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# Teichmüller curves, mainly from the viewpoint of algebraic geometry

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## 1. Introduction

Teichmüller curves are algebraic curves $C \to \mathcal{M}_g$ in the moduli space of curves that are totally geodesic for the Teichmüller metric. They are generated by very special flat surfaces, i.e. compact Riemann surfaces with a flat metric (up to finitely many cone-type singularities), called Veech surfaces. Studying the geometry and dynamics of Veech surfaces and connection to billiards was the original motivation to introduce Teichmüller curves. Here we consider Teichmüller curves as intrinsically interesting curves and relate their geometry to the geometry of $\mathcal{M}_g$.

If we take a more distant look and consider just any curve $C$ in $\mathcal{M}_g$ (or $\overline{\mathcal{M}}_g$) there are various quantities that one can associate algebraically with such a curve. Cohomologically, one can decompose the variation of Hodge structures of the family over $C$ into its irreducible summands. Intersection theory with various natural divisors (and line bundles) on $\mathcal{M}_g$ gives a collection of numbers that one can attach to $C$. The most prominent of them in algebraic geometry is the slope. Finally, dynamics attaches to a Teichmüller curve some characteristic numbers, namely Lyapunov exponents.

The purpose of these lecture notes is to relate the quantities 'decomposition of the variation of Hodge structures', 'slope' and 'Lyapunov exponents' on Teichmüller curves. Of course, Teichmüller curves are not just arbitrary curves in $\mathcal{M}_g$ and their origin from flat geometry allows special techniques, thus giving e.g. restrictions on their slope. Towards the end of these lecture notes we broaden the picture and highlight that all these quantities in fact make sense for any curve in $\mathcal{M}_g$ and most do even for any curve in the moduli space of abelian varieties. They hence deserve to be studied also in this broadened context!

We outline three guiding questions of the field: Can one classify Teichmüller curves, in particular those that are primitive, i.e. that do not arise via covering constructions? If one performs covering constructions of a surface generating a Teichmüller curve, how do the above quantities 'slope' and 'Lyapunov exponents' change? What can one say about the values for slopes and Lyapunov exponents

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appearing for the Teichmüller curves with fixed type (e.g. of singularities)? More detailed versions of these questions appear in the open problem sections.

The main results presented in this text are the following. First, the variation of Hodge structures over a Teichmüller curve decomposes into $r$ rank two pieces, where $r \leq g$ is a field extension degree. One of this pieces has 'maximal degree' and the presence of a Hodge decomposition with such a maximal piece characterizes Teichmüller curves (Theorem 5.5 and Theorem 5.6). This decomposition implies real multiplication on an $r$-dimensional abelian subvariety of the family of Jacobians and has proven to be a major tool towards classification of Teichmüller curves. Second, to a Teichmüller curve and to a generic flat surface one can associate the Lyapunov exponents (Theorem 6.1). The sum of Lyapunov exponents for Teichmüller curves can be calculated as slopes by a Riemann-Roch argument (Proposition 6.4). Finally, we show how the geometry of the moduli space allows to calculate the sum of Lyapunov exponents for all Teichmüller curves in low genus (Theorem 6.8).

As guideline for the reader, we indicate that Section 2, Section 3, and Section 4 contain background material. The core of what has been indicated above starts with the definition of Teichmüller curves in Section 5. To get started there we suggest to have read Section 2 and Section 3.1 and to jump back to the other introductory sections on demand.

**Plan of the lecture series:** The first lecture will introduce Teichmüller curves and state the VHS decomposition. Proofs of this decomposition along with some background on Hodge theory are part of the second lecture. The third lecture introduces Lyapunov exponents showing how their sum can be related to quantities that are well-known in algebraic geometry. The fourth lecture introduces the slope of a (one-parameter) family of curves and how properties of the slope lead to non-varying results for sums of Lyapunov exponents. Moreoever, it connects to some open research problems, in particular characterizing other curves in moduli space, such as Shimura curves, by Lyapunov exponents.

## 2. Flat surfaces and $\mathrm{SL}_2(\mathbb{R})$-action

### 2.1. Flat surfaces and translation structures

Let $\mathcal{M}_g$ be the moduli space of genus $g$ compact, smooth and connected algebraic curves. Sometimes we also need the moduli space $\mathcal{M}_{g,n}$ parametrizing curves with $n$ ordered marked points and we let $\mathcal{M}_{g,[n]}$ be the moduli space of curves with $n$ unordered marked points. Let $\Omega\mathcal{M}_g$ denote the vector bundle of holomorphic one-forms over $\mathcal{M}_g$ minus the zero section. Points in $\Omega\mathcal{M}_g$, called *flat surfaces*, are usually written as a pair $(X, \omega)$ for a holomorphic one-form $\omega$ on the Riemann surface $X$.

We use the terminology (algebraic) curve and (Riemann or flat) surface interchangeably. When talking of objects of complex dimension two we emphasize this by speaking of 'complex surfaces'. Fibered surfaces introduced in Section 3.1 are by definition of complex dimension two.

A *translation structure* on a Riemann surface $X^0$ is an atlas of complex charts $\{(U_\alpha, g_\alpha)\}_{\alpha \in I}$, all whose transition functions are locally translations. The group $\mathrm{SL}_2(\mathbb{R})$ acts on the set of translation structures by postcomposing the chart maps $g_\alpha$ with the linear map (thereby identifying $\mathbb{C}$ with $\mathbb{R}^2$). Since $\mathrm{SL}_2(\mathbb{R})$ normalizes the subgroup of translations within the affine group of $\mathbb{R}^2$, this action is well-defined.

For a flat surface $(X, \omega)$ we let $Z(\omega)$ be the set of zeros of $\omega$. From the viewpoint of translation structures it is natural to call $Z(\omega)$ the set of *singularities* of $(X, \omega)$. Note that the algebraic curve $X$ is non-singular at $Z(\omega)$.

The proof of the next proposition is straightforward, given that a translation structure knows about the winding number of a loop around a singularity.

**Proposition 2.1.** *If $(X, \omega)$ is a flat surface then $X \setminus Z(\omega)$ has a translation structure. Conversely, suppose $X^0$ is a Riemann surface obtained by removing from a compact topological surface $X$ a finite number of points. If $X^0$ has a translation structure, such that $X$ is the completion of $X^0$ with respect to the flat metric, then there is a flat surface $(X, \omega)$, such that $X^0 = X \setminus Z(\omega)$ and such that the translation structure associated with $(X, \omega)$ is just the given translation structure.*

**Corollary 2.2.** *There is an action of $\mathrm{SL}_2(\mathbb{R})$ on the moduli space of flat surfaces $\Omega\mathcal{M}_g$. This action preserves the stratification by the number and type of zeros.*

Proof. To define the $\mathrm{SL}_2(\mathbb{R})$-action we remove $Z(\omega)$, use the action of $\mathrm{SL}_2(\mathbb{R})$ on translation surfaces and use the converse statement in the previous proposition to glue the missing points back in. It is immediate to check that for any given $A \in \mathrm{SL}_2(\mathbb{R})$ the hypothesis on the metric completion still holds. □

Various one-parameter subgroups of $\mathrm{SL}_2(\mathbb{R})$ thus define flows on $\Omega\mathcal{M}_g$. The diagonal subgroup $g_t = \mathrm{diag}(e^t, e^{-t})$ is in fact the *Teichmüller geodesic flow* and the action of $h_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ defines a flow that is called *horocyclic flow*.

**Proposition 2.3.** *The $\mathrm{SL}_2(\mathbb{R})$-action preserves the subvariety of hyperelliptic flat surfaces.*

Proof. The hyperelliptic involution acts as $(-1)$ on all one-forms, hence on $\omega$. In the flat coordinates of $X$ given by $\Re(\omega)$ and $\Im(\omega)$, the hyperelliptic involution acts by the matrix $-\mathrm{Id}$. Since $-\mathrm{Id}$ is in the center of $\mathrm{SL}(2, \mathbb{R})$, we conclude that if $(X, \omega)$ admits a hyperelliptic involution, so does $A \cdot (X, \omega)$ for any $A \in \mathrm{SL}(2, \mathbb{R})$. □

**Some flat geometry.** The holomorphic one-form $\omega$ on $X$ defines a flat metric and consequently for any given direction $\theta$ we can define the straight line flow $\phi_t^\theta(x)$ in the direction $\theta$ starting at $x \in X$, where $t$ denotes the flow distance in the metric $|\omega|$. Consider the geodesics on a flat surface $(X, \omega)$ in a fixed direction $\theta$. If such a geodesic $\gamma$ ends forward and backward in finite time in a singularity (i.e. in $Z(\omega)$), then $\gamma$ is called a *saddle connection*. If in a given direction $\theta$ all geodesics are either periodic or saddle connections, then $\theta$ is called a periodic direction. A maximal union of homotopic closed geodesics on $(X, \omega)$ is called a *cylinder*. The *width* is the length of a core geodesic, the *height* is the length of a straight segment perpendicular to a core geodesic crossing the cylinders once. We define the *modulus* of a cylinder to be the ratio height over width. If the moduli of all cylinders in a periodic direction are commensurable, the direction is called *parabolic*. (The reason for this terminology will become apparent once we define affine groups.)

Every flat surface has a saddle connection, more precisely, the set of directions of saddle connection vectors is dense in $S^1$. Here and in the sequel we consider the torus with one artificial singularity. Moreover, the set of saddle connection vectors (with multiplicities) is discrete in $\mathbb{R}^2$. In particular, there exist finitely many shortest saddle connections on every flat surface. Proofs of these elementary, but fundamental properties can be found in [**Vor96**] and in [**MT02**].

**Half-translation structures and quadratic differentials.** A *half-translation structure* on a surface is an atlas whose transition functions are compositions of $\pm\mathrm{Id}$ and translations. Similarly to translation structures, one can set up a correspondence between half-translation structures and quadratic differentials. Except for hyperelliptic loci (defined below) we will disregard here the question whether the results on translation surfaces extend to half-translation surfaces or not. Occasionally this is easy, but often there are problems. We sometimes give references to the corresponding results for quadratic differentials.

