brill-Noether divisor and suppose moreover that X is irreducible first. We have $\operatorname{div}(q) = 6z_1 + 2z_2 \sim \omega_X^{\otimes 2}$ and $3z_1 + z_2 \not\sim \omega_X$ for $z_1 \neq z_2$. If $h^0(X, 2z_1 + z_2) = 2$, then $\omega_X \sim 2z_1 + z_2 + r$ and $2z_1 \sim 2r$ for some $r \neq z_1$, hence X is hyperelliptic and z_1, z_2 are Weierstrass points. This contradicts the assumption that C lies in the non-hyperelliptic component and Proposition 4.10.

When X is reducible, by Corollaries 4.5 and 4.6, it consists of a one-nodal union of two components X_1 and X_2 with genus 1 and 2, respectively, where $z_1 \in X_2$ and $z_2 \in X_1$. Denote the node by t. If $h^0(X, 2z_1 + z_2) = 2$, by analyzing the associated triple admissible cover, we conclude that z_1 and t are both Weierstrass points in X_2. Since the restrictions of ω_X to X_1 and X_2 are $\mathcal{O}_{X_1}(t)$ and $\mathcal{O}_{X_2}(2z_1+t)$ respectively, squaring it and comparing to $6p_1 + 2p_2$, we conclude that $2t \sim 2z_2$ on X_1 and $2t \sim 2z_1$ on X_2. We thus obtain that X is contained in the closure of hyperelliptic curves and z_1, z_2 are both ramification points of the double admissible cover. It contradicts again the assumption that C lies in the non-hyperelliptic component.

The proofs for $\mathcal{Q}(5,3)$, $\mathcal{Q}(5,2,1)$ and $\mathcal{Q}(7,2,-1)$ are completely analogous, so we prove for $\mathcal{Q}(5,3)$ only. Suppose that X is in the intersection with the Brill-Noether divisor. We have $\operatorname{div}(q) = 5z_1 + 3z_2 \sim \omega_X^{\otimes 2}$. By Corollaries 4.5 and 4.6, X is irreducible. If $h^0(X, 2z_1 + z_2) = 2$, we would have $\omega_X \sim 2z_1 + z_2 + r$ for some $r \in X$. It follows that $2r \sim z_1 + z_2$, hence X is hyperelliptic and z_1 and z_2 are conjugate. This implies that $\omega_X^{\otimes 2} \sim 4z_1 + 4z_2 \sim 5z_1 + 3z_2$, hence $z_1 \sim z_2$, impossible for $z_1 \neq z_2$.

The proofs for $\mathcal{Q}(4,4)$, $\mathcal{Q}(4,3,1)$ and $\mathcal{Q}(5,4,-1)$ are similar, so we prove for $\mathcal{Q}(4,4)$ only. Suppose a half-translation surface (X, q) parameterized by $\overline{C}$ also lies in $BN^1_{3,(2,1)}$. We have $\operatorname{div}(q) = 4z_1 + 4z_2 \sim \omega_X^{\otimes 2}$. Moreover, $h^0(X, 2z_1 + z_2) = 2$ and $\omega_X \not\sim 2z_1 + 2z_2$. First consider the case X is irreducible. We have $\omega_X \sim 2z_1 + z_2 + r$ for some $r \neq z_2$ and $2z_2 \sim 2r$. Hence X is hyperelliptic and z_2, r are both Weierstrass points. But this is impossible.

If X is reducible, by Corollaries 4.5 and 4.6, it consists of two irreducible components X_1 and X_2 of genus g_1 and g_2, containing z_1 and z_2, respectively, joined at n nodes $t_1, \ldots, t_n$ such that $2g_i - 2 + n = 2$ for $i = 1, 2$. Moreover, (g_1, g_2, n) is either $(1,1,2)$ or $(0,0,4)$. Below we show that both cases are impossible.

For the former case, analyzing the triple cover given by $|2z_1 + z_2|$ as shown in Figure 3, we conclude that $t_1 + t_2 \sim 2z_1$ in X_1. Consider the canonical model of a generic curve X' in C degenerating to X. By the description in Section 5.1, if X' is non-hyperelliptic, the log surface (S, X') degenerates to $(S_{2,2}, X)$, where S is the

Veronese $\mathbb{P}^2$ and $S_{2,2}$ is a cone over the one-nodal union of two conics in $\mathbb{P}^5$. If X' is hyperelliptic, the log surface (S_4, X') degenerates to $(S_{2,2}, X)$, where S_4 is a cone over a rational normal quartic in $\mathbb{P}^5$. The two components S_1, S_2 of $S_{2,2}$ contain X_1, X_2, respectively. The common ruling of S_1, S_2 contains the two nodes t_1, t_2. There exists a hyperplane section Q' of S or S_4 such that $Q \cdot X' = 4z_1' + 4z_2'$. The limit of Q' is a hyperplane section Q of S satisfying $Q \cdot X = 4z_1 + 4z_2$. Suppose that Q_1 and Q_2 are the components of Q lying in S_1 and S_2, respectively. Then $Q_i \cdot X_i = 4z_i$ for $i = 1, 2$. Since $2z_1 \sim t_1 + t_2$, the ruling $L_1 = \overline{vp_1}$ is tangent to X_1 at z_1. If Q_1 is smooth, i.e. if the ruling does not pass through v, then Q_1 has L_1 as its tangent line at z_1. But $Q_1 \cdot L_1 = 1$, leading to a contradiction. Hence we conclude that $Q_1 = 2L_1$ is a double ruling. The other conic Q_2 now also passes through the vertex v, because the hyperplane cutting out Q does. Hence Q_2 consists of two rulings in S_2. Since $Q_2 \cdot X_2 = 4z_2$, the only possibility is that Q_2 is a double ruling $2L_2$, where $L_2 = \overline{vp_2}$ is tangent to X_2 at z_2. We thus conclude that $t_1 + t_2 \sim 2z_2$ in X_2. Therefore, $2z_1 + 2z_2$ is a section of ω_X. Consequently the quadratic differential q is a global square, contradicting Proposition 4.9.

For the latter case, both components of X are $\mathbb{P}^1$. First assume that X is not hyperelliptic. Since X is 4-connected, its canonical embedding consists of two conics X_1, X_2 in $\mathbb{P}^2$, intersecting at four nodes. The condition $h^0(X, 2z_1 + z_2) = 2$ implies that the line $L = \overline{z_1 z_2}$ is tangent to X_1 at z_1. Consider the conic Q satisfying $Q \cdot X = 4z_1 + 4z_2$. If Q is smooth, then the intersection $Q \cdot L$ contains $2z_1 + z_2$, contradicting Bézout's theorem. Then Q must be a double line $2L_1$ with $L_1 \cdot X = 2z_1 + 2z_2$, contradicting that q is not a global square based on Proposition 4.9. If X is hyperelliptic, consider the log surface (S_4, X). Then both X_1 and X_2 have class equal to $4f$, where f is the ruling class. The condition $h^0(X, 2z_1 + z_2) = 2$ implies that there exists a ruling L such that $L \cdot X = 2z_1$ or $z_1 + z_2$. Consider the limiting hyperplane section Q in S_4 such that $Q \cdot X_1 = 4z_1$ and $Q \cdot X_2 = 4z_2$. Since $Q \cdot L = 1$, if Q is smooth, the only possibility is that $L \cdot X = z_1 + z_2$. Since $\mathcal{O}_X(2L) \sim \omega_X$, this implies that q is a global square, contradicting Proposition 4.9. If Q contains L as a component, then it passes through v, hence as a consequence $Q = 4L$, which again contradicts that q is not a global square.

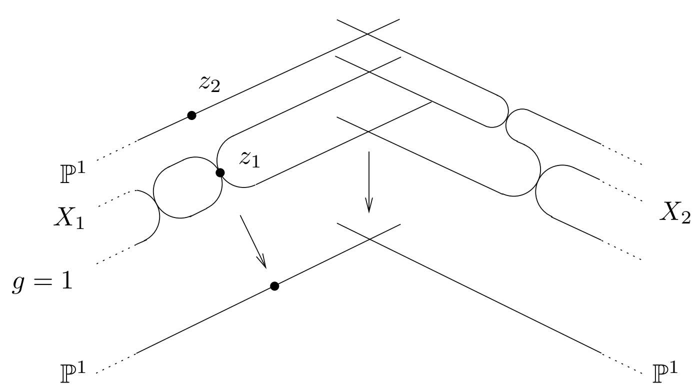

FIGURE 3. Admissible double cover for a reducible degeneration of type $(g_1, g_2, n) = (1, 1, 2)$

For $\mathcal{Q}(5,3,1,-1)$, the proof simply combines that of $\mathcal{Q}(6,2)$ and of $\mathcal{Q}(4,4)$, since the degeneration types of X are covered in those two strata. Finally for $\mathcal{Q}(7,2,-1)$ and $\mathcal{Q}(7,3,-1,-1)$, an argument similar to that of $\mathcal{Q}(6,2)$ works without any further complication. □

Based on the argument above, for $\mathcal{Q}(6,2)^{\text{non-hyp}}$ the only possible intersection of the lift of C to $\overline{\mathcal{M}}_{3,2}$ with a component of the boundary over δ_1 is $\delta_{1,\{2\}}$. This boundary term does not appear in the divisor class (8) of $BN^1_{3,(2,1)}$. More generally, by Proposition 4.8 the Teichmüller curve hits none of the boundary terms in the presentation (8) of $BN^1_{3,(2,1)}$. From the lemma we can thus calculate $L^+(C) = 5/4$. The same reasoning applies to the other strata listed in the lemma and one can quickly deduce the corresponding values of L^+.

10.3. **The Brill-Noether divisor $BN^1_{3,(1,1,1)}$.**

Lemma 10.4. *For a Teichmüller curve C generated by a half-translation surface (X,q) in one of the strata $\mathcal{Q}(4,2,2)$, $\mathcal{Q}(3,3,2)^{\text{non-hyp}}$, $\mathcal{Q}(4,3,2,-1)$, $\mathcal{Q}(3,2,2,1)$ and $\mathcal{Q}(3,3,1,1)^{\text{non-hyp}}$, lift C to $\overline{\mathcal{M}}_{3,3}$ using the first three zeros of q. Then $\overline{C}$ does not intersect the Brill-Noether divisor $BN^1_{3,(1,1,1)}$ in $\overline{\mathcal{M}}_{3,3}$.*

Proof. For $\mathcal{Q}(4,2,2)$, suppose that a half-translation surface X parameterized by $\overline{C}$ also lies in $BN^1_{3,(1,1,1)}$. We have $\operatorname{div}(q) = 4z_1 + 2z_2 + 2z_3 \sim \omega_X^{\otimes 2}$. The linear system $|z_1 + z_2 + z_3|$ yields a g^1_3 for X. First consider the case when X is irreducible. The associated triple cover implies that $\omega_X \sim z_1 + z_2 + z_3 + r$ for some $r \in X$ and we conclude that $2z_1 \sim 2r$. If $r = z_1$, then $\omega_X \sim 2z_1 + z_2 + z_3$. Consequently q is a global square, contradicting Proposition 4.9. If $r \neq z_1$, then X is hyperelliptic and z_1, r are both Weierstrass points. But $\omega_X \sim z_1 + z_2 + z_3 + r$ is still impossible for $z_2, z_3 \neq z_1$.

If X is reducible, by Corollaries 4.5 and 4.6, X consists of two irreducible components X_1 and X_2 of genus g_1 and g_2, containing z_1 and z_2, z_3 respectively, meeting at n nodes $t_1, \ldots, t_n$ such that $2g_i - 2 + n = 2$ for $i = 1, 2$. Moreover, (g_1, g_2, n) is either $(1,1,2)$ or $(0,0,4)$. Below we show that both cases are impossible.

For the former case, analyzing the triple cover, we have $z_2 + z_3 \sim t_1 + t_2$ in X_2. Restricting q to X_1, we also have $2t_1 + 2t_2 \sim 4z_1$ in X_1. Consider the canonical model of a generic curve X' in C degenerating to X. By the description in Section 5.1, the log surface (S, X') degenerates to $(S_{2,2}, X)$. The image of S in $\mathbb{P}^5$ is the Veronese $\mathbb{P}^2$ if X' is non-hyperelliptic, or it is a cone S_4 over a rational normal quartic if X' is hyperelliptic, and $S_{2,2}$ is a cone over the one-nodal union of two conics. Denote by S_1 and S_2 the two components of S and let v be the vertex. Note that X_i embeds into S_i for $i = 1, 2$. The common ruling of S_1, S_2 contains the two nodes t_1, t_2. For X', there exists a hyperplane section Q' of S such that $Q' \cdot X' = 4z'_1 + 2z'_2 + 2z'_3$. The limiting hyperplane section Q of $S_{2,2}$ satisfies $Q \cdot S_{2,2} = 4z_1 + 2z_2 + 2z_3$. Let Q_1 and Q_2 be the components of Q lying in S_1 and S_2, respectively. Then $Q_1 \cdot X_1 = 4z_1$ and $Q_2 \cdot X_2 = 2z_2 + 2z_3$. Since $z_2 + z_3 \sim t_1 + t_2$, the ruling $L_{23} = \overline{z_2 z_3}$ passes through v and consequently Q_2 is the double ruling $2L_{23}$. Then Q_1 also passes through v, hence it consists of two rulings. Since $Q_1 \cdot X_1 = 4z_1$, it must be the double ruling $2L_1$, where L_1 is tangent to X_1 at z_1. Then $L_1 + L_{23}$ cuts out $2z_1 + z_2 + z_3$ in X, hence it is a section of ω_X. Then we conclude that q is a global square, contradicting Proposition 4.9.

If both components of X are $\mathbb{P}^1$, first assume that X is not hyperelliptic. Since X is 4-connected, its canonical embedding consists of two conics in $\mathbb{P}^2$ intersecting at four nodes. The assumption $h^0(X, z_1 + z_2 + z_3) = 2$ implies that z_1, z_2, z_3 are contained in a line L. Consider the conic Q that cuts out $4z_1 + 2z_2 + 2z_3$ in X. If Q is smooth, then the intersection $Q \cdot L$ contains $z_1 + z_2 + z_3$, contradicting Bézout's theorem. Hence Q must be a union of two lines L_1 and L_2 such that $L_1 \cdot X = 2z_1 + 2z_2$, $L_2 \cdot X = 2z_1 + 2z_3$ or $L_1 \cdot X = L_2 \cdot X = 2z_1 + z_2 + z_3$. The former case is impossible, because L_1, L_2 would be the same line tangent to X at z_1. The latter is also impossible, because it would imply that q is a global square, contradicting Proposition 4.9. Now consider the case when X is hyperelliptic. We use the log surface $(S_{0,4}, X)$. Both X_1 and X_2 have class equal to $4f$, where f is the class of a ruling. There exists a ruling L such that $L \cdot X = z_1 + z_2$ (up to relabeling z_2, z_3). Since $\omega_X \sim \mathcal{O}_X(2L)$, we conclude that $4z_1 + 4z_2 \sim 4z_1 + 2z_2 + 2z_3$, hence $2z_2 \sim 2z_3$. Then the ruling L through z_2 is tangent to X at z_2, contradicting that $L \cdot X = z_1 + z_2$.

For $\mathcal{Q}(3,3,2)^{\text{non-hyp}}$, suppose that X is in the intersection with the Brill-Noether divisor. We have $\operatorname{div}(q) = 3z_1 + 3z_2 + 2z_3 \sim \omega_X^{\otimes 2}$. Let us deal with irreducible X first. If $h^0(\mathcal{O}_X(z_1 + z_2 + z_3)) = 2$, then $\omega_X \sim p_1 + p_2 + p_3 + r$ and $p_1 + p_2 \sim 2r$, which contradicts that it is non-hyperelliptic together with Proposition 4.10.

If X is reducible, by Corollaries 4.5 and 4.6, it consists of two irreducible components X_1 and X_2 of genus 1 and 2, containing z_3 and z_1, z_2 respectively, meeting at a node t. If $h^0(X, z_1 + z_2 + z_3) = 2$, by analyzing the triple admissible cover, we have $z_1 + z_2 \sim 2t$ in X_2, hence t is a Weierstrass point and z_1, z_2 are conjugate in X_2. Since $3z_1 + 3z_2 + 2z_3 \sim \omega_X^{\otimes 2}$, we conclude that $3z_1 + 3z_2 \sim 6t$ in X_2 and $2z_3 \sim 2t$. Therefore, X admits a double admissible cover with z_3 as a ramification point and z_1, z_2 being conjugate. It contradicts the non-hyperelliptic assumption.

The proofs for the other strata listed in the lemma are completely analogous to either one of the above two. $\square$

By Proposition 4.8 a Teichmüller curve in one of the above strata hits none of the boundary terms in the presentation (7) of $BN^1_{3,(1,1,1)}$. From the above lemma we can thus calculate the values of L^+ accordingly.

11. Genus four: non-exceptional strata

Theorem 11.1. *In genus four, besides the components of hyperelliptic flat surfaces and those non-varying exceptional components, the following strata are non-varying:*

Stratum	L^+	Stratum	L^+
$\mathcal{Q}(13,-1)$	$4/3$	$\mathcal{Q}(7,5)$	$32/21$
$\mathcal{Q}(11,1)$	$20/13$	$\mathcal{Q}(8,3,1)$	$8/5$
$\mathcal{Q}(10,2)^{\text{non-hyp}}$	$3/2$	$\mathcal{Q}(7,3,2)$	$47/30$
$\mathcal{Q}(8,4)$	$23/15$	$\mathcal{Q}(5,4,3)$	$167/105$

All the other strata in genus four of dimension less than or equal to nine are varying.

Examples of square-tiled surfaces certifying the varying strata are listed in the appendix.

11.1. **The Weierstrass divisor.**

Lemma 11.2. *For a Teichmüller curve C generated by (X, q) in one of the strata $\mathcal{Q}(13, -1)$ or $\mathcal{Q}(11, 1)$, lift C to $\overline{\mathcal{M}}_{4,1}$ using the first zero of q. Then $\overline{C}$ does not intersect the Weierstrass divisor W.*

Proof. Suppose that (X, q) is in the intersection. In the case of $\mathcal{Q}(13, -1)$ we have $\mathrm{div}(q) = 13z_1 - p_1 \sim \omega_X^{\otimes 2}$. If z_1 is a Weierstrass point, then $\omega_X \sim 4z_1 + r + s$ for some r and s, hence $5z_1 \sim p_1 + 2r + 2s$. If z_1 is a base point of $|5z_1|$, then $r = z_1$ (up to relabeling r, s) and $3z_1 \sim p_1 + 2s$. By Riemann-Roch, we get $h^0(X, 2z_1 + s) = 2$, hence $s = z_1$ and $z_1 \sim p_1$, impossible. If z_1 is not a base point of $|5z_1|$, we have $h^0(X, 5z_1) = h^0(X, 4z_1) + 1 = 3$. By Riemann-Roch, we conclude that $h^0(X, r + s - z_1) = 1$, hence $r = z_1$. Then $\omega_X \sim 5z_1 + s$ and $3z_1 \sim p_1 + 2s$. Arguing as above, we deduce the same contradiction.

For $\mathcal{Q}(11, 1)$, we have $\mathrm{div}(q) = 11z_1 + z_2 \sim \omega_X^{\otimes 2}$. If z_1 is a Weierstrass point, then $\omega_X \sim 4z_1 + r + s$ for some r and s, hence $3z_1 \sim z_2 + r + s$. Then $z_1 + r + s$ gives rise to a g_3^1 which is different from the one given by $z_2 + r + s$. If X is not hyperelliptic, using its canonical embedding in a quadric surface in $\mathbb{P}^3$, there do not exist two different rulings passing through r and s or both tangent to X at r (in the case $r = s$), leading to a contradiction. If X is hyperelliptic, since z_1 is a Weierstrass point, we have $12z_1 \sim \omega_X^{\otimes 2}$, hence $z_1 \sim z_2$, impossible. □

Since the stable curves parameterized by $\overline{C}$ in the above strata are irreducible, Theorem 11.1 follows in this case from this lemma together with (5) and Proposition 4.2.

11.2. The Brill-Noether divisors $BN^1_{4,(3,1)}$, $BN^1_{4,(2,2)}$ and $BN^1_{4,(2,1,1)}$.

Lemma 11.3. *For a Teichmüller curve C generated by (X, q) in one of the strata $\mathcal{Q}(10, 2)^{\text{non-hyp}}$, $\mathcal{Q}(8, 4)$ or $\mathcal{Q}(8, 3, 1)$, lift C to $\overline{\mathcal{M}}_{4,2}$ using the first two zeros of q. Then $\overline{C}$ does not intersect the Brill-Noether divisor $BN^1_{4,(3,1)}$.*

Proof. Suppose that (X, q) is in the intersection of $\overline{C}$ with $BN^1_{4,(3,1)}$. In the case of $\mathcal{Q}(10, 2)^{\text{non-hyp}}$ we know that $\mathrm{div}(q) = 10z_1 + 2z_2 \sim \omega_X^{\otimes 2}$. From $h^0(X, 3z_1 + z_2) = 2$ we deduce that $\omega_X \sim 3z_1 + z_2 + r + s$, hence $4z_1 \sim 2r + 2s$ for some r and s. Then $\omega_X \sim 4z_1 + u + v \sim 3z_1 + z_2 + r + s$. If these are the same divisor, we have $u = z_2$ and $r = z_1$, hence $2z_1 \sim 2s$. Note that $s \neq p_1$, for otherwise q would be a global square. Then we conclude that X is hyperelliptic, z_1 is a Weierstrass point and $s = p_2$ is also a Weierstrass point, contradicting that (X, q) is non-hyperelliptic and Proposition 4.10.

If $4z_1 + u + v \sim 3z_1 + z_2 + r + s$ are two different divisors, then $z_2 + r + s$ yields a g_3^1. By Riemann-Roch, $3z_1$ also yields a g_3^1. By $h^0(X, 4z_1) = h^0(X, 3z_1) = 2$, z_1 is a base point of $|4z_1|$. But $2r + 2s$ is a section, hence $r = z_1$. Then we still conclude that X is hyperelliptic and z_1, z_2 are Weierstrass points, leading to the same contradiction.

The proofs for $\mathcal{Q}(8, 4)$ and $\mathcal{Q}(8, 3, 1)$ are completely analogous. □

Lemma 11.4. *For a Teichmüller curve C generated by (X, q) in $\mathcal{Q}(7, 5)$, lift C to $\overline{\mathcal{M}}_{4,2}$ using the two zeros of q. Then the intersection with the Brill-Noether divisor $BN^1_{4,(2,2)} \cdot \overline{C} = 0$.*

Proof. The proof of Lemma 11.3 can be copied almost verbatim, replacing coefficients $(3, 1)$ by $(2, 2)$ at the appropriate places in the above. □

Lemma 11.5. *For a Teichmüller curve C generated by (X, q) in $\mathcal{Q}(7,3,2)$ or $\mathcal{Q}(5,4,3)$, lift C to $\overline{\mathcal{M}}_{4,3}$ using the three zeros of q. Then $\overline{C}$ does not intersect the Brill-Noether divisor $BN^1_{4,(2,1,1)}$.*

Proof. The proof of this lemma is completely parallel to Lemma 11.3. □

By Proposition 4.8 the Teichmüller curves in these strata hit none of the boundary terms in the presentation (10) of $BN^1_{4,(3,1)}$, $BN^1_{4,(2,2)}$ or $BN^1_{4,(2,1,1)}$, respectively. From the lemma the claim on the sum of Lyapunov exponents follows.