### 2.2. Affine groups and the trace field

One of the basic invariants of a flat surface is the *affine group* $\mathrm{SL}(X,\omega)$ (also called *Veech group*) defined as follows. Let $\mathrm{Aff}^+(X,\omega)$ be the group of orientation-preserving homeomorphisms of $X$, that are affine diffeomorphisms on $X \setminus Z(\omega)$ with respect to the charts defined by integrating $\omega$. (We will abusively call elements of $\mathrm{Aff}^+(X,\omega)$ *affine diffeomorphisms*, although they are not differentiable at the zeros of $\omega$.) The matrix part of the affine map is independent of the charts and provides a map

$$D : \mathrm{Aff}^+(X,\omega) \to \mathrm{SL}_2(\mathbb{R}).$$

The image of $D$ is called the affine group $\mathrm{SL}(X,\omega)$.

**Proposition 2.4.** *The image of $D$ is a discrete group that is never cocompact.*

Proof. Discreteness follows from the discreteness of the set of saddle connection vectors. If $\mathrm{SL}(X,\omega)$ was cocompact, suppose the horizontal direction has a saddle connection. Then there exists an unbounded sequence of times $t_n$ and elements $\varphi_n \in \mathrm{SL}(X,\omega)$ such that $\varphi_n(g_{t_n}(X,\omega))$ converges in $\mathbb{H}/\mathrm{SL}(X,\omega)$. On the limiting surface we obtain a contradiction to the above lower bound for norms of saddle connection vectors. $\square$

Let $K = \mathbb{Q}(\mathrm{tr}(\varphi), \varphi \in \mathrm{SL}(X,\omega))$ denote the *trace field* of the affine group. The field extension $K/\mathbb{Q}$ has a priori no reason to be Galois and we let $L/\mathbb{Q}$ be the Galois closure of $K/\mathbb{Q}$.

An important restriction on the trace field is the following bound.

**Proposition 2.5.** *The degree of the trace field of the affine group $\mathrm{SL}(X,\omega)$ is bounded by the genus $g(X)$.*

Proof. Write $K = \mathbb{Q}(t)$ for some $t = \sum a_i \mathrm{tr}(D\varphi_i)$ with $a_i \in \mathbb{Q}$ and $\varphi_i \in \mathrm{SL}(X,\omega)$ using the theorem of the primitive element and the fact that $\mathrm{tr}(A)\mathrm{tr}(B) = \mathrm{tr}(AB) + \mathrm{tr}(AB^{-1})$ for $A$, $B \in \mathrm{SL}_2(\mathbb{R})$. Now consider

$$T = \sum a_i(\varphi_i^* + (\varphi_i^{-1})^*) \quad \in \mathrm{End}(H^1(X,\mathbb{Q})).$$

On the two-dimensional subspace $L = \langle \Re(\omega), \Im(\omega) \rangle$ we have $T|_L = t \cdot \mathrm{Id}$. Hence $t$ is an eigenvalue of $T$ of multiplicity at least two. The square of the characteristic polynomial of $t$ thus divides the polynomial $\det(xI_{2g} - T)$, which is of degree $2g$, since $\dim H^1(X,\mathbb{Q}) = 2g$. $\square$

The Veech group of a general flat surface in a given stratum is of order two or trivial, depending on whether the stratum is hyperelliptic or not ([**Möl09**]). We will eventually be most interested in surfaces where the opposite extreme holds, i.e. where the affine group is as large as possible.

We briefly recall Thurston's classification of surface homeomorphisms. A homeomorphism $\varphi$ of $X$ is called *elliptic*, if it is isotopic to a diffeomorphism of finite order. It is easy to see that an affine diffeomorphism $\varphi$ of $(X,\omega)$ is elliptic, if it is of finite order. In particular $D(\varphi)$ is of finite order. Conversely, if $D(\varphi)$ is of finite order, then $\varphi$ is of finite order, since $\mathrm{Ker}(D)$ consists of holomorphic diffeomorphisms of $X$ and consequently $\mathrm{Ker}(D)$ is finite by Hurwitz' theorem.

A diffeomorphism $\varphi$ is called *reducible*, if it is isotopic to a diffeomorphism fixing a (real) simple closed curve on $X$. If $\varphi$ is neither reducible nor elliptic, then $\varphi$ is called *pseudo-Anosov*. By [**HM79**] there exists a pair $(X,q)$ such that $\varphi$ is an affine diffeomorphism of $(X,q)$. As stated above, we will restrict to the case that $q=\omega^2$. Moreover, $(X,\omega)$ can be chosen such that $\varphi$ stretches the horizontal lines by some factor $\lambda>1$, called *dilatation coefficient*, and contracts the vertical lines by the same factor $\lambda$. Thus, $|\mathrm{tr}D(\varphi)|>2$ for an affine pseudo-Anosov diffeomorphism. The corresponding matrix $D(\phi)$ is called *hyperbolic*.

Consequently, an affine diffeomorphism $\varphi$ with $|\mathrm{tr}D(\varphi)|=2$, i.e. such that $D(\varphi)$ is *parabolic*, is a reducible affine diffeomorphism. We briefly recall the structure of such a *parabolic diffeomorphism*. Say the horizontal direction is the eigendirection of $D(\varphi)$. Then some power of $\varphi$ fixes all the finitely many horizontal saddle connections. The complement of these saddle connections has to consist of metric cylinders and $\varphi$ acts as (power of a) Dehn twist along the core curves of the cylinders. Since $\varphi$ has to be affine globally, the moduli have to be commensurable.

Conversely, composing Dehn twists in cylinders, we obtain the following proposition. We define the $\mathrm{lcm}(q_1,\ldots,q_n)$ for rational numbers $q_i$ to be the smallest positive rational number that is an integral multiple of all the $q_i$.

**Proposition 2.6.** *If the horizontal direction of $(X,\omega)$ decomposes into cylinders of moduli $m_i$ that are commensurable and $m=\mathrm{lcm}(m_1^{-1},\ldots,m_n^{-1})$, then there is an affine diffeomorphism $\varphi$ with* $D(\varphi)=\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix}$.

This idea is elaborated in the following construction.

**Thurston-Veech construction.** The following construction first appears in the famous 1976 preprint of Thurston ([**Thu88**]). See also [**Vee89**], [**HL06a**] and [**McM06a**] for recent versions and presentations.

A *multicurve* $A$ on a surface $\Sigma_g$ of genus $g$ is a union of disjoint essential simple closed curves, no two of which bound an annulus. A pair $(A,B)$ of multicurves *fills* (or *binds*) the surface if for each curve in $A$ and each curve in $B$ the geometric intersection number is minimal in their homotopy classes and if the complement $\Sigma_g\setminus(A\cup B)$ is a simply connected polygonal region with at least 4 sides. Such a set of curves is shown in Figure 1.

We index the components of $A$ and $B$ such that $A=\cup_{i=1}^{a}\gamma_i$ and $B=\cup_{i=a+1}^{a+b}\gamma_i$ and let $C$ be the (unsigned) intersection matrix of $A$ and $B$, i.e. for $i\neq j$ we have $C_{ij}=|\gamma_i\cap\gamma_j|$ and $C_{jj}=0$ for all $j$.

As additional input datum for the construction we fix a set of multiplicities $d_i\in\mathbb{N}$ for $i=1,\ldots,a+b$. Since $(A,B)$ fills $\Sigma_g$, the intersection graph is connected

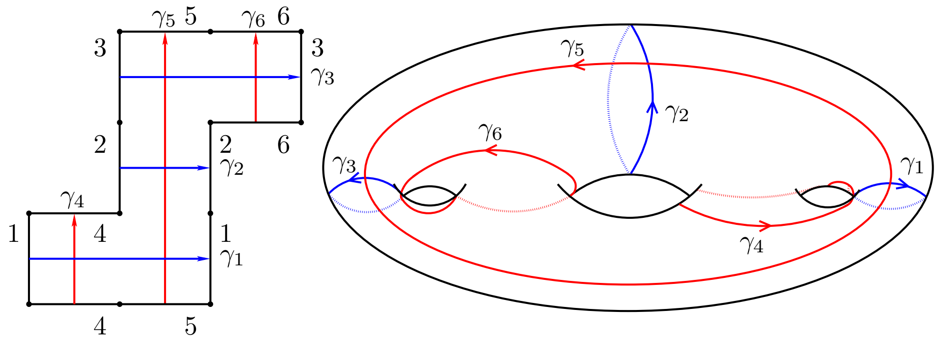


FIGURE 1. Binding curves and the surface resulting from the Thurston-Veech construction

and the matrix $(d_i C_{ij})$ is a Perron-Frobenius matrix. Hence there is a unique positive eigenvector $(h_i)$ up to scalar such that

$$\mu h_i = \sum_{j=1}^{j=a+b} d_i C_{ij} h_j$$

for some positive eigenvalue $\mu$.

We now glue a surface $X$ from rectangles $R_p = [0, h_i] \times [0, h_j] \subset \mathbb{C}$ for each intersection point $p \in \gamma_i \cap \gamma_j$. Namely, glue $R_p$ to $R_q$ along the vertical (resp. horizontal) sides whenever $p$ and $q$ are joined by an edge in $A$ (resp. $B$) of the graph $A \cup B$. The differentials $dz^2$ on each rectangle glue to a global quadratic differential $q$ on $X$. The resulting surface is also shown in Figure 1, where sides with the same label have to be identified by parallel translations.

Let $\tau_i$ be the Dehn twist around $\gamma_i$ and define

$$\begin{aligned} \tau_A &= \textstyle\prod_{i=1}^{a} \tau_i^{d_i} \\ \tau_B &= \textstyle\prod_{i=a+1}^{a+b} \tau_i^{d_i}. \end{aligned}$$

**Theorem 2.7** ([**Thu88**],[**Vee89**]). *The flat surface $(X, q)$ constructed above contains affine diffeomorphisms $\tau_A$ and $\tau_B$ with derivatives*

$$D\tau_A = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad D\tau_B = \begin{pmatrix} 1 & 0 \\ -\mu & 1 \end{pmatrix}.$$

*In particular the elements $\tau_A^n \tau_B$ are pseudo-Anosov diffeomorphisms for $n > 1$.*

PROOF. By construction the modulus $m_i$ of the cylinder with core curve $\gamma_i$ is $d_i/\mu$. Hence the powers of the Dehn twists occurring in the definition of $\tau_A$ and $\tau_B$ have linear part as claimed. They fix the boundary of the horizontal resp. vertical cylinders and together define affine diffeomorphisms.