Appendix A. Filtration of the Hodge bundle

There are three exceptional components $\mathcal{Q}(6,3,-1)^{\mathrm{irr}}$, $\mathcal{Q}(12)^{\mathrm{irr}}$ and $\mathcal{Q}(9,3)^{\mathrm{irr}}$, non-varying as claimed in Theorems 6.2 and 7.2, but the proof is not given yet. Since the lifts of these components are contained in the Brill-Noether divisors that were used to show non-varying for the regular components, respectively, it seems difficult to come up with some other divisors disjoint with the Teichmüller curves in these strata. Here we take an alternative approach, adapting the idea of [YZ12]. Let us first do some preparatory setup. Some of the lemmas below are also stated in [YZ12], but we include their proofs for completeness.

Let $f : \mathcal{X} \to C$ be the universal curve over a Teichmüller curve C. Let $S_1, \ldots, S_k$ be the disjoint sections specializing to the zeros $z_1, \ldots, z_n$ in each fiber. If $h^0(X, \sum_{i=1}^k a_i z_i) = n$ for every fiber X in $\mathcal{X}$, then $f_*\mathcal{O}_{\mathcal{X}}(\sum_{i=1}^k a_i S_i)$ is a vector bundle of rank n on C. Moreover, any subsheaf of a vector bundle on C is locally free.

Lemma A.1. *If $h^0(X, \sum_{i=1}^k d_i z_i)$ is the same for every fiber X and if*

$$h^0\Big(X, \sum_{i=1}^k d_i z_i\Big) = h^0\Big(X, \sum_{i=1}^k (d_i - a_i) z_i\Big) + \sum_{i=1}^k a_i,$$

then the first chern classes of the corresponding direct images are related by

$$c_1\Big(f_*\mathcal{O}_{\mathcal{X}}\Big(\sum_{i=1}^k d_i S_i\Big)\Big) = c_1\Big(f_*\mathcal{O}_{\mathcal{X}}\Big(\sum_{i=1}^k (d_i - a_i) S_i\Big)\Big) + \sum_{i=1}^k c_1(f_*\mathcal{O}_{a_i S_i}(d_i S_i)).$$

Proof. We have the exact sequence

$$0 \to f_*\mathcal{O}_{\mathcal{X}}\Big(\sum_{i=1}^k (d_i - a_i) S_i\Big) \to f_*\mathcal{O}_{\mathcal{X}}\Big(\sum_{i=1}^k d_i S_i\Big) \to \sum_{i=1}^k f_*\mathcal{O}_{a_i S_i}(d_i S_i)$$
$$\to R^1 f_*\mathcal{O}_{\mathcal{X}}\Big(\sum_{i=1}^k (d_i - a_i) S_i\Big) \to R^1 f_*\mathcal{O}_{\mathcal{X}}\Big(\sum_{i=1}^k d_i S_i\Big) \to 0.$$

All terms are locally free by assumption. The two $R^1 f_*$ terms have the same rank by Riemann-Roch, hence they are isomorphic and then the claim follows. □

Lemma A.2. *Suppose the a_i are positive integers. If $h^0(X, \sum_{i=1}^k a_i z_i) = 1$ for every fiber X, then $f_*\mathcal{O}_{\mathcal{X}}(\sum_{i=1}^k a_i S_i) = \mathcal{O}_C$.*

Proof. Consider the exact sequence

$$0 \to f_*\mathcal{O}_{\mathcal{X}}((a_1 - 1) S_1 + \sum_{i=2}^k a_i S_i) \to f_*\mathcal{O}_{\mathcal{X}}\Big(\sum_{i=1}^k a_i S_i\Big) \to f_*\mathcal{O}_{S_1}\Big(\sum_{i=1}^k a_i S_i\Big)$$

$$\to R^1 f_*\mathcal{O}_{\mathcal{X}}\Big((a_1-1)S_1+\sum_{i=2}^k a_iS_i\Big)\to R^1 f_*\mathcal{O}_{\mathcal{X}}\Big(\sum_{i=1}^k a_iS_i\Big)\to 0.$$

By assumption we have

$$h^0\Big(X,\sum_{i=1}^k a_iS_i\Big)=h^0\Big(X,(a_1-1)S_1+\sum_{i=2}^k a_iS_i\Big)=1,$$

hence the first two f_* terms are line bundles and the last two R^1f_* terms are vector bundles whose ranks differ by one. The middle term is also a line bundle isomorphic to $\mathcal{O}_{S_1}(a_1S_1)$, which cannot have a torsion subsheaf. Hence the first two line bundles are isomorphic. Now the claim follows by induction on a. $\square$

Lemma A.3. *Let S be a section in the above setting and $a\geq 1$. Define aS to be the subscheme of $\mathcal{X}$ whose ideal sheaf is $\mathcal{O}(-aS)$. Then for any integer b we have*

$$c_1(f_*\mathcal{O}_{aS}(bS))=\sum_{i=0}^{a-1}c_1(f_*\mathcal{O}_S((b-i)S)).$$

Proof. The claim holds trivially for $a=1$. Suppose it is true for all positive integers smaller than or equal to a. Treating aS as a subscheme of $(a+1)S$, we have

$$0\to\mathcal{I}_{aS/(a+1)S}\to\mathcal{O}_{(a+1)S}\to\mathcal{O}_{aS}\to 0.$$

The ideal sheaf $\mathcal{I}_{aS/(a+1)S}$ is isomorphic to $(N^*_{S/\mathcal{X}})^{\otimes a}$, where $N^*_{S/\mathcal{X}}$ is the conormal bundle isomorphic to $\mathcal{O}_S(-aS)$. Tensor the sequence with $\mathcal{O}_{\mathcal{X}}(bS)$ and apply f_*. We obtain that

$$0\to f_*\mathcal{O}_S((b-a)S)\to f_*\mathcal{O}_{(a+1)S}(bS)\to f_*\mathcal{O}_{aS}(bS)\to 0,$$

since R^1f_* is zero acting on any line bundle over S. Then the claim follows by induction. $\square$

Let ω be the relative dualizing sheaf, $\gamma=c_1(\omega)$ and η the nodal locus in $\mathcal{X}$. In order to deal with quadratic differentials, we need to express $c_1(f_*(\omega^{\otimes 2}))$ by tautological classes on the moduli space of curves. An introduction to the calculations of the following type can be found in [HM98, Chapter 3.E].

Since all the higher direct images of $\omega^{\otimes 2}$ are zero, by Grothendieck-Riemann-Roch we have

$$\begin{aligned}
\mathrm{ch}(f_*(\omega^{\otimes 2})) &= f_*\Big(\mathrm{ch}(\omega)^2\cdot\Big(1-\frac{\gamma}{2}+\frac{\gamma^2+\eta}{12}\Big)\Big)\\
&= f_*\Big(\Big(1+\gamma+\frac{\gamma^2}{2}\Big)^2\cdot\Big(1-\frac{\gamma}{2}+\frac{\gamma^2+\eta}{12}\Big)\Big)\\
&= f_*\Big(1+\frac{3}{2}\gamma+\frac{13\gamma^2+\eta}{12}\Big)\\
&= (3g-3)+\Big(\frac{13}{12}\kappa+\frac{1}{12}\delta\Big)\\
&= (3g-3)+(\lambda+\kappa),
\end{aligned}$$

where $\kappa=f_*(\gamma^2)$. Hence we conclude that

$$c_1(f_*(\omega^{\otimes 2}))=\lambda+\kappa.$$

This formula was previously obtained by Trapani [Tra92, Remark 5.7] using analytic techniques.

Recall the notation and the intersection calculation in Proposition 4.2. We have

$$\omega^{\otimes 2} = f^*\mathcal{F} \otimes \mathcal{O}_{\mathcal{X}}\Big(\sum_{j=1}^{n} d_j S_j\Big).$$

By the projection formula, we get

$$f_*(\omega^{\otimes 2}) = \mathcal{F} \otimes f_*\mathcal{O}_{\mathcal{X}}\Big(\sum_{j=1}^{n} d_j S_j\Big).$$

Note that $f_*\mathcal{O}_{\mathcal{X}}(\sum_{j=1}^{n} d_j S_j)$ is a vector bundle of rank $3g-3$, whose fibers are $H^0(X, \omega_X^{\otimes 2})$ for X parameterized in C. Then we conclude that

$$\begin{aligned} \frac{c_1(f_*\mathcal{O}_{\mathcal{X}}(\sum_{j=1}^{n} d_j S_j))}{\chi} &= \frac{C\cdot\lambda}{\chi} + \frac{C\cdot\kappa}{\chi} - (3g-3) \\ &= \frac{1}{2}L^+(C) + 6\kappa_\mu - (3g-3), \end{aligned}$$

where $\chi = \deg(\mathcal{F})$ and $\kappa_\mu = \frac{1}{24}\big(\sum_{j=1}^{n} \frac{d_j(d_j+4)}{d_j+2}\big)$ for the signature $\mu = (d_1, \ldots, d_n)$. Therefore, we obtain the following expression:

$$L^+(C) = (6g-6) + 2\cdot \frac{c_1(f_*\mathcal{O}_{\mathcal{X}}(\sum_{j=1}^{n} d_j S_j))}{\chi} - 12\kappa_\mu. \tag{21}$$

A.1. **The component $\mathcal{Q}(6,3,-1)^{\mathrm{irr}}$.** Let (X,q) be a half-translation surface parameterized by a Teichmüller curve C in $\mathcal{Q}(6,3,-1)^{\mathrm{irr}}$. We have $6z_1+3z_2-p \sim \omega_X^{\otimes 2}$. Moreover, by the parity we know $h^0(X, 2z_1+z_2) = 2$.

Lemma A.4. *We have $h^0(X, 3z_1+z_2) = 2$ and $h^0(X, z_1+z_2) = 1$ for all X in C.*

Proof. Consider first the case when X is irreducible. If $h^0(X, 3z_1+z_2) \geq 3$, then $\omega_X \sim 3z_1+z_2$ and $6z_1+2z_2 \sim 6z_1+3z_2-p$. It implies that $z_2 \sim p$, impossible. If $h^0(X, z_1+z_2) \geq 2$, then X is hyperelliptic and z_1, z_2 are conjugate. Then we have $\omega_X \sim 2z_1+2z_2$ and $2z_1 \sim z_2+p$, contradicting that z_1, z_2 are conjugate.

Next, suppose X is reducible. By Corollaries 4.5 and 4.6, the only degenerate type is that X consists of X_1 and X_2 of genus 1 and 2, respectively, joined at a node t such that X_1 contains z_2, p and X_2 contains z_1. By $h^0(X_2, 3z_1) = 2$ and $h^0(X_1, z_2) = 1$, it implies that $h^0(X, 3z_1+z_2) = 2$, since a section on X_1 and a section on X_2 need to have the same value if they can be glued to form a global section on X. Similarly by $h^0(X_2, z_1) = 1$ and $h^0(X_1, z_2) = 1$, we conclude that $h^0(X, z_1+z_2) = 1$. $\square$

Now we are ready to calculate L^+. Let S_1, S_2 and S_3 be the sections in $\mathcal{X}$ corresponding to the loci of z_1, z_2 and p, respectively. By the exact sequence

$$0 \to f_*\mathcal{O}_{\mathcal{X}}(6S_1+3S_2-S_3) \to f_*\mathcal{O}_{\mathcal{X}}(6S_1+3S_2) \to f_*\mathcal{O}_{S_3} \to 0,$$

we have

$$c_1(f_*\mathcal{O}_{\mathcal{X}}(6S_1+3S_2-S_3)) = c_1(f_*\mathcal{O}_{\mathcal{X}}(6S_1+3S_2)).$$

Since $R^1f_*\mathcal{O}_X(3S_1+S_2) = 0$ by the above lemma, we obtain that

$$\begin{aligned} c_1(f_*\mathcal{O}_{\mathcal{X}}(6S_1+3S_2)) &= c_1(f_*\mathcal{O}_{\mathcal{X}}(3S_1+S_2)) + c_1(f_*\mathcal{O}_{3S_1}(6S_1)) + c_1(f_*\mathcal{O}_{2S_2}(3S_2)) \\ &= c_1(f_*\mathcal{O}_{\mathcal{X}}(3S_1+S_2)) + 15S_1^2 + 5S_2^2, \end{aligned}$$

where Lemma A.3 is used in the last equality. Similarly using Lemmas A.1 and A.2 one can show that

$$\begin{aligned} c_1(f_*\mathcal{O}_{\mathcal{X}}(3S_1+S_2)) &= c_1(f_*\mathcal{O}_{\mathcal{X}}(2S_1+S_2)) \\ &= c_1(f_*\mathcal{O}_{\mathcal{X}}(S_1+S_2)) + 2S_1^2 \\ &= 2S_1^2. \end{aligned}$$

Then we obtain that

$$c_1(f_*\mathcal{O}_{\mathcal{X}}(6S_1+3S_2-S_3)) = 17S_1^2 + 5S_2^2 = -\frac{25}{8}\chi,$$

where we use the self-intersection formula of the S_j in Proposition 4.2. Finally by the equality (21), we conclude that $L^+(C) = 7/5$.

A.2. **The component $\mathcal{Q}(12)^{\mathrm{irr}}$.** Let (X,q) be a half-translation surface parameterized by a Teichmüller curve C in $\mathcal{Q}(12)^{\mathrm{irr}}$. We have $12z \sim \omega_X^{\otimes 2}$. By the parity we know $h^0(X,4z) = 2$.

Lemma A.5. *We have $h^0(X,6z) = 3$, $h^0(X,5z) = 2$ and $h^0(X,3z) = 1$ for all X in C.*

Proof. Since q has a unique zero, X is irreducible. If $h^0(X,6z) \geq 4$, we have $\omega_X \sim 6z$ and q is a global square, impossible. If $h^0(X,5z) \geq 3$, then $\omega_X \sim 5z+r$ for some $r \neq z$. We have $10z+2r \sim 12z$, hence $2r \sim 2z$, X is hyperelliptic and z is a Weierstrass point, still contradicting that q is not a global square. If $h^0(X,3z) \geq 2$, X cannot be Gieseker-Petri special, for otherwise q would be a global square. Then $|3z|$ gives rise to a g^1_3 and suppose $|z+r+s|$ is the other g^1_3. We have $\omega_X \sim 4z+r+s$ and $2r+2s \sim 4z$. But z is a base point of $|4z|$, hence $r = z$ and $\omega_X \sim 5z+s$, contradicting that $h^0(X,5z) = 2$ as shown before. □

Let S be the section of zeros of q in $\mathcal{X}$. By the above lemma, $R^1 f_*\mathcal{O}_X(5S) = 0$, hence we conclude that

$$\begin{aligned} c_1(f_*\mathcal{O}_{\mathcal{X}}(12S)) &= c_1(f_*\mathcal{O}_{\mathcal{X}}(5S)) + c_1(f_*\mathcal{O}_{7S}(12S)) \\ &= c_1(f_*\mathcal{O}_{\mathcal{X}}(5S)) + 63S^2. \end{aligned}$$

Similarly we can show that

$$\begin{aligned} c_1(f_*\mathcal{O}_{\mathcal{X}}(5S)) &= c_1(f_*\mathcal{O}_{\mathcal{X}}(4S)) \\ &= c_1(f_*\mathcal{O}_{\mathcal{X}}(3S)) + 4S^2 \\ &= 4S^2. \end{aligned}$$

Then we obtain that

$$c_1(f_*\mathcal{O}_{\mathcal{X}}(12S)) = 67S^2 = -\frac{67}{14}\chi.$$

By the identity (21), we conclude that $L^+(C) = 11/7$.

A.3. **The component $\mathcal{Q}(9,3)^{\mathrm{irr}}$.** Let (X,q) be a half-translation surface parameterized by a Teichmüller curve C in $\mathcal{Q}(9,3)^{\mathrm{irr}}$. We have $9z_1+3z_2 \sim \omega_X^{\otimes 2}$. By the parity we know $h^0(X,3z_1+z_2) = 2$.

Lemma A.6. *We have $h^0(X,4z_1+z_2) = 2$ and $h^0(X,2z_1+z_2) = 1$ for all X parameterized in C.*

Proof. By Corollaries 4.5 and 4.6, X is irreducible. If $h^0(X, 4z_1 + z_2) \geq 3$, then $\omega_X \sim 4z_1 + z_2 + r$, hence $2r \sim z_1 + z_2$, X is hyperelliptic and z_1, z_2 are conjugate. Then $\omega_X \sim 3z_1 + 3z_2$, hence $3z_1 \sim 3z_2$, which implies that z_1, z_2 are both Weierstrass points, contradicting that they are conjugate. If $h^0(X, 2z_1 + z_2) \geq 2$, X cannot be Gieseker-Petri special, for otherwise $\omega_X \sim 4z_1 + 2z_2$ and $z_1 \sim z_2$, impossible. Consequently $|2z_1 + z_2|$ provides a g^1_3 and suppose that $|z_1 + r + s|$ is the other g^1_3. We have $\omega_X \sim 3z_1 + z_2 + r + s$, hence $2r + 2s \sim 3z_1 + z_2$. Note that z_1 is a base point of $|3z_1 + z_2|$, hence $r = z_1$ and we thus obtain $\omega_X \sim 4z_1 + z_2 + s$, contradicting that $h^0(X, 4z_1 + z_2) = 2$ as shown before. $\square$

By the above lemma $R^1 f_* \mathcal{O}_X(4S_1 + S_2) = 0$ and consequently

$$\begin{aligned} c_1(f_*\mathcal{O}_{\mathcal{X}}(9S_1 + 3S_2)) &= c_1(f_*\mathcal{O}_{\mathcal{X}}(4S_1 + S_2)) + c_1(f_*\mathcal{O}_{5S_1}(9S_1)) + c_1(f_*\mathcal{O}_{2S_2}(3S_2)) \\ &= c_1(f_*\mathcal{O}_{\mathcal{X}}(4S_1 + S_2)) + 35S_1^2 + 5S_2^2. \end{aligned}$$

Similarly one can show that

$$\begin{aligned} c_1(f_*\mathcal{O}_{\mathcal{X}}(4S_1 + S_2)) &= c_1(f_*\mathcal{O}_{\mathcal{X}}(3S_1 + S_2)) \\ &= c_1(f_*\mathcal{O}_{\mathcal{X}}(2S_1 + S_2)) + 3S_1^2 \\ &= 3S_1^2. \end{aligned}$$

Then we obtain that

$$c_1(f_*\mathcal{O}_{\mathcal{X}}(9S_1 + 3S_2)) = 38S_1^2 + 5S_2^2 = -\frac{49}{11}\chi.$$

Finally by (21), we conclude that $L^+(C) = 92/55$.

Remark A.7. Once we understand the boundary behavior as well as $h^0(X, \sum a_i z_i)$ for all X in a Teichmüller curve C, apparently the above method may provide a parallel proof for many other non-varying strata of quadratic differentials. For instance, consider a Teichmüller curve C generated by (X, q) in $\mathcal{Q}(8)$ with $\mathrm{div}(q) = 8z$. By the proof of Lemma 10.2, we know $h^0(X, 3z) = 1$ and $h^1(X, 3z) = 0$ for all X in $\overline{C}$. Then we obtain that

$$\begin{aligned} c_1(f_*\mathcal{O}_{\mathcal{X}}(8S)) &= c_1(f_*\mathcal{O}_{\mathcal{X}}(3S)) + c_1(f_*\mathcal{O}_{5S}(8S)) \\ &= 30S^2 = -3\chi. \end{aligned}$$

By (21) we conclude that $L^+(C) = 6/5$ as in Theorem 10.1.

A.4. **The missing non-varying strata of Abelian differentials.** In our earlier work [CM12] three strata of Abelian differentials $\Omega\mathcal{M}_4(4,2)^{\mathrm{odd}}$, $\Omega\mathcal{M}_4(4,2)^{\mathrm{even}}$ and $\Omega\mathcal{M}_5(6,2)^{\mathrm{odd}}$ were predicted to be non-varying based on the computer data of Zorich and Delecroix, but no proof was given there. In [YZ12] an argument for their non-varying property was found using the filtration of the Hodge bundle. For completeness we include a detailed proof in this section.

Let (X, ω) be a flat surface generating a Teichmüller curve C in a stratum of Abelian differentials. Use z_i to denote the zeros of ω.

Lemma A.8. *For all X in C, we have the following results.*
If C is in $\Omega\mathcal{M}_4(4,2)^{\mathrm{odd}}$, then $h^0(X, 2z_1 + z_2) = 1$.
If C is in $\Omega\mathcal{M}_4(4,2)^{\mathrm{even}}$, then $h^0(X, 2z_1 + z_2) = 2$ and $h^0(X, z_1 + z_2) = 1$.
If C is in $\Omega\mathcal{M}_5(6,2)^{\mathrm{odd}}$, then $h^0(X, 3z_1 + z_2) = 1$.

Proof. The reader may refer to [CM12, Section 4] for properties of Teichmüller curves generated by Abelian differentials. All the claims in the lemma follow directly from the spin parity except that $h^0(X, z_1+z_2) = 1$ for the stratum $\Omega\mathcal{M}_4(4,2)^{\text{even}}$. Suppose on the contrary $h^0(X, z_1+z_2) \geq 2$. Then X is in the hyperelliptic locus and z_1, z_2 are conjugate. It implies that $\omega_X \sim 3z_1+3z_2 \sim 4z_1+2z_2$, hence $z_1 \sim z_2$, contradicting that $z_1 \neq z_2$. It holds even if X is reducible. In that case each component X_i of X has to contain a zero z_i. By $h^0(X_i, z_i) = 1$ for $i = 1, 2$, we conclude that $h^0(X, z_1+z_2) = 1$, because gluing sections on the X_i to form a global section on X imposes an additional condition. □

Let S_i be the section in $\mathcal{X}$ corresponding to the zero z_i of order m_i. Let ω be the relative dualizing sheaf of $f: \mathcal{X} \to C$. We have

$$\omega = f^*\mathcal{L} \otimes \mathcal{O}_{\mathcal{X}}\Big(\sum_{i=1}^k m_i S_i\Big),$$

where $\mathcal{L}$ is the line bundle on C corresponding to the generating Abelian differential and $\deg(\mathcal{L}) = \chi/2$. The projection formula implies that

$$f_*\omega = \mathcal{L} \otimes f_*\Big(\mathcal{O}_{\mathcal{X}}\Big(\sum_{i=1}^k m_i S_i\Big)\Big).$$

Since $f_*(\mathcal{O}_{\mathcal{X}}(\sum_{i=1}^k m_i S_i))$ is a vector bundle of rank g whose fibers are $H^0(X, \omega_X)$, we have

$$\begin{aligned} C \cdot \lambda &= c_1(f_*\omega) \\ &= g \cdot \frac{\chi}{2} + c_1\Big(f_*\Big(\mathcal{O}_{\mathcal{X}}\Big(\sum_{i=1}^k m_i S_i\Big)\Big)\Big). \end{aligned}$$

By [CM12, Proposition 4.5] we conclude that

$$\begin{aligned} L(C) &= 2 \cdot \frac{C \cdot \lambda}{\chi} \\ &= g + 2 \cdot \frac{c_1(f_*(\mathcal{O}_{\mathcal{X}}(\sum_{i=1}^k m_i S_i)))}{\chi}. \end{aligned} \tag{22}$$

Theorem A.9. *Let C be a Teichmüller curve generated by a flat surface in one of the strata $\Omega\mathcal{M}_4(4,2)^{\text{odd}}$, $\Omega\mathcal{M}_4(4,2)^{\text{even}}$ or $\Omega\mathcal{M}_5(6,2)^{\text{odd}}$. Then the sum of Lyapunov exponents $L(C)$ equals $29/15$, $32/15$ or $46/21$, respectively.*