In order to check the last claim, one has to recall that an affine diffeomorphism is pseudo-Anosov if and only if the absolute value of its trace is greater than two. □

Since we are dealing exclusively with flat surfaces in the sequel, we remark that the quadratic differential has a square root, i.e. $q = \omega^2$ if and only if for a suitable orientation of the $\gamma_i$ their geometric and algebraic intersection numbers coincide.

**Totally real fields**. A field is called totally real if all its complex embeddings factor through $\mathbb{R}$. Totally real trace fields play an important role for the classification of Teichmüller curves. The following result was first established for Teichmüller curves in [**Möl06b**]. Later Hubert and Lanneau gave the following short proof, showing that trace fields are totally real in a much more general context.

**Theorem 2.8** ([**HL06a**]). *Let $(X, \omega)$ be a flat surface. If $\mathrm{SL}(X, \omega)$ contains a parabolic and a hyperbolic element, then the trace field of $\mathrm{SL}(X, \omega)$ is totally real.*

PROOF. In the first step we show that such a surface $(X, \omega)$ arises via the Thurston-Veech construction. Take the curves in $A$ to be the core curves of the cylinders of the parabolic element $P$. Let $H$ be the hyperbolic element. Then $HPH^{-1}$ is again parabolic and its fixed direction is different from the fixed direction of $P$. Take $B$ to be the core curves of the cylinders of $HPH^{-1}$. (We may conjugate $\mathrm{SL}(X, \omega)$ within $\mathrm{SL}_2(\mathbb{R})$ so that these fixed directions become horizontal and vertical.) Then $(A, B)$ fills the surface. Take the $d_i$ for $i = 1, \ldots, |A|$ resp. for $i = |A|+1, \ldots, |A \cup B|$ to be the least common multiples of the ratios of the moduli of the cylinders in the horizontal resp. vertical direction. Then $(X, \omega)$ is just the surface obtained by the Thurston-Veech construction using the data $(A, B, d_i)$.

In the second step we show that any surface that arise via the Thurston-Veech construction has totally real trace field. We continue to use the notation of that section of Theorem 2.7. Let $D$ be the diagonal matrix with entries $d_i$. The square of the largest eigenvalue of the matrix $C$ is the largest eigenvalue of the matrix $C^2$. Hence we have to show that all the eigenvalues of $(DC)^2$ are real.

Suppose first for simplicity $d_i = 1$ for all $i$. Since for some matrix $C_0$ we have

$$DC = C = \begin{pmatrix} 0 & C_0 \\ C_0^T & 0 \end{pmatrix}, \quad \text{hence} \quad (DC)^2 = C^2 = \begin{pmatrix} C_0 C_0^T & 0 \\ 0 & C_0^T C_0 \end{pmatrix}.$$

Since $C^2$ is symmetric, all its eigenvalues are real. Thus $\mathbb{Q}(\mu^2)$ is totally real.

If the $d_i$ are no longer identically one, $(DC)^2$ is still similar to a symmetric matrix: Split $D$ into two pieces $D'$ and $D''$ of size $a$ resp. $b$ and let $D'_{\surd}$ resp. $D''_{\surd}$ denote the diagonal matrix with entries $\sqrt{d_i}$. Then

$$(DC)^2 = \begin{pmatrix} D' C_0 D'' C_0^T & 0 \\ 0 & D'' C_0^T D' C_0 \end{pmatrix}.$$

The upper block decomposes as

$$D' C_0 D'' C_0^T = D'_{\surd} (D'_{\surd} C_0 D''_{\surd})(D'_{\surd} C_0 D''_{\surd})^T (D'_{\surd})^{-1}$$

and for the lower block the same trick works. The above conclusion about the eigenvalues thus still holds. □

**Example:** The affine group of a regular $n$-gon is a triangle group generated by a rotation around the center of the $n$-gon and a parabolic element ([**Vee89**]). There are examples of surfaces whose Veech group contains pseudo-Anosovs but no parabolic elements, in particular this pseudo-Anosov does not arise from the Thurston-Veech construction ([**HL06a**]). A survey of examples of affine groups that can appear is [**Möl09**].

**2.3. Strata of $\Omega\mathcal{M}_g$ and hyperelliptic loci**

The space $\Omega\mathcal{M}_g$ parameterizing flat surfaces is stratified according to the zeros of one-forms. For $m_i \geq 1$ and $\sum_{i=1}^k m_i = 2g-2$, let $\Omega\mathcal{M}_g(m_1,\ldots,m_k)$ denote the stratum parameterizing one-forms that have $k$ distinct zeros of order $m_1,\ldots,m_k$.

For $d_i \geq -1$ and $\sum_{i=1}^s d_i = 4g-4$, let $\mathcal{Q}(d_1,\ldots,d_s)$ denote the *moduli space of quadratic differentials* that have $s$ distinct zeros or poles of order $d_1,\ldots,d_s$. For $d_i \geq 0$ this is a subset of the vector bundle of holomorphic quadratic differentials over $\mathcal{M}_g$. The condition $d_i \geq -1$ ensures that the quadratic differentials in $\mathcal{Q}(d_1,\ldots,d_s)$ have at most simple poles and thus finite volume. Namely, $\mathcal{Q}(d_1,\ldots,d_s)$ parametrizes pairs $(X,q)$ of a Riemann surface $X$ and a meromorphic section $q$ of $\omega_X^{\otimes 2}$ with the prescribed type of zeros and poles.

If the quadratic differential is not a global square of a one-form, there is a natural double covering $\pi : Y \to X$ such that $\pi^* q = \omega^2$. This covering is ramified precisely at the zeros of odd order of $q$ and at its poles. It gives a map

$$\phi : \mathcal{Q}(d_1,\ldots,d_s) \to \Omega\mathcal{M}_g(m_1,\ldots,m_k),$$

where the signature $(m_1,\ldots,m_k)$ is determined by the ramification type (see [**KZ03**] for more details).

If the domain and the range of the map $\phi$ have the same dimension for some signature, we call the image a *component of hyperelliptic flat surfaces* of the corresponding stratum. This can only happen, if the domain of $\phi$ parametrizes genus zero curves, thus justifying the terminology. More generally, if the domain of $\phi$ parametrizes genus zero curves, we call the image a *locus of hyperelliptic flat surfaces* in the corresponding stratum. These loci are often called hyperelliptic loci, e.g. in [**KZ03**] and [**EKZ11**]. We prefer to reserve the expression *hyperelliptic locus* for the subset of $\mathcal{M}_g$ (or its closure in $\overline{\mathcal{M}_g}$) parameterizing hyperelliptic curves and thus specify with 'flat surfaces' if we speak of subsets of $\Omega\mathcal{M}_g$.

**2.4. Spin structures and connected components of strata**

A *spin structure* (or *theta characteristic*) on a smooth curve $X$ is a line bundle $\mathcal{L}$ whose square is the canonical bundle, i.e. $\mathcal{L}^{\otimes 2} \sim K_X$. The *parity of a spin structure* is given by $\dim H^0(X,\mathcal{L}) \mod 2$. This parity is well-known to be a deformation invariant. The *moduli space of spin curves* $\mathcal{S}_g$ parametrizes pairs $(X,\eta)$, where $\eta$ is a theta characteristic of $X$. It has two components $\mathcal{S}_g^-$ and $\mathcal{S}_g^+$ distinguished by the parity of the spin structure. The spin structures on stable curves are defined such that the morphisms $\pi : \mathcal{S}_g^- \to \overline{\mathcal{M}_g}$ and $\pi : \mathcal{S}_g^+ \to \overline{\mathcal{M}_g}$ are finite of degree $2^{g-1}(2^g-1)$ and $2^{g-1}(2^g+1)$, respectively. (The compactification is need for intersection theory only, i.e. for some cases of Theorem 6.8 that are not treated in details in these notes. The reader is referred to [**Cor89**] for spin structures on stable curves.) If $(X,\omega) \in \Omega\mathcal{M}_g(2\ell_1,\ldots,2\ell_k)$ with zeros of $\omega$ being $P_1,\ldots,P_k$, then the line bundle $\mathcal{L} = \mathcal{O}_X(\sum_{i=1}^k \ell_i P_i)$ naturally defines a spin structure.

We can now recall the classification of connected components of strata in $\Omega\mathcal{M}_g$.

**Theorem 2.9** ([**KZ03**]). *The strata of $\Omega\mathcal{M}_g$ have up to three connected components, distinguished by the parity of the spin structure and by being hyperelliptic or not. For $g \geq 4$, the strata $\Omega\mathcal{M}_g(2g-2)$ and $\Omega\mathcal{M}_g(2\ell,2\ell)$ with an integer $\ell = (g-1)/2$ have three components, the component of hyperelliptic flat surfaces and two components with odd or even parity of the spin structure but not consisting exclusively of hyperelliptic curves.*

*The stratum $\Omega\mathcal{M}_3(4)$ has two components, $\Omega\mathcal{M}_3(4)^{\mathrm{hyp}}$ and $\Omega\mathcal{M}_3(4)^{\mathrm{odd}}$. The stratum $\Omega\mathcal{M}_3(2,2)$ also has two components, $\Omega\mathcal{M}_3(2,2)^{\mathrm{hyp}}$ and $\Omega\mathcal{M}_3(2,2)^{\mathrm{odd}}$.*

*Each stratum $\Omega\mathcal{M}_g(2\ell_1,\ldots,2\ell_r)$ for $r\geq 3$ or $r=2$ and $\ell_1\neq(g-1)/2$ has two components determined by even and odd spin structures.*

*Each stratum $\Omega\mathcal{M}_g(2\ell-1,2\ell-1)$ for $\ell\geq 2$ has two components, the component of hyperelliptic flat surfaces $\Omega\mathcal{M}_g(2\ell-1,2\ell-1)^{\mathrm{hyp}}$ and the other component $\Omega\mathcal{M}_g(2\ell-1,2\ell-1)^{\mathrm{non-hyp}}$.*

*In all the other cases, the stratum is connected.*

Consider the partition $(2,\ldots,2)$. For $(X,\omega)\in\Omega\mathcal{M}_g(2,\ldots,2)^{\mathrm{odd}}$ with $\mathrm{div}(\omega)=2\sum_{i=1}^{g-1}p_i$, the line bundle $\eta=\mathcal{O}_X(\sum_{i=1}^{g-1}p_i)$ is an odd theta characteristic. Therefore, we have a natural morphism

$$f:\Omega\mathcal{M}_g(2,\ldots,2)^{\mathrm{odd}}/\mathbb{C}^*\to\mathcal{S}_g^-\,.$$

Note that $f$ contracts the locus where $h^0(\eta)>1$. Note also that for $g=3$ Clifford's theorem implies that $h^0(\eta)=1$, i.e. $f$ is an isomorphism in this case.

### 2.5. Stable differentials and Deligne-Mumford compactification

The space $\Omega\mathcal{M}_g$ is not compact for two reasons. First, it is a vector bundle (minus the zero section), so we should rather use the bundle of projective spaces $\mathbb{P}\Omega\mathcal{M}_g=\Omega\mathcal{M}_g/\mathbb{C}^*$. (This projective space bundle is also useful when dealing with Teichmüller curves, since they will be naturally (complex) curves in $\mathbb{P}\Omega\mathcal{M}_g$, whereas in $\Omega\mathcal{M}_g$ they are objects of real dimension four.)

Second, the moduli space of curves itself is not compact. Denote by $\overline{\mathcal{M}}_g$ the Deligne-Mumford compactification of $\mathcal{M}_g$. Points in the boundary of $\overline{\mathcal{M}}_g$ are *stable curves*, i.e. projective connected algebraic curves with at most nodes as singularities and such that each irreducible component isomorphic to a projective line has at least three nodes.

The bundle of holomorphic one-forms extends over $\overline{\mathcal{M}}_g$, parameterizing *stable one-forms* or equivalently sections of the dualizing sheaf. We denote the total space of this extension by $\Omega\overline{\mathcal{M}}_g$. The stable one-forms are holomorphic except for simple poles at the nodes with the conditions that residues at the two branches of the node add up to zero. An example of a stable one-form is the form $dz/z$ on the projective line $\mathbb{P}^1$ with puncture at zero, at $\infty$ and maybe some other points. If we view the projective line $\mathbb{P}^1$ as an infinitely high cylinder with waist curve of length one (i.e. obtained by identifying $\Re(z)=0$ with $\Re(z)=1$ in the complex plane) and with the points at $i\infty$ and $-i\infty$ glued together, this one-form becomes $dz$. This metric picture of one infinite cylinder (or two half-cylinders if we moreover cut along the real axis) should be kept in mind when understanding the boundary points of Teichmüller curves.

For a stable curve $X$, denote the dualizing sheaf by $\omega_X$. We will stick to the notation that points in $\Omega\overline{\mathcal{M}}_g$ are given by a pair $(X,\omega)$ with $\omega\in H^0(X,\omega_X)$, although this notation may result in confusion since dropping the subscript $X$ drastically changes the meaning.

## 3. Curves and divisors in $\mathcal{M}_g$

The aim of this section is a short introduction to the Picard group of $\mathcal{M}_g$. This will be used to attach to any curve or divisor in $\overline{\mathcal{M}}_g$ the quantity slope.

### 3.1. Curves and fibered surfaces

The aim of this section is to show how to associate with a map $C \to \overline{\mathcal{M}}_g$ from a smooth algebraic curve $C$ to the moduli space a fibered surface and to discuss various models of that fibered surface.

Let $f : \mathcal{X} \to C$ be a smooth family of curves of genus $g$ over the smooth curve $C$, i.e. a smooth morphism with connected fibers, which are smooth curves of genus $g$. By definition of the moduli space, such a family yields a moduli map $m : C \to \mathcal{M}_g$, but the converse does not quite hold. So we will pass to finite unramified covers. More precisely, let $\mathcal{M}_g^{[n]}$ be a finite cover, that is isomorphic to the quotient of Teichmüller space by some subgroup $\Gamma_g^{[n]}$ of finite index in the mapping class group without torsion elements. Technically the notation $\mathcal{M}_g^{[n]}$ refers to level-$n$-structure and the claim on torsion elements holds for any $n \geq 3$, but we will not need the precise definition.

The moduli space $\mathcal{M}_g^{[n]}$ carries a universal family $f_{\rm univ} : \mathcal{X}_{\rm univ} \to \mathcal{M}_g^{[n]}$ that we can pullback via any map $B \to \mathcal{M}_g^{[n]}$ to obtain a family of curves $f : \mathcal{X} \to B$.

If we start with $C \to \mathcal{M}_g$, we can take $B$ to be the preimage of $C$ in $\mathcal{M}_g^{[n]}$, or rather if the map is not an embedding we obtain $B$ as the fiber product $C \times_{\mathcal{M}_g} \mathcal{M}_g^{[n]}$. In general this is not an unramified cover of $C$, but if we provide $C$ with an orbifold structure such that $C \to \mathcal{M}_g$ factors through the moduli stack, then $B \to C$ is unramified. This will happen in all the cases we need in the sequel.

Given a (not necessarily compact) curve $C$ (or $B$) as above, we denote by $\overline{C}$ (resp. $\overline{B}$) its closure, i.e. the corresponding smooth projective curve. Since $\overline{\mathcal{M}}_g$ is projective, there is a map $\overline{m} : \overline{C} \to \overline{\mathcal{M}}_g$ extending the map $m : C \to \mathcal{M}_g$. We denote by $\Delta \subset \overline{C}$ (or $\Delta_C$ and $\Delta_B$ if we need to distinguish) the preimage of the boundary of $\overline{\mathcal{M}}_g$.

The stable reduction theorem states that after a further covering of $\overline{B}$, unramified outside $\Delta$ the pullback of the map $f : \mathcal{X} \to B$ can be completed to a family of stable curves. Since $B$ is already such kind of covering of $C$, we stick to the notation of $B$ for the base curve and denote by $f : \overline{\mathcal{X}} \to \overline{B}$ this family of stable curves. Moreover we can suppose that the monodromy around the cusps of $B$ (see Section 4) is unipotent after a further finite base change unramified outside $\Delta$. Again, we assume this in the sequel but stick to the letter $B$ for the base curve.

The total space $\overline{\mathcal{X}}$ is, in general, not a smooth complex surface. Indeed it has singularities at some of the singular points of the singular fibers of $\overline{f}$. If the node in the fiber is given by the equation $x \cdot y = 0$ and $t$ is a local parameter on the base $B$, then the local equation of $\overline{\mathcal{X}}$ is $x \cdot y = t^n$ for some $n$, and this is smooth if and only if $n = 1$. In general this is a singularity of type $A_{n-1}$ ([**BHPVdV04**]).

One can resolve these singularities and obtain a smooth surface $\widetilde{\mathcal{X}}$ together with a birational map $\widetilde{\mathcal{X}} \to \overline{\mathcal{X}}$. The price we pay for that is that the induced fibration $\tilde{f} : \widetilde{\mathcal{X}} \to \overline{B}$ has no longer stable but only semistable fibers. The fiber of the map $\widetilde{\mathcal{X}} \to \overline{\mathcal{X}}$ over a singular point of type $A_{n-1}$ is a chain of $n-1$ rational curves. See [**HM98**] Proposition 3.47 and Proposition 3.48 for an algorithm how to compute the stable and semi-stable models and the references in loc. cit. for a general proof.

With a view towards Teichmüller curves, the advantage of the stable model $\overline{\mathcal{X}}$ is its direct relation to the geometry of flat surfaces (see Section 5.4), whereas calculations of intersection numbers work without correction terms on $\widetilde{\mathcal{X}}$ only.

### 3.2. Picard groups of moduli spaces

We write $\mathrm{Pic}(\cdot)$ for the rational Picard group $\mathrm{Pic}_{\mathrm{fun}}(\cdot)_{\mathbb{Q}}$ of a moduli stack (see [**HM98**] for more details). Since the quantities we are interested in, the sum of Lyapunov exponents and slopes, are invariant under coverings unramified in the interior of $\overline{\mathcal{M}}_g$, this is the group we want to calculate intersections with, not the Picard group of the coarse moduli space.

We fix some standard notation for elements in the Picard group. Let $\lambda$ denote the first Chern class of the Hodge bundle. Let $\delta_i$, $i = 1, \ldots, \lfloor g/2 \rfloor$ be the boundary divisor of $\overline{\mathcal{M}}_g$ whose generic element is a smooth curve of genus $i$ joined at a node to a smooth curve of genus $g-i$. The generic element of the boundary divisor $\delta_0$ is an irreducible nodal curve of geometric genus $g-1$. In the literature sometimes $\delta_0$ is denoted by $\delta_{\mathrm{irr}}$. We write $\delta$ for the total boundary class.

For moduli spaces with marked points we denote by $\omega_{\mathrm{rel}}$ the relative dualizing sheaf of $\overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$ and $\omega_{i,\mathrm{rel}}$ its pullback to $\overline{\mathcal{M}}_{g,n}$ via the map forgetting all but the $i$-th marked point. For a set $S \subset \{1, \ldots, n\}$ we let $\delta_{i;S}$ denote the boundary divisor whose generic element is a smooth curve of genus $i$ joined at a node to a smooth curve of genus $g-i$ and the sections in $S$ lying on the first component.

**Theorem 3.1** ([**AC87**])**.** *The rational Picard group of $\overline{\mathcal{M}}_g$ is generated by $\lambda$ and the boundary classes $\delta_i$, $i = 0, \ldots, \lfloor g/2 \rfloor$.*

*More generally, the rational Picard group of $\overline{\mathcal{M}}_{g,n}$ is generated by $\lambda$, $\omega_{i,\mathrm{rel}}$, $i = 1, \ldots, n$, by $\delta_0$ and by $\delta_{i;S}$, $i = 0, \ldots, \lfloor g/2 \rfloor$, where $|S| > 1$ if $i = 0$ and $1 \in S$ if $i = g/2$.*

Alternatively, we define $\psi_i \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$ to be the class with value $-\pi_*(\sigma_i^2)$ on the universal family $\pi : \mathcal{X} \to C$ with section $\sigma_i$ corresponding to the $i$-th marked point. We have the relation

$$\omega_{i,\mathrm{rel}} = \psi_i - \sum_{i \in S} \delta_{0;S}.$$

Consequently, a generating set of $\mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$ can also be formed by the $\psi_i$, $\lambda$ and boundary classes.