Proof. For the stratum $\Omega\mathcal{M}_4(4,2)^{\text{odd}}$, using the preceding Lemma to verify the hypothesis of the Lemmas A.1, A.2 and A.3, we conclude

$$\begin{aligned} c_1(f_*\mathcal{O}_{\mathcal{X}}(4S_1+2S_2)) &= c_1(f_*\mathcal{O}_{\mathcal{X}}(2S_1+S_2)) + c_1(f_*\mathcal{O}_{2S_1}(4S_1)) + c_1(f_*\mathcal{O}_{S_2}(2S_2)) \\ &= 7S_1^2 + 2S_2^2 \\ &= -\frac{31}{30} \cdot \chi, \end{aligned}$$

where the self-intersection formula of the S_i in [CM12, Proposition 4.5] is used in the last step. By the relation (22), we finally obtain that

$$L(C) = 4 - 2 \cdot \frac{31}{30} = \frac{29}{15}.$$

For the stratum $\Omega\mathcal{M}_4(4,2)^{\mathrm{even}}$, by the above lemmas we have

$$\begin{aligned}
c_1(f_*\mathcal{O}_{\mathcal{X}}(4S_1+2S_2)) &= c_1(f_*\mathcal{O}_{\mathcal{X}}(3S_1+S_2)) + c_1(f_*\mathcal{O}_{S_1}(4S_1)) + c_1(f_*\mathcal{O}_{S_2}(2S_2)) \\
&= c_1(f_*\mathcal{O}_{\mathcal{X}}(3S_1+S_2)) + 4S_1^2 + 2S_2^2 \\
&= c_1(f_*\mathcal{O}_{\mathcal{X}}(2S_1+S_2)) + 4S_1^2 + 2S_2^2 \\
&= c_1(f_*\mathcal{O}_{S_1}(2S_1)) + 4S_1^2 + 2S_2^2 \\
&= 6S_1^2 + 2S_2^2 \\
&= -\frac{14}{15}\cdot\chi.
\end{aligned}$$

Then by the equality (22), we conclude that

$$L(C) = 4 - 2\cdot\frac{14}{15} = \frac{32}{15}.$$

For the stratum $\Omega\mathcal{M}_6(6,2)^{\mathrm{odd}}$, we have

$$\begin{aligned}
c_1(f_*\mathcal{O}_{\mathcal{X}}(6S_1+2S_2)) &= c_1(f_*\mathcal{O}_{\mathcal{X}}(3S_1+S_2)) + c_1(f_*\mathcal{O}_{3S_1}(6S_1)) + c_1(f_*\mathcal{O}_{S_2}(2S_2)) \\
&= c_1(f_*\mathcal{O}_{\mathcal{X}}(3S_1+S_2)) + 15S_1^2 + 2S_2^2 \\
&= 15S_1^2 + 2S_2^2 \\
&= -\frac{59}{42}\cdot\chi.
\end{aligned}$$

Finally by (22), we conclude that

$$L(C) = 5 - 2\cdot\frac{59}{42} = \frac{46}{21}.$$

□

Appendix B. Local calculations for the exceptional strata

B.1. **Genus three.** We begin with the proof of Proposition 6.3, which is decomposed into the following lemmas. We follow the notation of Section 6.

Lemma B.1. *If X is a non-hyperelliptic, smooth genus three curve, then for a generic choice of a line passing through p the parity plane cubic E is irreducible.*

Proof. Suppose that E consists of three lines. Then one of them passes through r_1,r_2 and r_3 but none of the p_i. Hence the other two lines intersect X at 9 points (counting with multiplicity), leading to a contradiction.

Next, suppose that E consists of a line L and an irreducible conic Q. By the same argument, L has to contain precisely one of the r_i, say r_1. We now discuss case by case.

In the case $\mathcal{Q}(9,-1)$ it follows that $L\cdot X = 3z_1 + r_1$ and $Q\cdot X = 6z_1 + r_2 + r_3$. Since $Q\sim 2L$ as divisor classes, we have $2r_1 \sim r_2 + r_3$ on X, which implies that X is hyperelliptic, contradicting the hypothesis.

In the case $\mathcal{Q}(6,3,-1)$ we conclude that $L\cdot X$ is one of the divisors

$$r_1 + 3z_1,\quad r_1 + 2z_1 + z_2,\quad r_1 + z_1 + 2z_2 \quad\text{or}\quad r_1 + 3z_2.$$

In the first case, $Q\cdot X = r_2 + r_3 + 3z_1 + 3z_2$. This implies $3z_2 + r_2 + r_3 - r_1 \sim r_1 + r_2 + r_3 + p$ as sections of $\mathcal{O}_X(1)$. Then $3z_2 \sim 2r_1 + p$. Since $3z_2 - p$ is determined by q, there are at most finitely many choices for such r_1. One can choose L away from these r_1. In the second case, $Q\cdot X = r_2 + r_3 + 4z_1 + 2z_2$. Using $Q\sim 2L$, we conclude that $2r_1 \sim r_2+r_3$, contradicting the non-hyperelliptic assumption. In the third case, we have $r_1+z_1+2z_2 \sim r_1+r_2+r_3+p$ as sections of $\mathcal{O}_X(1)$. Hence $r_2+r_3 \sim z_1+2z_2-p$. Since z_1+2z_2-p is determined by q, for such r_2 and r_3 there are finitely many choices, since otherwise X would be hyperelliptic. In the last case, we have $r_1+3z_2 \sim r_1+r_2+r_3+p$, hence $r_2+r_3 \sim 3z_2-p$ which restricts r_2 and r_3 to finitely many choices.

In the case $\mathcal{Q}(3,3,3,-1)$ the intersection $L \cdot X$ could be

$$r_1+3z_1, \quad r_1+2z_1+z_2 \quad \text{or} \quad r_1+z_1+z_2+z_3.$$

In the first case, $3z_1 \sim r_2+r_3+p$, hence the choices of r_2 and r_3 are limited to a finite number, since X is not hyperelliptic. In the second case, $2z_1+z_2 \sim r_2+r_3+p$ and the same argument applies. Finally, in the last case since $Q\cdot X = r_2+r_3+2z_1+2z_2+2z_3$, we have $2r_1 \sim r_2+r_3$ on X, impossible for X being non-hyperelliptic. □

Next two preparatory lemmas imply that the z_i are contained in the smooth locus of E.

Lemma B.2. *Let E be an irreducible plane cuspidal cubic with z_1 as its cusp. If a plane quartic X has intersection multiplicity $(X\cdot E)_{z_1} \geq 4$, then X is singular at z_1.*

Proof. Without loss of generality, let $y^2-x^3=0$ be the defining equation of E (in affine coordinates) and $z_1=(0,0)$. Suppose that the quartic X is defined by

$$f(x,y) = \sum_{i+j\leq 4} a_{ij}x^iy^j.$$

Then we have

$$\dim_{\mathbb{C}} \mathbb{C}[[x,y]]/(y^2-x^3, f(x,y)) \geq 4.$$

We now use the rational parameterization of E by setting $y=t^3$ and $x=t^2$. Then

$$f(t) = \sum_{i+j\leq 4} a_{ij}t^{2i+3j}$$

and

$$\dim_{\mathbb{C}} \mathbb{C}[[t^2,t^3]]/(f(t)) \geq 4.$$

Since X contains z_1, we know that $a_{00}=0$. If $a_{10}\neq 0$, then $f(t)$ is proportional to $t^2(1+b_1t+b_2t^2+\cdots)$. The vector space $\mathbb{C}[[t^2,t^3]]/(f(t))$ can be generated by $1,t^2,t^3$, since $1+b_2t^2+\cdots$ is invertible in $\mathbb{C}[[t^2,t^3]]$. Consequently, $\dim_{\mathbb{C}} \mathbb{C}[[t^2,t^3]]/(f(t)) \leq 3$, contradicting the assumption. Next, if $a_{01}\neq 0$, then $f(t)$ is proportional to $t^3(1+c_1t+c_2t^2+\cdots)$. By the same token, $\mathbb{C}[[t^2,t^3]]/(f(t))$ can be generated by $1,t^2,t^4$, hence $\dim_{\mathbb{C}} \mathbb{C}[[t^2,t^3]]/(f(t)) \leq 3$, leading to a contradiction. We thus conclude that $f\in (x,y)^2$, hence X is singular at $z_1=(0,0)$. □

Lemma B.3. *Let E be an irreducible plane rational nodal cubic with z_1 as its node. Suppose that a plane quartic X has intersection multiplicity $(X\cdot E)_{z_1}\geq 4$ and that X is smooth at z_1. Then z_1 is not a flex of X.*

Proof. Without loss of generality, let $y^2-x^2-x^3=0$ be the equation of E and $z_1=(0,0)$. Then the two branches at the node have tangent lines $L^-: x-y=0$ and $L^+: x+y=0$, respectively. Since X is smooth at z_1, it intersects one branch, say the one tangent to L^+, transversality, and intersects the other with multiplicity ≥ 3. We use a local rational parameterization of E by setting $x=s(s+2)$ and

$y = s(s+1)(s+2)$. Suppose $f(x,y) = \sum_{i+j\leq 4} a_{ij}x^i y^j$ is the defining equation of X. Then we have

$$f(s) = \sum_{i+j\leq 4} a_{ij} s^{i+j}(s+2)^{i+j}(s+1)^j,$$

and $\dim_{\mathbb{C}} \mathbb{C}[[s]]/(f(s)) \geq 3$. Writing out the coefficients we see that

$$a_{00} = 0, \quad a_{10} + a_{01} = 0, \quad a_{01} + 2(a_{20} + a_{11} + a_{02}) = 0.$$

Now suppose that z_1 is a flex of X. Then L^- is the corresponding flex line. The condition $\dim_{\mathbb{C}} \mathbb{C}[[x,y]]/(f(x,y), x-y) \geq 3$ implies that

$$a_{00} = 0, \quad a_{10} + a_{01} = 0, \quad a_{20} + a_{11} + a_{02} = 0.$$

Combining the above equations, we conclude that

$$a_{00} = a_{10} = a_{01} = 0,$$

hence $f \in (x,y)^2$ and X is singular at z_1, contradicting the assumption. □

Proof of Proposition 6.3. It remains to show that the z_i are located at non-singular points of E. In the case of $\mathcal{Q}(9,-1)$ the possibility that z_1 is a singular point of E_1 can be ruled out by Lemmas B.2. In the case of $\mathcal{Q}(6,3,-1)$ the fact that the 6-fold zero z_1 is a smooth point can also be verified by Lemmas B.2 and B.3. Smoothness at z_2 is clear because the line $\overline{z_1 z_2}$ intersects E at z_2 with multiplicity one. The case $\mathcal{Q}(3,3,3,-1)$ is dealt with by the same argument, using the line through z_1, z_2 and z_3 instead. □

We conclude this section by showing that an exceptional component in genus three cannot be contained in the hyperelliptic locus.

Proof of Lemma 6.7. Suppose on the contrary that such a component exists. In Section 6.2 we added a line L in $\mathbb{P}^2$ to the given bicanonical divisor in order to work with the effective divisor $\mathcal{L}(X,q) + L \cdot X$. Such a line corresponds to a conic in $\mathbb{P}^5$ under the Veronese embedding. Here we use the hyperelliptic assumption and add a line in $\mathbb{P}^5$ to get an effective divisor. More precisely, we first start with $(X,q) \in \mathcal{Q}(k_1,\ldots,k_n,-1)$ and X hyperelliptic. As in Section 5.1, let S be a cone over a rational normal quartic in $\mathbb{P}^5$, which is the image of the ruled surface F_4. On F_4, the class of X is $2e+8f$ and $\omega_X \sim \mathcal{O}_X(2f)$. Take a ruling f_p passing through the unique pole p of $\operatorname{div}(q)$. Let $p' = (f_p \cdot X) - p$ be the conjugate of p in X. Let $D = \operatorname{div}(q) + p$ be a degree 9 divisor. A bicanonical divisor is a section of $\mathcal{O}_X(e+4f)$, so D is a section of $\mathcal{O}_X(e+5f)$. By the exact sequence

$$0 \to \mathcal{O}_{F_4}(-e-3f) \to \mathcal{O}_{F_4}(e+5f) \to \mathcal{O}_X(e+5f) \to 0,$$

we know that $D + p'$ is cut out by a unique rational curve R of class $e+5f$. The pair $(R, D+p')$ takes the role of $(E, \sum_{i=1}^n k_i z_i + r_1 + r_2 + r_3)$. Then we can mimic the dimension count in Proposition 6.5.