For a divisor class $D = a\lambda - \sum_{i=0}^{\lfloor g/2 \rfloor} b_i \delta_i$ in $\mathrm{Pic}(\overline{\mathcal{M}}_g)$, define its *slope* to be

$$s(D) = \frac{a}{b_0}. \tag{1}$$

### 3.3. Special divisors on moduli spaces

In the application for Teichmüller curves we do not care about the coefficients of $\delta_i$ for $i \geq 1$ in the divisor classes in $\mathrm{Pic}(\overline{\mathcal{M}}_g)$, since Teichmüller curves do not intersect these components (see Corollary 5.11). As shorthand, we use $\delta_{\mathrm{other}}$ to denote some linear combination of $\delta_i$ for $i \geq 1$. Similarly, in $\overline{\mathcal{M}}_{g,n}$ we use $\delta_{\mathrm{other}}$ to denote some linear combination of all boundary divisors but $\delta_0$. For the same reason we do not distinguish between $\omega_{i,\mathrm{rel}}$ and $\psi_i$ for a divisor class, since they only differ by boundary classes in $\delta_{\mathrm{other}}$.

*The hyperelliptic locus in $\overline{\mathcal{M}}_3$.* Denote by $H \subset \overline{\mathcal{M}}_g$ the closure of locus of genus $g$ hyperelliptic curves. We call $H$ the hyperelliptic locus in $\overline{\mathcal{M}}_g$. Note that $H$ is a divisor if and only if $g = 3$. A stable curve $X$ lies in the boundary of $H$ if there is an *admissible cover* of degree two $\tilde{X} \to \mathbb{P}^1$, for some nodal curve $\tilde{X}$ whose stabilization is $X$. We refer to [**HM98**] for the definition of admissible covers.

The class of the hyperelliptic locus $H \subset \overline{\mathcal{M}}_3$ calculated e.g. in [**HM98**, (3.165)] is given as follows:

$$H = 9\lambda - \delta_0 - 3\delta_1, \tag{2}$$

hence it has slope $s(H) = 9$.
*The Brill-Noether divisors.* A good reference for the material quickly recalled here is [**ACGH85**]. For a divisor $D$ of degree $d$ on a curve $X$, denote by

$$|D| = \{\mathrm{div}(s) + D \mid s \in H^0(X, \mathcal{L}(D)) \setminus \{0\}\}$$

the set of all effective divisors linearly equivalent to $D$. $|D|$ naturally has the structure of a projective space. A $g^r_d$ on $X$ is a projective linear subspace of $|D|$ of dimension $r$. A $g^1_d$ is called a *pencil*.

The Brill-Noether locus $BN^r_d$ in $\overline{\mathcal{M}}_g$ parametrizes curves $X$ that possesses a $g^r_d$. If the Brill-Noether number

$$\rho(g, r, d) = g - (r + 1)(g - d + r) = -1,$$

then $BN^r_d$ is indeed a divisor.

There are pointed versions of this divisor. Let $\underline{w} = (w_1, \ldots, w_n)$ be a tuple of integers. Let $BN^r_{d,\underline{w}}$ be the locus in $\overline{\mathcal{M}}_{g,n}$ of pointed curves $(X, p_1, \ldots, p_n)$ with a line bundle $\mathcal{L}$ of degree $d$ such that $\mathcal{L}$ admits a $g^r_d$ and $h^0(\mathcal{L}(-\sum w_i p_i)) \geq r$. This Brill-Noether locus is a divisor, if the generalized Brill-Noether number

$$\rho(g, r, d, \underline{w}) = g - (r + 1)(g - d + r) - r(|\underline{w}| - 1) = -1.$$

The hyperelliptic divisor and e.g. the Weierstrass divisor in $\overline{\mathcal{M}}_{g,1}$ can also be interpreted as Brill-Noether divisors.

The class of these pointed divisors has been calculated in many special cases, in particular in [**Log03**] and later in [**Far09b**]. We give two examples. The class of the classical Brill-Noether divisor was calculated in [**HM82**], in particular

$$BN^1_3 = 8\lambda - \delta_0 - \delta_{\mathrm{other}} \quad \text{for} \quad g = 5. \tag{3}$$

If all $w_i = 1$ and $n = r + 1$ the Brill-Noether divisor specializes to the divisor Lin calculated in [**Far09b**, Sec. 4.2]. In particular [**Far09b**, Thm. 4.6] gives

$$\mathrm{Lin}^1_3 = BN^1_{3,(1,1)} = -\omega_{1,\mathrm{rel}} - \omega_{2,\mathrm{rel}} + 8\lambda - \delta_0 - \delta_{\mathrm{other}} \quad \text{for} \quad g = 4. \tag{4}$$

We will illustrate the method of *test curves* for calculating the class of a divisor (see e.g. [**HM98**] for more examples). For instance, using certain test curves including a Teichmüller curve we can determine (partially, but sufficiently for our purposes) the class of $\mathrm{Lin}^1_3$. We have to use some terminology that is introduced below. The reader is invited to skip over it at a first reading and later check that we do not use circular reasoning.

**Proposition 3.2.** *The class of* $\mathrm{Lin}^1_3$ *equals*

$$\mathrm{Lin}^1_3 = k(-\omega_{1,\mathrm{rel}} - \omega_{2,\mathrm{rel}} + 8\lambda - \delta_0 - \delta_{\mathrm{other}})$$

*for some constant $k$.*

Before proceeding to the proof, we recall some facts from algebraic geometry. If $X$ is a non-hyperelliptic curve of genus 4, then its canonical image in $\mathbb{P}^3$ is contained in a unique irreducible quadric. Up to isomorphism there are two types of quadrics in $\mathbb{P}^3$: smooth quadrics, e.g. $xw = yz$ and singular quadrics, e.g. $xy = z^2$. In particular, in the smooth case, the quadric $Q$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and its Picard group is therefore isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. It is generated by the class $(1,0) = \{\mathrm{pt}\} \times \mathbb{P}^1$ and the class $(0,1) = \mathbb{P}^1 \times \{\mathrm{pt}\}$. The intersection product of two divisors $D_1 = (a,b)$ and $D_2 = (a',b')$ is therefore

$$D_1 \cdot D_2 = ab' + a'b.$$

Note also that the canonical divisor $K_Q$ has class $(-2,-2)$.

Proof. Suppose that

$$\mathrm{Lin}_3^1 = a_1\omega_{1,\mathrm{rel}} + a_2\omega_{2,\mathrm{rel}} + b\lambda - c\delta_0 - \delta_{\mathrm{other}}$$

for some unknown coefficients $a_1, a_2, b, c$. By symmetry we have $a_1 = a_2$.

To construct the first test curve, we start with a general pencil $B$ in the complete linear system $|(3,3)|$ on a smooth quadric $Q \subset \mathbb{P}^3$. Note that $|(3,3)|$ is a projective space of dimension 15. Via the isomorphism $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, its elements can be viewed as polynomials $f(x_0, x_1, y_0, y_1)$ which are bihomogenuous of degree 3 in the $x$ and $y$-coordiantes. Therefore a basis of the underlying vector space is given by $x_0^i x_1^{3-i} y_0^j y_1^{3-j}$ for $i, j = 0, \ldots, 3$. The elements of $B$ are curves $X_t$ on $Q$ cut out by an equation of the form $f_t = \sum_{i,j}(a_{ij} + tb_{ij}) x_0^i x_1^{3-i} y_0^j y_1^{3-j} = 0$ where $t \in \mathbb{P}^1$.

First note that without loss of generality, the generic member of $B$ is non-singular and irreducible. In fact, the singular curves in $|(3,3)|$ form a subset of projective dimension 14. Likewise, irreducibility is an open condition, and there is at least one irreducible curve in $|(3,3)|$. For a smooth projective curve $X$ on a surface, we can compute its genus using the adjunction formula

$$2g(X) - 2 = X \cdot (X + K_Q) = (3,3) \cdot ((3,3) + (-2,-2)) = 6,$$

hence $g(X) = 4$.

Recall that a base point of the linear system $B$ is a point $x \in Q$ that is contained in all divisors in $B$. We wish to determine the number of base points. Take two curves $X_0$ and $X_t$ in $B$. They intersect in $(3,3) \cdot (3,3) = 18$ points, counted with multiplicity. Take any of the intersection points $x_t$ on $X_0$. Now let $t$ vary and trace the image of $x_t \in X_0 \cap X_t$. If $x_t$ moved then we would obtain a non-constant map $\mathbb{P}^1 \to X_0$. But if we choose a smooth irreducible member of $B$ as $X_0$ then $g(X_0) = 4$, which yields a contradiction.