Suppose that we have shown that R is irreducible. The dimension of the linear system containing R equals $\dim \mathbb{P}H^0(F_4, e+5f) = 7$. Hence the parameter space for $(R, D+p')$ has dimension $n+8$. From

$$0 \to \mathcal{O}_{F_4}(e+3f) \to \mathcal{O}_{F_4}(2e+8f) \to \mathcal{O}_R(2e+8f) \to 0$$

we deduce that for fixed $(R, D+p')$ the space of X with $X \cdot R = D+p'$ has dimension $h^0(F_4, e+3f) = 4$. From the triple $(X, R, D+p')$ we recover $\operatorname{div}(q) = X \cdot R - X \cdot f_{p'}$. Altogether, since the automorphism group of F_4 has dimension 9, the space of the triples $(X, R, D + p')$ has dimension $n + 3$, smaller than $\dim \mathbb{P}\mathcal{Q}(k_1, \ldots, k_n, -1) = n + 4$.

Finally, we deal with the case when R is reducible. In this case R consists of a fibers along with a component of class $e + (5 - a)f$. Since $e^2 = -4$, we have $a = 5$ or $a = 1$, for otherwise R would not be effective.

In the first case we can disregard e, since $e \cdot X = 0$. Let f be the fiber containing the pole p. It contains also some zero z_i, different from p. This implies that $z_i = p'$, hence $f \cdot X = 2z_i$ and z_i is a Weierstrass point, contradicting that $p \neq z_i$.

Now consider $a = 1$. In the case $\mathcal{Q}(9, -1)$ we have $f \cdot X = 2z_1$ by the same argument, hence $10z_1 \sim 9z_1 + p'$ and $z_1 \sim p'$, leading to the same contradiction. In the remaining cases $f \cdot X = 2z_1$ with z_1 a 6-fold zero or $f \cdot X = z_1 + z_2$ runs into a similar contradiction. We thus may suppose that $f \cdot X = 2z_i$ with z_i a 3-fold zero. In this case we count the dimension as above.

The component of class $e + 4f$ moves in a linear system of dimension 5. For the stratum $\mathcal{Q}(6, 3, -1)$ the choice of z_1, z_2 and p gives 3 more parameters. Now the choice of X accounts for $h^0(F_4, e + 4f) = 6$ parameters, and quotienting by $\mathrm{Aut}(F_4)$ we obtain a parameter space of dimension 5, less than $\dim \mathbb{P}\mathcal{Q}(6, 3, -1) = 6$. For the stratum $\mathcal{Q}(3, 3, 3, -1)$ the parameter space has one more dimension stemming from the choice of z_3, but since the dimension of this stratum is one larger than $\dim \mathcal{Q}(6, 3, -1)$ we deduce the same contradiction. □

B.2. **Genus four.** We prove here the technical lemmas for the exceptional strata in genus four. Although this may seem a tedious case distinction, it cannot be circumvented in some form, since the hyperelliptic components appear from our perspective for some type of reducible parity curves.

Proof of Lemma 7.3. The elliptic curve E has class $(2, 2)$ on Q. For the stratum $\mathcal{Q}(12)$ the curve E is indeed irreducible and reduced. If it is not the case, suppose first that E contains a conic C_1 of class $(1, 1)$. The the other component of E is also a conic C_2 of class $(1, 1)$. Both C_1 and C_2 cut out $6z_1$ in X, hence q is a global square, leading to a contradiction. The other possibility is that E consists of a line L of class $(1, 0)$ union a curve R of class $(1, 2)$, we have $L \cdot X = 3z_1$ and $R \cdot X = 9z_1$. Then $h^0(X, 3z_1) = 2$, hence $3z_1$ and $z_1 + r + s$ both admit a g^1_3, where r, s are points in X not equal to z_1. Then $\omega_X \sim 4z_1 + r + s$ and $4z_1 \sim 2r + 2s$. Since $h^0(X, 4z_1) = h^0(X, 3z_1) = 2$, we conclude that p is a base point of $|4z_1|$. But $2r + 2s$ is a section of this linear series, which contradicts that $r, s \neq z_1$.

For all the other strata we first suppose that E is not reduced. If $E = 2C$ with C a conic of class $(1, 1)$, then q is a global square, contradicting the standing assumption. If $E = 2L + C$ with L a line of class $(1, 0)$ and C a curve of class $(0, 2)$, then C has to be a union of two distinct lines L_1, L_2 of class $(0, 1)$. Each L_i cuts out a degree 3 divisor in X. Note that $L_1 \cdot X$ and $L_2 \cdot X$ are disjoint, because they belong to two different rulings in the same g^1_3. Moreover, if L and L_i meet X at the same z_j, then one of them intersects X transversely at z_j. Using these observations we can easily rule out $\mathcal{Q}(9, 3)$ and $\mathcal{Q}(6, 6)$. For $\mathcal{Q}(6, 3, 3)$, the only possibility is that $L \cdot X = 3z_1$, $L_1 \cdot X = 3z_2$ and $L_2 \cdot X = 3z_3$. It implies that $\omega_X \sim 3z_1 + 3z_2$, hence $3z_1$ gives a g^1_3 and $3z_2 \sim 3z_3$ gives the other. This is possible, but we claim that then $h^0(X, 2z_1 + z_2 + z_3) = 1$. Otherwise by Riemann-Roch we have $h^0(X, z_1 + 2z_2 - z_3) = 1$, hence $z_1 + 2z_2$ also provides a g^1_3, leading to a contradiction. For $\mathcal{Q}(3, 3, 3, 3)$, the only possibility is that $L \cdot X = z_1 + z_2 + z_3$

(up to relabeling the z_i). Then $(L_1 + L_2) \cdot X = z_1 + z_2 + z_3 + 3z_4$ and one of them, say L_1, has to contain two points among z_1, z_2, z_3, which contradicts that $L_1 \neq L$.

Now assume that E is reduced, but not sufficiently smooth and moreover that $\dim H^0(X, \operatorname{div}(q)/3) = 2$. Below we will seek a contradiction. The assumption implies the existence of a hyperplane H such that $H \cdot E = \operatorname{div}(q)/3$. Suppose first that $E = L + R$ for some line L, say of class $(1, 0)$, necessarily contained in H. Then $H \cdot Q = L + L'$ with L' of class $(0, 1)$.

If R decomposes further, say it contains another line L_2 of class $(1, 0)$ and a curve C of class $(0, 2)$. From $H \cdot E = \operatorname{div}(q)/3$ we deduce that $L \cdot L' = z_1$, $L_2 \cdot L' = z_2$ and $L \cdot C = z_3 + z_4$ are the zeros of q (for some choice of numbering). But then $X \cdot (L + L_2 + C)$ has multiplicity at most one at z_1, contradicting that it is at least three. If R contains a line L_2 of class $(0, 1)$ the intersection point of L and L' provides again the contradiction.

Suppose now that R is irreducible. We let again $L \cdot L' = z_1$ and obtain the same contradiction as above unless R passes through z_1. Then, if z_2 denotes the second intersection point of L and R, we obtain $H \cdot E = (L + L') \cdot (L + R) = 3z_1 + z_2$. Consequently, we deal with the stratum $\mathcal{Q}(9, 3)$. Since $L' \cdot E = 2z_1$, a section of one of the two g^1_3 corresponding to L' must be $2z_1 + t$ for some t. Then $\omega_X \sim (L+L') \cdot X = (z_1 + 2z_2) + (2z_1 + t)$, since L and L' belong to different ruling classes due to $H \cdot Q$ being of class $(1, 1)$. Doubling it, we obtain $6z_1 + 4z_2 + 2t \sim 9z_1 + 3z_2$, hence $3z_1 \sim z_2 + 2t$ is also a g^1_3. But one checks that it can neither be equivalent to $z_1 + 2z_2$ nor to $2z_1 + t$.

The second case is that E consists of two irreducible conics C_1 and C_2. If the supports of $X \cdot C_1$ and $X \cdot C_2$ are disjoint, then E is sufficiently smooth. If these supports are equal, then q is a global square. These supports can consist of at most three points, since otherwise C_1 and C_2 have three points in common, contradicting the intersection degree. The same argument applies to counting a common tangent, e.g. for $\mathcal{Q}(6, 3, 3)$ and $X \cdot C_1 = 4z_1 + 2z_2$. If $X \cdot C_1 - X \cdot C_2$ is the difference of two effective divisors of degree two, then X is hyperelliptic. Using these arguments, only the following cases remain.

Case $\mathcal{Q}(9, 3)$ with $X \cdot C_1 = 6z_1$ and $X \cdot C_2 = 3z_1 + 3z_2$. This implies $3z_1 \sim 3z_2$, hence the ruling L tangent to X at z_1 is a flex line. But then $C_1 = 2L$, contradicting its irreducibility.

Case $\mathcal{Q}(6, 6)$ with $X \cdot C_1 = 5z_1 + z_2$ and $X \cdot C_2 = z_1 + 5z_2$. The existence of a hyperplane H with $H \cdot X = 2z_1 + 2z_2$ implies $\omega_X \sim 5z_1 + z_2 \sim z_1 + 5z_2 \sim 2z_1 + 2z_2 + r + s$ for some r, s. Hence $h^0(X, 3z_1) = h^0(X, 3z_2) = 2$. Consequently, we also have $h^0(X, 2z_1 + z_2) = h^0(X, z_1 + 2z_2) = 2$. Since X is not hyperelliptic, the ruling through z_1 and z_2 is tangent both at z_1 and z_2, a contradiction.

Case $\mathcal{Q}(6, 3, 3)$ with $X \cdot C_1 = 5z_1 + z_2$ and $X \cdot C_2 = z_1 + 2z_2 + 3z_3$. The existence of a hyperplane H with $H \cdot X = 2z_1 + z_2 + z_3$ implies

$$\omega_X \sim 5z_1 + z_2 \sim z_1 + 2z_2 + 3z_3 \sim 2z_1 + z_2 + z_3 + r + s$$

for some r, s. Since X is not hyperelliptic, $h^0(\omega_X(-z_1 - z_2)) \leq 2$, hence $z_2 + z_3 = r + s$, which leads to the contradiction $z_1 \sim z_2$.

Case $\mathcal{Q}(3, 3, 3, 3)$ with $X \cdot C_1 = 3z_1 + 2z_2 + z_3$ and $X \cdot C_2 = z_2 + 2z_3 + 3z_4$. The existence of a hyperplane H with $H \cdot X = z_1 + z_2 + z_3 + z_4$ implies that $h^0(X, 2z_1 + z_2) = h^0(X, z_1 + z_2 + z_3) = 2$. This is a contradiction, since they both contain z_1 and z_2. $\square$

Proof of Lemma 7.7. This follows from the two lemmas below, given that the strata are smooth. □

Lemma B.4. *No component of a stratum in $\mathcal{E}_4$ except the hyperelliptic components lies entirely in the hyperelliptic locus.*

Proof. By Section 5.2, a hyperelliptic curve X of genus four lies in $S_{1,6}$, which is the image of the Hirzebruch surface F_5 in $\mathbb{P}^8$. Let R be a rational curve of class $e+6f$. Since $e \cdot X = 0$, we have $\mathcal{O}_X(R) \sim \mathcal{O}_X(6f) \sim \omega_X^{\otimes 2}$. By the exact sequence

$$0 \to \mathcal{O}_{F_5}(-e-4f) \to \mathcal{O}_{F_5}(e+6f) \to \mathcal{O}_X(e+6f) \to 0,$$

there exists a unique section R in $|e+6f|$ that cuts out $D = \mathrm{div}(q)$ in X.

The pair (R, D) now takes the role of (E, D) and we mimic the dimension count in Proposition 7.5. Suppose that R is irreducible. The parameter space for R has dimension equal to $\dim \mathbb{P}H^0(F_5, e+6f) = 8$. Hence the parameter space for (R, D) has dimension $8+n$. From

$$0 \to \mathcal{O}_{F_5}(e+4f) \to \mathcal{O}_{F_5}(2e+10f) \to \mathcal{O}_R(2e+10f) \to 0,$$

we deduce that for fixed (R, D), the space of (necessarily hyperelliptic) genus four curves X with $X \cdot R = D$ has dimension $h^0(F_5, e+4f) = 5$. From (X, R, D) we recover $\mathrm{div}(q) = X \cdot R$. Since the automorphism group of F_5 is 10-dimensional, the space of triples (X, R, D) has dimension $n+3$, smaller than $\dim \mathbb{P}\mathcal{Q}(k_1, \ldots, k_n) = n+5$.

Finally, we have to treat the case when R is reducible. Suppose R consists of a fibers f along with a component of class $e+(6-a)f$. Since $e^2 = -5$ we have $a = 6$ or $a = 1$, since otherwise R would not be effective.

We first deal with the case $a = 1$ and perform a dimension count similar to the above. The parameter space for the component of class $e+5f$ has dimension equal to $\dim \mathbb{P}H^0(F_5, \mathcal{O}_{F_5}(e+5f)) = 6$ and the parameter space for (R, D) has dimension $6+n$. The other component of R of class f intersects X in $2z_i$ or in $z_i + z_j$. In any case, it is determined up to finitely many choices by R and D. Now the space of genus four curves X with $X \cdot R = D$ has dimension $h^0(F_5, e+5f) = 7$. As above, since the automorphism group of F_5 is 10-dimensional, the space of triples (X, R, D) has dimension $n+3$, smaller than the dimension of the corresponding stratum.

Finally, we deal with the case $a = 6$ to retrieve the hyperelliptic components. Now R is the divisor e union several (possibly non-reduced) fibers.

For $\mathcal{Q}(12)$, the fiber has to have multiplicity 6 and z_1 has to be a Weierstrass point. But then q would be a global square, contradicting the hypothesis.

For $\mathcal{Q}(9,3)$ both k_i are odd, hence it is impossible that $9z_1 + 3z_3$ is cut out by fibers only.

For $\mathcal{Q}(6,6)$ and $\mathrm{div}(q) = 6z_1 + 6z_2$, there are two possibilities. First R may contain a 6-fold fiber f with $f \cdot X = z_1 + z_2$. But then the z_i are Weierstrass points and q is a global square, contradicting the hypothesis. The other case is that R contains two 3-fold fibers f_1 and f_2 with $f_i \cdot X = 2z_i$. Then both z_i are Weierstrass points. We recover in this way the hyperelliptic component of this stratum.

For $\mathcal{Q}(6,3,3)$, there is only one possibility, namely R contains $3f_1 + 3f_2$, where $f_1 \cdot X = 2z_1$ and $f_2 \cdot X = z_2 + z_3$. Hence z_1 is a Weierstrass point and z_2, z_3 are conjugate. We thus recover the hyperelliptic component.

For $\mathcal{Q}(3,3,3,3)$ there is again only one possibility, namely that R contains $3f_1 + 3f_2$, with $f_1 \cdot X = z_1 + z_2$ and $f_2 \cdot X = z_3 + z_4$ for an appropriate ordering of the zeros. In this way we recover again the hyperelliptic component of this stratum. □

Lemma B.5. *No component of a stratum in $\mathcal{E}_4$ is contained entirely in the Gieseker-Petri locus but is not entirely contained in the hyperelliptic locus.*

Proof. By Section 5.2 we work on the Hirzebruch surface F_2 with R a rational curve of class $2e+4f$ and X of class $3e+6f$. Since $e \cdot X = 0$, we have $\mathcal{O}_X(R) \sim \mathcal{O}_X(6f) \sim \omega_X^{\otimes 2}$. The argument is parallel to the hyperelliptic case. By the exact sequence

$$0 \to \mathcal{O}_{F_2}(-e-2f) \to \mathcal{O}_{F_2}(2e+4f) \to \mathcal{O}_X(2e+4f) \to 0,$$

there exists a unique section R in $|2e+4f|$ that cuts out $D = \operatorname{div}(q)$ in X.

Suppose that R is irreducible. The parameter space for R has dimension equal to $\dim \mathbb{P}H^0(F_2, 2e+4f) = 8$. Hence the parameter space for (R, D) has dimension $8+n$. From

$$0 \to \mathcal{O}_{F_2}(e+2f) \to \mathcal{O}_{F_2}(3e+6f) \to \mathcal{O}_R(3e+6f) \to 0,$$

we deduce that for fixed (R, D), the space of (necessarily hyperelliptic) genus four curves X with $X \cdot R = D$ has dimension $h^0(F_2, e+2f) = 2$. From (X, R, D) we recover $\operatorname{div}(q) = X \cdot R$. Since the automorphism group of F_2 is 7-dimensional, the space of triples (X, R, D) has dimension $n+3$, smaller than $\dim \mathbb{P}\mathcal{Q}(k_1, \ldots, k_n) = n+5$.

If R is reducible, there are three possible decompositions: first, two components R_1 and R_2, both of class $e+2f$; second, two components of class e together with four fibers and third, $R_1 = e + bf$ with $b \geq 2$ together with a component of class e and $4-b$ fibers.

The components R_1 and R_2 in the first case cannot be identical, otherwise q would be a global square. Consequently, $X \cdot R_i = 6$ and $R_1 \cdot R_2 = 2$. If the divisor $X \cdot R = D$ is supported away from the singular locus R_{sing} of R, the same dimension count as above goes through. If D meets R_{sing} and moreover the components R_1 and R_2 intersect transversely at two nodes p_1 and p_2, at each node p_i the curve X can only be tangent to one of the two branches. This involves a choice of two possibilities, but once we choose one of the two, i.e. once we know which branch is tangent to X, then the above fiber dimension count is still fine. More generally, if R_1 and R_2 are tangent at p and if D contains p with multiplicity n, then suppose $(X \cdot R_i)_p = n_i$ with $n_1 + n_2 = n$ and $n_i > 1$. Once we specify the pair (n_1, n_2), which again amounts to finitely many choices, the Cartier divisor D is determined and a similar fiber dimension count goes through.

In the second case, if the four fibers are all distinct, $X \cdot R = D$ is located away from R_{sing} and the dimension count is the same. If the configuration is $2f_1+f_2+f_3$, then $X \cdot R = 2D_1 + D_2 + D_3$ with disjoint divisors D_i. The only possible stratum is $(6,3,3)$ with $D_i = 3z_i$. Since $2f_1$ is a double ruling, the parameter space of X for a given (R, D) increases dimension by three compared to the above count, but the base dimension drops by three, so the dimension argument is fine. If the configuration is $2f_1 + 2f_2$ or $4f_1$, then q is a global square, impossible. Finally, if the configuration is $3f_1 + f_2$, we are in the stratum $\mathcal{Q}(9,3)$. Then the parameter space of X for a given (R, D) increases dimension by six compared to the count in the irreducible case, but the base dimension drops by seven, so the dimension argument still goes through.

In the third case, suppose first that R consists of an irreducible component R_1 of class $e+3f$ together with a ruling f_1 besides e. Then $R_1 \cdot f_1 = 1$ and they intersect transversely. So we can apply the above argument by specifying X tangent to R_1 or f_1, if $X \cdot R$ contains the node. Next, suppose that R consists of R_1 of class $e+2f$ and two rulings f_1, f_2 besides e. First suppose f_1 and f_2 are distinct. If X is disjoint from R_{sing}, we are done. If X intersects R_{sing}, since $f_i \cdot R_1 = 1$, this is also fine. Now suppose $f_1 = f_2$ is a double ruling and $X \cdot R$ contains the node $p = f_1 \cdot R_1$. Write $(X \cdot R)_p = n$, $(X \cdot R_1)_p = m$ and we have $(X \cdot f_1)_p = k < 4$ with $m + 2k = n$. This amounts to a finite case distinction. Since R_1 and f_1 intersect transversely at p, we have $m = 1$ or $k = 1$. If $m = 1$, the condition imposed to X is $k = (n-1)/2$ instead of n as the expected fiber codimension. But the locus of such $(R, 2f_1)$ in the linear system $|\mathcal{O}(2)|$ has dimension 4, i.e. codimension 5, which is higher than $n - k = k + 1$ since $k < 4$. So the total dimension of the parameter space is not enough for being a component. If $k = 1$, then the number of conditions imposed to X is $m = n - 2$. The parameter space for X has dimension two larger than in the irreducible case, but the base dimension drops by more than that. $\square$

Appendix C. Varying strata: examples

In this section we collect all data of half-translation surfaces indicating that beyond the cases discussed in our main results, most of the other strata are varying. We emphasize though, that we do not dispose of any proof that all strata beyond a certain dimension are varying. Below we give lists of explicit surfaces, discussing one stratum at a time.

Almost all the half-translation surfaces were calculated using a computer program by Vincent Delecroix, built on its predecessor by Anton Zorich. These programs were designed for square-tiled surfaces. We thus list below the monodromy permutations of the canonical double cover of a half-translation surface we are interested in. The program gives the sum L of Lyapunov exponents for the double cover, but using (3) we can also compute the quantity L^+.

In the table 'Index' refers to the index of the Veech group in $\mathrm{SL}_2(\mathbb{Z})$.

The exceptional strata. We give examples completing Theorems 6.2 and 7.2, proving that certain components of the exceptional strata are varying:

Stratum	Monodromy	Index	L	L^+
$\mathcal{Q}(3,3,3,-1)^{\mathrm{irr}}$	$r = (2,3)(4,5,6,7)$ $(9,10,11,12)(14,15)$ $u = (1,2)(3,4,8,9,5,10)$ $(6,11,7,13,12,14)(15,16)$	200	3	$\frac{13}{10}$
$\mathcal{Q}(3,3,3,-1)^{\mathrm{irr}}$	$r = (3,4)(5,6,7,8)$ $(10,11,12,13)(15,16)$ $u = (1,2,3)(4,5,9,10,6,11)$ $(7,12,8,14,13,15)(16,17,18)$	2350	$\frac{150}{47}$	$\frac{328}{235}$
$\mathcal{Q}(6,6)^{\mathrm{irr}}$	$r = (2,3)(4,5)(7,8)$ $(10,11)(12,13)(15,16)$ $u = (1,2,4,6,5,10,12)$ $(3,7,9,8,11,14,15)$	768	$\frac{103}{32}$	$\frac{19949}{12015}$
$\mathcal{Q}(6,6)^{\mathrm{irr}}$	$r = (2,3,4,5)(8,9,10,11)$ $u = (1,2,6,4)(3,7,8,12)$ $(5,13,10,15)(9,14,11,16)$	48	$\frac{7}{2}$	$\frac{7}{4}$
$\mathcal{Q}(6,3,3)^{\mathrm{irr}}$	$r = (2,3)(4,5)(6,7,8,9)$ $(10,11)(12,13,14,15)(17,18)$ $u = (1,2,4,6,10,12,5)(7,15)$ $(3,14,18,11,16,17,8)(9,13)$	9420	$\frac{5593}{1570}$	$\frac{1359}{785}$
$\mathcal{Q}(6,3,3)^{\mathrm{irr}}$	$r = (2,3)(5,6,7)(8,9)(10,11)$ $(12,13)(15,16,17)$ $u = (1,2,4,5,8,10,12,6)$ $(3,13,16,18,11,14,15,9)$	3480	$\frac{514}{145}$	$\frac{999}{580}$
$\mathcal{Q}(3,3,3,3)^{\mathrm{irr}}$	$r = (2,3)(4,5)(6,7,8,9)$ $(11,12,13,14)(16,17)(18,19)$ $u = (1,2,4,6,10,11,3,14,7,5)$ $(8,15,13,18,17,20,19,12,9,16)$	3390	$\frac{2121}{565}$	$\frac{1004}{565}$
$\mathcal{Q}(3,3,3,3)^{\mathrm{irr}}$	$r = (2,3)(5,6)(7,8,9,10)$ $(11,12,13,14)(15,16)(17,18)$ $u = (1,2,4,5,7,3,9,6)(8,11)$ $(10,13)(12,15,14,17,19,16,20,18)$	690	$\frac{441}{115}$	$\frac{209}{115}$