There are 18 base points in the pencil. Choose two of them as the marked points $p, q$. Since $B$ is general, $p, q$ are not contained in any ruling of $Q$, namely there is no section of a linear series $g_3^1$ that contains both $p$ and $q$. It implies that $B$ and $\mathrm{Lin}_3^1$ are disjoint in $\overline{\mathcal{M}}_{4,2}$, i.e.

$$B \cdot \mathrm{Lin}_3^1 = 0.$$

Blowing up the 18 base points, we obtain a surface $S \subset \mathbb{P}^1 \times Q$, which is a one-parameter family of genus 4 curves over $B \cong \mathbb{P}^1$. Let $\chi_{\mathrm{top}}(\cdot)$ denote the topological Euler characteristic. We have

$$\chi_{\mathrm{top}}(S) = \chi_{\mathrm{top}}(\mathbb{P}^1) \cdot \chi_{\mathrm{top}}(X) + B \cdot \delta_0,$$

where $X$ has genus equal to 4. We know that

$$\chi_{\mathrm{top}}(\mathbb{P}^1) = 2, \quad \chi_{\mathrm{top}}(X) = -6,$$

$$\chi_{\mathrm{top}}(S) = \chi_{\mathrm{top}}(Q) + 18 = \chi_{\mathrm{top}}(\mathbb{P}^1) \cdot \chi_{\mathrm{top}}(\mathbb{P}^1) + 18 = 22.$$

All together it implies there are 34 irreducible nodal curves in the family $B$, namely,

$$B \cdot \delta_0 = 34.$$

Let $\omega_{S/B}$ denote the relative dualizing sheaf of $S$ over $B$. Since $S$ has class $(1;3,3)$ in the Picard group of $\mathbb{P}^1 \times Q$, one checks that

$$\begin{aligned}\omega_{S/B} &= \omega_S - f^*\omega_B \\ &= (\omega_{\mathbb{P}^1\times Q} + S)|_S - f^*\omega_B \\ &= (-2;-2,-2) + (1;3,3) - (-2;0,0) \\ &= (1;1,1)\end{aligned}$$

on $S$, where $f : S \to B$ is the projection. Then $f_*(c_1^2(\omega_{S/B}))$ on $B$ is equal to the top intersection

$$(1;1,1)\cdot(1;1,1)\cdot(1;3,3) = 14$$

on $\mathbb{P}^1 \times Q$. Using the Noether formula $12\lambda = f_*(c_1^2(\omega_{S/B})) + \delta$, we get

$$B\cdot\lambda = \frac{1}{12}(34+14) = 4.$$

Moreover, let $\Gamma_p$ and $\Gamma_q$ be the exceptional curves corresponding to the blow-up of the two marked points. For $i = 1,2$ we have $B \cdot \omega_{i,\mathrm{rel}} = -1$, since this is the self-intersection of $\Gamma_p$ (resp. $\Gamma_q$). Note that $B$ does not intersect any boundary divisors except for $\delta_0$. Putting the above intersection numbers together, we obtain a relation

$$-a_1 + 2b + 17c = 0.$$

As the second test curve, we take a Teichmüller curve $C$ generated by a flat surface in the stratum $\Omega\mathcal{M}_4(3,3)^{\mathrm{non-hyp}}$, e.g. the square-tiled surface given by the permutations ($\pi_r = (123456789\,10), \pi_u = (19568)$). Using the algorithm in [**EKZ11**] along with equation (12) we find for this particular curve that the sum of Lyapunov exponents equals $L(C) = 2$, i.e. the slope $s(C) = 33/4$. Using Proposition 5.12 with $\kappa = 5/8$, we obtain another relation

$$-a_1 + 4b + 33c = 0.$$

The two relations imply that

$$b = -8c, \ a_1 = c$$

and this concludes the proof. □

### 3.4. Slopes of divisors and of curves in $\overline{\mathcal{M}}_g$

We summarize in this section some results on slopes of divisors and of curves. With the exception of one consequence of the Noether formula that we prove below, they are not strictly needed in the sequel. They are meant to compare the interest in calculating (the sum of) Lyapunov exponents below with a topic that is classic in algebraic geometry.

For $g \geq 4$ the canonical bundle of $\overline{\mathcal{M}}_g$ has class

$$K_{\overline{\mathcal{M}}_g} = 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \cdots - 2\delta_{\lfloor g/2\rfloor}.$$

If there is an effective divisor $D$ in $\overline{\mathcal{M}}_g$ with slope $s(D) < 13/2 = s(K_{\overline{\mathcal{M}}_g})$, then one can show that $\overline{\mathcal{M}}_g$ is of general type. This observation on the birational geometry of $\overline{\mathcal{M}}_g$ initiated the quest for divisors of low slope and led Harris-Morrison to conjecture

$$s(D) \geq 6 + \frac{12}{g+1}$$

for all effective divisors in $\overline{\mathcal{M}}_g$. They could show that this bound holds in small genera, but the conjecture is known to be false ([**FP05**]). Still there is no effective divisor with $s(D) \le 6$ known and the best known lower bounds for the slope are in the order of $1/g$ for large $g$.

We now turn our attention to curves and fibered surfaces. Let $s(C)$ be the *slope* of a curve $C \to \mathcal{M}_g$ defined by

$$s(C) = \frac{\overline{C} \cdot \delta}{\overline{C} \cdot \lambda},$$

where $\delta = \sum_{i=0}^{\lfloor g/2 \rfloor} \delta_i$ is the total boundary. (This is not exactly dual to the standard definition of the slopes of divisors in (1), but because of Corollary 5.11 the difference does not matter for the applications we have in mind.)

There are numerous results on the slope of a curve in the moduli space of curves. We will later be interested in slopes of Teichmüller curves in connection with dynamical properties. Just to put this into the right perspective we cite some results on slopes of curves from a geometric perspective.

**Proposition 3.3.** *Let $f : \mathcal{X} \to C$ be a family of curves with smooth fibers over a smooth curve $C$. The corresponding curve $C \to \mathcal{M}_g$ satisfies the slope inequality*

$$s(C) \le 12.$$

PROOF. On the smooth minimal model $\tilde{f} : \widetilde{\mathcal{X}} \to \overline{C}$ we have the Noether equality

$$12\chi(\mathcal{O}_{\widetilde{\mathcal{X}}}) - c_2(\omega_{\widetilde{\mathcal{X}}}) = c_1(\omega_{\widetilde{\mathcal{X}}})^2. \tag{5}$$

For a fibered surface with fiber genus $g$ and base genus $b$ we have by Riemann-Roch and the Leray spectral sequence (see e.g. [**Xia85**])

$$\begin{array}{rcl} \chi(\mathcal{O}_{\widetilde{\mathcal{X}}}) & = & \deg \tilde{f}_*\omega_{\widetilde{\mathcal{X}}/\overline{C}} + (g-1)(b-1), \\ c_2(\omega_{\widetilde{\mathcal{X}}}) & = & \sum_{F \text{ sing.}} \Delta\chi_{\text{top}}(F) + 4(g-1)(b-1) \\ c_1(\omega_{\widetilde{\mathcal{X}}})^2 & = & \omega^2_{\widetilde{\mathcal{X}}/\overline{C}} + 8(g-1)(b-1), \end{array},$$

where $\Delta\chi_{\text{top}}(F)$ denotes the differences of the topological Euler characteristics of the given singular fiber $F$ and a smooth fiber. We will use the equality in the form

$$12 \deg \tilde{f}_*\omega_{\widetilde{\mathcal{X}}/\overline{C}} - \sum_{F \text{ sing.}} \Delta\chi_{\text{top}}(F) = \omega^2_{\widetilde{\mathcal{X}}/\overline{C}}. \tag{6}$$

Since $\omega^2_{\widetilde{\mathcal{X}}/\overline{C}}$ is nef (Arakelov's theorem, see e.g. [**Deb82**]), its self-intersection is non-negative and we only have to check that $C \cdot \delta = \sum_{F \text{ sing.}} \Delta\chi_{\text{top}}(F)$. Both sides are additive, so we can check the contribution for each singular fiber and each node of such a fiber individually. A local equation $xy = t^n$ gives a contribution of $n$ to the intersection number. To resolve the singularity we have to replace the node by a chain of $n-1$ rational curves. The Euler characteristic of a nodal curve of geometric genus $g-1$ with such a chain differs by $n$ from the Euler characteristic of a smooth curve of genus $g$ and this proves the claim. $\square$

Slope bounds in general are studied in [**Xia87**]. Note that Xiao uses a related ratio, namely

$$\widetilde{s}(C) = \frac{\omega^2_{\widetilde{\mathcal{X}}/\overline{C}}}{\overline{C} \cdot \lambda} = 12 - s(C)$$

that he also calls slope. In our slope convention he obtains the following upper sharper bound, that was independently obtained in [**CH88**].

**Theorem 3.4.** *Let $f : \mathcal{X} \to C$ be a family of smooth curves, giving rise to a curve $C \to M_g$. Then the slope satisfies the inequality*

$$0 \leq s(C) \leq 12 - 4\frac{g-1}{g}.$$

*The lower bound is attained if and only if every fiber is smooth and irreducible. The upper bound is attained for a family of hyperelliptic curves.*

A curve is called *trigonal*, if it admits a degree three map to a projective line. For curves of genus $\leq 4$ all curves are trigonal, but from genus $g \geq 5$ on this is no longer the case. For $g = 5$ the locus of trigonal curves is the divisor $BN_3^1$ discussed in Section 3.3.

**Theorem 3.5** ([**SF00**])**.** *Suppose that $f : \mathcal{X} \to C$ is a family of trigonal curves. Then we have the slope bound*

$$s(C) \leq \frac{36(g+1)}{5g+1},$$

*which is attained for certain families of trigonal curves whose fibers are all irreducible.*

**References:** The recent survey by Farkas ([**Far09a**]) summarizes what is known on the slopes of divisors on the moduli space of curves and its consequences for the birational classification of $\overline{\mathcal{M}}_g$. The slope conjecture appears first in [**HM98**], and the first counterexample appears in [**FP05**].

One can also use Teichmüller curves to prove slope estimates for divisors. The bounds presently obtained in that way are as good (and as weak) as for 'moving' families (compare [**Che10a**] to [**Mor09**]).

## 4. Variation of Hodge structures and real multiplication

The abstract concept of a variation of Hodge structure should be viewed as a formalism of how the cohomology of a variety or a family of varieties looks like. We indicate how the weight one situation, the most important for us, mimics the situation of (families of) abelian varieties. The reader may as well think of families of curves and their Jacobians. The advantage of the abstract concept is that one can handle multilinear operators (such as dual and tensor products) easily. Even if our main interest is weight one only, we need to consider endomorphisms of those (variations of) Hodge structures and thus need the general concept.

**Hodge structures.** For any field $K \subset \mathbb{R}$, we define a (weight $k$) *$K$-Hodge structure* on the $K$-vector space $V$ to be a decomposition

$$V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=k} V^{(p,q)}$$

into $\mathbb{C}$-vector spaces, such that $\overline{V^{(p,q)}} = V^{(q,p)}$. We say that $V$ is a *$\mathbb{Z}$-Hodge structure*, if $V$ is a $\mathbb{Q}$-Hodge structure and $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. (Sometimes $\Lambda$ is called the *integral lattice*.) A *polarization* of a $K$-Hodge structure is a $\mathbb{C}$-valued bilinear form $Q$ on $V_{\mathbb{C}}$, such that the generalization of the Riemann bilinear relations hold, i.e.

$Q(V^{(p,q)}, V^{(r,s)}) = 0$ unless $p = s$ and $q = r$ and $i^{p-q}Q(\cdot, \bar{\cdot})$ is positive definite. For $\mathbb{Z}$-Hodge structures we require moreover, that the polarization is $\mathbb{Z}$-valued on $\Lambda$.

This definition is motivated by the fact that an abelian variety $A = \mathbb{C}^g/\Lambda$ gives rise to a polarized $\mathbb{Z}$-Hodge structure of weight one. In fact we define $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ and Hodge theory of complex tori states that $V = V^{(1,0)} \oplus V^{(0,1)}$ with $V^{(1,0)} = H^0(A, \Omega^1_A)$ and $V^{(0,1)} = H^1(A, \mathcal{O}_A) = \overline{V^{(1,0)}}$. The polarization is just the Hermitian form and the required conditions are met because of the Riemann bilinear relations.

Conversely, given a polarized weight one $\mathbb{Z}$-Hodge structure $(V, \Lambda, Q)$, we let $A = V^{(0,1)}/\Lambda$. The complex conjugation condition guarantees that this is a complex torus and the polarizations is a Hermitian form which is the first Chern class of a positive line bundle. Hence the complex torus is an abelian variety.

The following alternative viewpoint will generalize to families. Giving a Hodge structure is the same as giving a decreasing filtration

$$\ldots \supseteq F^p(V) \supseteq F^{p+1}(V) \supseteq \ldots$$

of $V_{\mathbb{C}}$, such that $V_{\mathbb{C}} = F^p(V) \oplus \overline{F^{k-p+1}}$. For the above decomposition one obtains the filtration by

$$F^p(V) = \bigoplus_{i \geq p} V^{(i,k-i)}$$

and conversely, the filtration determines the decomposition by

$$V^{(p,q)} = F^p(V) \cap \overline{F^q(V)}.$$

**Period domains.** From the filtration viewpoint it is obvious that the collection of all possible Hodge structures on a fixed vector space $V$ form a complex subvariety $D^\vee$ of a product of Grassmann varieties. The polarization imposes a further positivity condition and the collection of all possible polarized Hodge structures form a domain $D$ in $D^\vee$.

We specialize to the case of weight one. Then

$$D = \mathbb{H}_g = \{Z \in \mathbb{C}^{g \times g} : \; Z^T = Z \text{ and } \Im(Z) > 0\}$$

is the Siegel upper half space. Even more special, for $g = 1$, i.e. for polarized rank two weight one variations of Hodge structures the period domain is just the upper half plane $\mathbb{H}$.

**Variation of Hodge structures** The filtration viewpoint of Hodge structures generalizes to families so as to maintain the correspondence of weight one with abelian varieties. Recall that (for $K$ a field or $\mathbb{Z}$) a $K$-local system $\mathbb{V}$ on a base $B$ with fiber a $K$-vector space $V$ is just a representation of $\pi_1(B) \to \mathrm{GL}(V)$. Equivalently, one may view a local system as a vector bundle over $B$ whose transition functions are locally constant, or yet equivalently, as a vector bundle over $B$ with a flat connection $\nabla$. For families of curves (or abelian varieties), the local system will be given by the monodromy representation of $\pi_1(B)$ on the first cohomology of some fiber. Equivalently, the connection is the Gauss-Manin connection given by parallel transport of (e.g. singular) cohomology classes. Given a local system $\mathbb{V}$ on $B$, and suppose that the base is the complement of a normal crossing divisor $\Delta$ in a smooth, projective complex variety $\overline{B}$. Then there is a natural 'Deligne' extension of the vector bundle $\mathbb{V} \otimes \mathcal{O}_B$ to a vector bundle $\mathcal{V}$ on the whole base $\overline{B}$. We will not discuss the details of how to construct this extension and refer to [**Del70**, Chapter II.5] for a thorough definition.

For any field $K \subset \mathbb{R}$ and any complex ('base') manifold $B$, we define a $K$-*variation of Hodge structures* of weight $k$ (VHS for short) to be a $K$-local system $\mathbb{V}$ together with a filtration of $\mathcal{V}$

$$F^\bullet(\mathcal{V}) = (\ldots \supseteq F^p(\mathcal{V}) \supseteq F^{p+1}(\mathcal{V}) \supseteq \ldots)$$

with the following properties: i) For every point $b \in B$ the stalks of the filtration form a weight $k$ $K$-Hodge structure on the stalk $\mathbb{V}_b$. ii) Griffiths' transversality holds, i.e.

$$\nabla(F^p(\mathcal{V})) \subset F^{p-1}(\mathcal{V}) \otimes \Omega^1_{\overline{B}}.$$

We define a $\mathbb{Z}$-*variation of Hodge structures* to be a $\mathbb{Q}$-VHS $(\mathbb{V}_\mathbb{Q}, F^\bullet(\mathcal{V}))$ together with a $\mathbb{Z}$-local system $\mathbb{V}_\mathbb{Z}$ with the property that $\mathbb{V}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q} = \mathbb{V}_\mathbb{Q}$. A *polarization* of a $K$-VHS is a locally constant $\mathbb{C}$-bilinear map $Q : \mathbb{V}_\mathbb{C} \otimes \mathbb{V}_\mathbb{C} \to \mathbb{C}_B$ to the constant (rank one) local system $\mathbb{C}$ on $B$, whose stalks at every point $b \in B$ give a polarization of the induced Hodge structure on $\mathbb{V}_b$. Again, for $\mathbb{Z}$-variation of Hodge structures we require moreover, that the image $Q(\mathbb{V}_\mathbb{Z}, \mathbb{V}_\mathbb{Z})$ lies in the constant (rank one) local system $\mathbb{Z}_B$.

Recall the setup in Section 3.1 how we associated to a map $C \to \overline{\mathcal{M}}_g$ a family of curves $f : \mathcal{X} \to B$. The corresponding family of Jacobians has a weight one $\mathbb{Z}$-VHS that we can define explicitly as follows. Let $X$ be a fiber over some point $b \in B$. The local system is given by the monodromy representation of $\pi_1(B)$ on $H^1(X, \mathbb{Z})$, in sheaf theory language as the higher direct image $R^1 f_* \mathbb{Z}$.

A map between VHS is a linear map of the underlying local systems that is compatible with the filtrations. The notion of VHS obviously admits all kinds of operations of multilinear algebra. In particular, the dual of a $K$-VHS of weight $k$ is a $K$-VHS of weight $-k$. The tensor product of two $K$-VHS of weight $k_1$ and $k_2$ is a $K$-VHS of weight $k_1 + k_2$. The reader should keep in mind the particular case that $\mathbb{E}\mathrm{nd}(\mathbb{V}) = \mathbb{V}^\vee \otimes \mathbb{V}$ carries a weight zero VHS.

The most important and remarkable theorem is Deligne's semisimplicity result.

**Theorem 4.1** (Deligne ([**Del74**]),[**Sch73**, Theorem 7.25])**.** *Let $B$ be complex manifold, complement of a normal crossing divisor $\Delta$ in an algebraic manifold $\overline{B}$. If $(\mathbb{V}, F^\bullet(\mathcal{V}), Q)$ is a polarized $K$-VHS over $B$, then the monodromy representation*

$$\pi_1(B, b) \to \mathrm{GL}(\mathbb{V}_b)$$

*is completely reducible into VHS, i.e. any $\mathbb{R}$-subrepresentation has a $\pi_1(B)$-invariant complement and inherits from $F^\bullet$ a filtration that makes it into a sub-$\mathbb{R}$-VHS.*

One may want to decompose a local system into $\mathbb{C}$-irreducible pieces. Deligne defined in [**Del87**] the corresponding notion of $\mathbb{C}$-VHS and the semisimplicity theorem still holds. Since this is much less common, we chose to avoid this concept here at the cost of decomposing VHS over $\mathbb{R}$ only. This will be sufficient for the main properties of Teichmüller curves thanks to Theorem 2.8.

In Section 5 we will prove a special instance of Theorem 4.1 with an application to Teichmüller curves. We give here the necessary prerequisites that we want to reduce the theorem to.

**Theorem 4.2** ([**Sch73**, Theorem 7.22])**.** *If $B$ is a complex manifold, complement of a normal crossing divisor $\Delta$ in an algebraic manifold $\overline{B}$, and $(\mathbb{V}_K, F^\bullet(\mathcal{V}), Q)$ is* *a polarized $K$-VHS of weight $k$ over $B$, then for any flat section $e = \sum_p e_p$ the Hodge components $e_p \in V^{(p,k-p)}$ are also flat.*

INGREDIENTS OF THE PROOF: Let $\ell$ be the least integer such that $e_p = 0$ for $p > \ell$. Then one uses the following properties of period domains and period mappings. The function $\varphi = i^{2\ell-k}Q(e_\ell, e_\ell)$ is plurisubharmonic for any VHS ([**Sch73**, Lemma 7.19], see also [**CMSP03**]). It is bounded above by a consequence of the nilpotent orbit theorem, hence constant by the hypothesis on $B$. The curvature calculation that gives plurisubharmonicity can now be revisited to show that $e_\ell$ is flat. The proof concludes by induction on $\ell$. □

**Period mapping.** Suppose that locally on $U \subset B$ we have trivialized the local system $\mathbb{V}$ underlying a VHS. Then we may associate with $b \in U$ the point in the period domain determined by the filtration $F^\bullet(\mathbb{V}_b)$. This defines a map $U \to D$, if the VHS is polarized, or from the universal covering $\widetilde{B}$ of $B$ to $D$. This map is called period map, it is known to be holomorphic ([**Gri68**]).