In genus one we give a pair of examples justifying that $\mathcal{Q}(4,2,-1^6)$ is varying:

Stratum	Monodromy	Index	L	L^+
$\mathcal{Q}(4,2,-1^6)$	$r = (1,2,3,4,5,6)(7,8,9,10,11,12)$ $u = (1,9,4,12)(2,8,5,11)(3,7,6,10)$	36	$\frac{5}{3}$	0
$\mathcal{Q}(4,2,-1^6)$	$r = (1,2,3,4,5,6,7,8)$ $(9,10,11,12,13,14,15,16)$ $u = (1,9,6,10,4,14,8,12)$ $(2,16,5,13)(3,15,7,11)$	148	$\frac{13}{6}$	$\frac{1}{4}$

In genus two all the strata of dimension at most seven except for those listed in Theorem 9.1 are varying. We give examples in the two cases, with largest and smallest possible orders of zeros:

Stratum	Monodromy	Index	L	L^+
$\mathcal{Q}(8,-1^4)$	$r=(2,3)(4,5)(6,7)$ $(9,10)(11,12)(13,14)$ $u=(1,2,4,6)(3,5,8,9)$ $(7,11,13)(10,12,14)$	160	$\frac{11}{5}$	$\frac{3}{5}$
$\mathcal{Q}(8,-1^4)$	$r=(2,3)(4,5)(6,7)$ $(9,10)(11,12)(13,14)$ $u=(1,2,4,6,7,10,12)$ $(3,5,8,9,11,13,14)$	105	$\frac{13}{5}$	$\frac{4}{5}$
$\mathcal{Q}(2^2,1^2,-1^2)$	$r=(2,3)(6,7,8)(9,10,11)(13,14)$ $u=(1,2,4,3,5,6)(7,9)$ $(8,10)(11,12,13,15,14,16)$	192	3	$\frac{7}{6}$
$\mathcal{Q}(2^2,1^2,-1^2)$	$r=(2,3,4)(6,7,8)$ $(9,10,11)(12,13,14)$ $u=(1,2,5,6,9,12,4,7)$ $(3,8,11,15,13,16,10,14)$	522	$\frac{220}{87}$	$\frac{27}{29}$

In genus three all the strata of dimension at most eight except for those listed in Theorem 10.1 are varying. We give examples in the two cases, with largest and smallest possible orders of zeros:

Stratum	Monodromy	Index	L	L^+
$\mathcal{Q}(11,-1^3)$	$r=(2,3)(4,5)(6,7)(9,10)$ $(11,12,13)(14,15,16)$ $u=(1,2,4,6)(3,5,8,9)$ $(7,11,13,15)(10,12,14,16)$	9828	$\frac{6388}{2457}$	$\frac{173}{189}$
$\mathcal{Q}(11,-1^3)$	$r=(3,4)(5,6)(7,8)(11,12)$ $(13,14,15)(16,17,18)$ $u=(1,2,3,5,7)(4,6,9,10,11)$ $(8,13,15,17)(12,14,16,18)$	78390	$\frac{101078}{39195}$	$\frac{2728}{3015}$
$\mathcal{Q}(2,2,2,2)$	$r=(2,3)(5,6)(9,10)(12,13)$ $u=(1,2,4,5,7,3,8,9)$ $(6,11,12,14,10,15,13,16)$	24	$\frac{5}{2}$	$\frac{5}{4}$
$\mathcal{Q}(2,2,2,2)$	$r=(2,3,4)(6,7,8)$ $(9,10,11)(12,13,14)$ $u=(1,2,5,6,9,8,10,7)$ $(3,12,4,14,16,11,15,13)$	36	$\frac{8}{3}$	$\frac{4}{3}$

In genus four all the strata of dimension at most nine except for those listed in Theorem 11.1 are varying. We give examples in the two cases, with largest and smallest possible orders of zeros:

Stratum	Monodromy	Index	L	L^+
$\mathcal{Q}(14,-1^2)^{\rm nh}$	$r=(2,3)(5,6)(7,8)(9,10,11)$ $(13,14)(16,17,18)$ $u=(1,2,4,5,7,9)(3,10)$ $(6,8,12,13,15,16)(14,18)$	381280	$\frac{138559}{47660}$	$\frac{114729}{95320}$
$\mathcal{Q}(14,-1^2)^{\rm nh}$	$r=(3,4)(6,7)(8,9)(10,11,12)$ $(14,15)(18,19,20)$ $u=(1,2,3,5,6,8,10)(4,11)$ $(7,9,13,14,16,17,18)(15,20)$	3454784	$\frac{626281}{215924}$	$\frac{518319}{431848}$
$\mathcal{Q}(4,4,4)$	$r=(2,3)(5,6)(8,9)$ $(10,11)(12,13)(16,17)$ $u=(1,2,3,4,5,7,8,10,9)$ $(6,12,14,11,15,16,17,18,13)$	6480	$\frac{139}{45}$	$\frac{139}{790}$
$\mathcal{Q}(4,4,4)$	$r=(2,3)(5,6)(8,9)$ $(10,11)(12,13)(16,17)$ $u=(1,2,4,5,7,8,10,3,9)$ $(6,12,14,11,15,16,18,13,17)$	180	$\frac{44}{15}$	$\frac{22}{15}$

We remark that showing a varying stratum, say for dimension greater nine in genus four, would heavily requires computer resources. For example, in dimension nine, the simplest pillow-case tiled surface in the stratum $(12,1,-1)$ has 9 tiles, whose Veech group has index 292824 in $\mathrm{SL}_2(\mathbb{Z})$ and $L^+ = 793/581$. The next example with 10 tiles has a Veech group of index 2635416 in $\mathrm{SL}_2(\mathbb{Z})$ and also $L^+ = 793/581 \approx 1.36488$. However, this stratum is varying, as an example with 11 tiles, a Veech group of index 13187664 and $L^+ = 851/623 \approx 1.3659$ shows.

In genus five even the smallest stratum is varying:

Stratum	Monodromy	Index	L	L^+
$\mathcal{Q}(16)$	$r=(2,3)(5,6)(7,8)(9,10,11)$ $(12,13)(16,17,18)$ $u=(1,2,4,5,7,9)(6,10)$ $(3,12,14,8,15,16)(13,18)$	648810	$\frac{39898}{12015}$	$\frac{19949}{12015}$
$\mathcal{Q}(16)$	$r=2,3)(5,6)(7,8,9)(10,11)$ $(12,13,14)(16,17)$ $u=(1,2,4,5)(3,7,10,12,14)$ $(6,13,17,9,8)(11,15,16,18)$	6480	$\frac{10}{3}$	$\frac{5}{3}$

References

[AC87] E. Arbarello and M. Cornalba. The Picard groups of the moduli spaces of curves. *Topology*, 26(2):153–171, 1987.

[AC98] E. Arbarello and M. Cornalba. Calculating cohomology groups of moduli spaces of curves via algebraic geometry. *Inst. Hautes Études Sci. Publ. Math.*, (88):97–127 (1999), 1998.

[Bai07] M. Bainbridge. Euler characteristics of Teichmüller curves in genus two. *Geom. Topol.*, 11:1887–2073, 2007.

[Ber74] L. Bers. Spaces of degenerating Riemann surfaces. In *Discontinuous groups and Riemann surfaces (Proc. Conf., Univ. Maryland, College Park, Md., 1973)*, pages 43–55. Ann. of Math. Studies, No. 79. Princeton Univ. Press, Princeton, N.J., 1974.

[BM10] I. Bouw and M. Möller. Teichmüller curves, triangle groups, and Lyapunov exponents. *Ann. of Math. (2)*, 172(1):139–185, 2010.

[Che11] D. Chen. Square-tiled surfaces and rigid curves on moduli spaces. *Adv. Math.*, 228(2):1135–1162, 2011.

[CM12] D. Chen and M. Moeller. Non-varying sums of Lyapunov exponents of Abelian differentials in low genus. *Geom. Topol.*, 16:2427–2479, 2012.

[Cuk89] F. Cukierman. Families of Weierstrass points. *Duke Math. J.*, 58(2):317–346, 1989.

[EKZ11] A. Eskin, M. Kontsevich, and A. Zorich. Sum of Lyapunov exponents of the Hodge bundle with respect to the Teichmüller geodesic flow, 2011, arXiv: math.AG/1112.5872.

[Far09] G. Farkas. Koszul divisors on moduli spaces of curves. *Amer. J. Math.*, 131(3):819–867, 2009.

[Has99] B. Hassett. Stable log surfaces and limits of quartic plane curves. *Manuscripta Math.*, 100(4):469–487, 1999.

[HM98] J. Harris and I. Morrison. *Moduli of Curves*, volume 187 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[Kon97] M. Kontsevich. Lyapunov exponents and Hodge theory. In *The mathematical beauty of physics (Saclay, 1996)*, volume 24 of *Adv. Ser. Math. Phys.*, pages 318–332. World Sci. Publishing, River Edge, NJ, 1997.

[KZ03] M. Kontsevich and A. Zorich. Connected components of the moduli spaces of Abelian differentials with prescribed singularities. *Invent. Math.*, 153(3):631–678, 2003.

[Lan04] E. Lanneau. Hyperelliptic components of the moduli spaces of quadratic differentials with prescribed singularities. *Comment. Math. Helv.*, 79(3):471–501, 2004.

[Lan08] E. Lanneau. Connected components of the strata of the moduli spaces of quadratic differentials. *Ann. Sci. Éc. Norm. Supér. (4)*, 41(1):1–56, 2008.

[Log03] A. Logan. The Kodaira dimension of moduli spaces of curves with marked points. *Amer. J. Math.*, 125(1):105–138, 2003.

[Mas82] H. Masur. Interval exchange transformations and measured foliations. *Ann. of Math. (2)*, 115(1):169–200, 1982.

[Mir89] R. Miranda. *The basic theory of elliptic surfaces*. Dottorato di Ricerca in Matematica. [Doctorate in Mathematical Research]. ETS Editrice, Pisa, 1989.

[Möl06] M. Möller. Variations of Hodge structures of a Teichmüller curve. *J. Amer. Math. Soc.*, 19(2):327–344 (electronic), 2006.

[Möl11] M. Möller. Teichmüller curves, mainly from the point of view of algebraic geometry, 2011, available on the author's web page, to appear as PCMI lecture notes.

[Tra92] S. Trapani. On the determinant of the bundle of meromorphic quadratic differentials on the deligne-mumford compactification of the moduli space of riemann surfaces. *Math. Ann.*, 293(4):681–705, 1992.

[Vee90] W. Veech. Moduli spaces of quadratic differentials. *J. Analyse Math.*, 55:117–171, 1990.

[Wol89] S. Wolpert. Riemann surfaces, moduli and hyperbolic geometry. In *Lectures on Riemann surfaces (Trieste, 1987)*, pages 48–98. World Sci. Publ., Teaneck, NJ, 1989.

[YZ12] F. Yu and K. Zuo. Weierstrass filtration on Teichmüller curve and Lyaponov exponent, 2012, preprint, arXiv:AG/1203.6053.

[Zor08] A. Zorich. Explicit jenkins-strebel representatives of all strata of abelian and quadratic differentials. *Journ. Modern Dyn.*, 2(1):139–185, 208.

Department of Mathematics, Boston College, Chestnut Hill, MA 02467, USA
E-mail address: `dawei.chen@bc.edu`

Institut für Mathematik, Goethe-Universität Frankfurt, Robert-Mayer-Str. 6-8, 60325 Frankfurt am Main, Germany
E-mail address: `moeller@math.uni-frankfurt.de`