Concretely, for a rank two weight one VHS $\mathbb{L}$, choose a basis $\{\alpha, \beta\}$ of $\mathbb{L}^\vee$ locally on $U$ (or on $\widetilde{B}$) and let $\omega(b)$ be a non-zero section of $\mathbb{L}^{(1,0)}$. Then the period map is given by

$$b \mapsto \langle \beta, \omega(b)\rangle / \langle \alpha, \omega(b)\rangle.$$

Less abstractly, take $\mathbb{L}$ to be the VHS of a family of elliptic curves, choose $\alpha, \beta$ generators of $H_1(E, \mathbb{Z})$, where $E$ is the elliptic curve in some fiber. Then the contraction $\langle \alpha, \omega(b)\rangle$ is just the integration $\int_\alpha \omega(b)$ of the one-form along some loop representing $\alpha$.

**Hodge norm.** In the section on Lyapunov exponents we want to measure sizes of cohomology classes in $H^1(X, \mathbb{R})$ where $X$ is a curve (or maybe also an abelian variety). We can build a norm on $H^1(X, \mathbb{R})$ using the polarization $Q$ as follows. Write $V = H^1(X, \mathbb{R}) \ni v = v^{(1,0)} + v^{(0,1)}$ and let $C$ be the linear map defined as $i^{p-q}$ on $V^{(p,q)}$. Then define $(v, w) = Q(Cv, \bar{w})$ and the *Hodge norm* is the associated norm $||v||^2 = (v, v)$.

Alternatively, for a curve $X$ define the Hodge-* operator as $*v = i(\overline{v^{(1,0)}} - \overline{v^{(0,1)}})$. Then $(v, w) = Q(v, *w)$ is a scalar product on $H^1(X, \mathbb{R})$ and the associated norm

$$||v||^2 = (v, v) = \frac{i}{2} \int_X v \wedge *v$$

is called the *Hodge norm* of $v$.

**Period coordinates.** As a motivation recall that the Torelli theorem states that the period map for a family of curves is locally an embedding outside the hyperelliptic locus ([**OS80**]). We may thus view a period matrix a 'coordinate system' for $\mathcal{M}_g$ outside the hyperelliptic locus, but we should keep in mind that the Torelli map is not (locally) onto, since $\dim \mathcal{M}_g < \dim \mathcal{A}_g$ for $g \geq 4$. We thus used quotation marks for this coordinate system. The period matrix entries use *all* holomorphic one-forms and all periods, but *only absolute* periods.

To contrast, for the strata $\Omega\mathcal{M}_g$ we do have *period coordinates* defined as follows. Fix locally around a given point $(X, \omega)$ a basis $\gamma_1, \ldots, \gamma_N$ of $H_1(X, Z(\omega), \mathbb{Z})$ and map a neighboring flat surface $(Y, \eta)$ to $(\int_{\gamma_i} \eta)_{i=1}^N \in \mathbb{C}^N$. This map is indeed a local isomorphism (see [**Vee90**] for a proof using flat surface geometry, [**HM79**] and [**Möl09**] for algebraic proofs). We emphasize that these period coordinates use *only one* of the one-forms on $X$ but *also relative* periods.

**4.1. Hilbert modular varieties and the locus of real multiplication**
Let $\mathcal{A}_g = \mathbb{H}_g/\mathrm{Sp}_{2g}(\mathbb{Z})$ be the moduli space of $g$-dimensional principally polarized abelian varieties, where $\mathbb{H}_g$ is the $g(g+1)/2$-dimensional Siegel upper half space. Most abelian varieties only have the endomorphisms given by multiplication by an integer. A locus where the endomorphism ring is strictly bigger will play a central role when analyzing the VHS of a Teichmüller curve. We denote by $\mathcal{R}A_{\mathfrak{o}} \subset \mathcal{A}_g$ the *locus of abelian varieties with real multiplication* by the order $\mathfrak{o}$, that we now define precisely.

Consider a totally real number field $F$ of degree $g$. A *lattice in* $F$ is a subgroup of the additive group of $F$ isomorphic to a rank $g$ free abelian group. An *order* in $F$ is a lattice which is also a subring of $F$ containing the identity element. The ring of integers in $F$ is the unique maximal order.

Let $A$ be a principally polarized $g$-dimensional abelian variety. We let $\mathrm{End}(A)$ be the ring of endomorphisms of $A$ and $\mathrm{End}^0(A)$ the subring of endomorphisms such that the induced endomorphism of $H_1(A;\mathbb{Q})$ is self-adjoint with respect to the symplectic structure defined by the polarization.

*Real multiplication* by $F$ on $A$ is a monomorphism $\rho\colon F \to \mathrm{End}^0(A)\otimes_{\mathbb{Z}}\mathbb{Q}$. The subring $\mathfrak{o} = \rho^{-1}(\mathrm{End}(A))$ is an order in $F$, and we say that $A$ has real multiplication by $\mathfrak{o}$.

The locus of real multiplication $\mathcal{R}A_{\mathfrak{o}}$ is the image of a union of Hilbert modular varieties, defined as follows. Choose an ordering $\iota_1,\dots,\iota_g$ of the $g$ real embeddings of $F$. We use the notation $x^{(i)} = \iota_i(x)$. The group $\mathrm{SL}_2(F)$ then acts on $\mathbb{H}^g$ by $A\cdot(z_i)_{i=1}^g = (A^{(i)}\cdot z_i)_{i=1}^g$, where $\mathrm{SL}_2(\mathbb{R})$ acts on the upper-half plane $\mathbb{H}$ by Möbius transformations in the usual way.

Given a lattice $M \subset F^2$, we define $\mathrm{SL}(M)$ to be the subgroup of $\mathrm{SL}_2(F)$ which preserves $M$. The *Hilbert modular variety* associated to $M$ is

$$X(M) = \mathbb{H}^g/\mathrm{SL}(M).$$

Given an order $\mathfrak{o} \subset F$, we define

$$X_{\mathfrak{o}} = \coprod_M X(M),$$

where the union is over a set of representatives of all isomorphism classes of proper rank two symplectic $\mathfrak{o}$-modules. If $\mathfrak{o}$ is a maximal order, then every rank two symplectic $\mathfrak{o}$-module is isomorphic to $\mathfrak{o}\oplus\mathfrak{o}^\vee$ (this also holds if $g=2$; see [**McM07**]), so in this case $X_{\mathfrak{o}}$ is connected. In general, $X_{\mathfrak{o}}$ is not connected, as there are non-isomorphic proper symplectic $\mathfrak{o}$-modules; see the Appendix of [**BM09**].

We now construct the map $X(M) \to \mathcal{A}_g$ in the simplified situation where $M = \mathfrak{o}\oplus\mathfrak{o}^\vee$. Pick a $\mathbb{Z}$-basis $\omega_1,\dots,\omega_g$ of $\mathfrak{o}$ and let $B = (\iota_j(\omega_k))_{j,k=1}^g$ and $A = B^{-1}$. Then the map

$$\psi : (\tau_1,\dots,\tau_g) \mapsto A\cdot \mathrm{diag}(\tau_1,\dots,\tau_g)\cdot A^T, \quad \mathbb{H}^g \mapsto \mathbb{H}_g$$

is equivariant with respect to the action of $\mathrm{SL}(\mathfrak{o}\oplus\mathfrak{o}^\vee)$ on $\mathbb{H}^g$ and its $\Psi$-image on $\mathbb{H}_g$, where $\Psi$ is defined as follows. For $a\in F$ let $a^*$ be the diagonal matrix with entries $\iota_k(a)$, write

$$\mathrm{diag}(\gamma) = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(\mathfrak{o} \oplus \mathfrak{o}^\vee)$$

and we let

$$\Psi : \gamma \mapsto \mathrm{diag}(A, (A^T)^{-1}) \cdot \mathrm{diag}(\gamma) \cdot \mathrm{diag}(A, (A^T)^{-1})^{-1}, \quad \mathrm{SL}(\mathfrak{o} \oplus \mathfrak{o}^\vee) \to \mathrm{Sp}(2g, \mathbb{Z}).$$

Consequently $\psi$ descends to the desired map $X(M) \to \mathcal{A}_g$. One easily checks that abelian varieties with the period matrix $(I_g,\ A\,\mathrm{diag}(\tau_1, \dots, \tau_g)A^T)$ or equivalently with period matrix $(B, \mathrm{diag}(B^T)^{-1})$ do indeed have real multiplication with $\mathfrak{o}$.

**A criterion for real multiplication.** The next theorem gives a criterion (see [**Del71**, Section 4.4]) how to detect from the decomposition of a weight one VHS that the corresponding family has real multiplication.

**Theorem 4.3.** *Let $\mathbb{V}$ be the weight one VHS associated with a family of abelian varieties $f : \mathcal{A} \to C$.*

*If the VHS $\mathbb{V}$ decomposes over $\mathbb{Q}$ as $\mathbb{V}_1 \oplus \mathbb{V}_2$, then the family of abelian varieties decomposes up to isogeny into two families of abelian varieties of dimension* $\mathrm{rk}(\mathbb{V}_j)/2$.

*If we have a decomposition*

$$\mathbb{V}_L = (\oplus_{\sigma \in \mathrm{Gal}(L/\mathbb{Q})/\mathrm{Gal}(L/F)} \mathbb{L}^\sigma)$$

*with the property that $\mathbb{L}^\sigma \cong \mathbb{L}^\tau$ if and only if $\sigma\tau^{-1}$ fixes $F$, then the family of abelian varieties has real multiplication by $F$.*

Proof. A homomorphism between two abelian varieties gives rise to a $\mathbb{C}$-linear map between their universal coverings and a homomorphism between their period lattices. Conversely, a homomorphism between the period lattices compatible with a $\mathbb{C}$-linear map of the universal coverings defines a homomorphism between two abelian varieties. Similar statements hold in families.

Let $\mathrm{End}(\mathbb{V}_\mathbb{Q})$ denote the global sections of the local system $\mathbb{E}\mathrm{nd}(\mathbb{V}_\mathbb{Q})$. We claim that an element of $\varphi \in \mathrm{End}(\mathbb{V}_\mathbb{Q}) \cap \mathbb{E}\mathrm{nd}(\mathbb{V})^{(0,0)}$ defines an element of $\mathrm{End}(\mathcal{A}) \otimes \mathbb{Q}$. In fact, a multiple $n\varphi$ will lie in $\mathrm{End}(\mathbb{V}_\mathbb{Z})$ and thus defines a self-map between the local system (of 'period lattices'). Lying in $\mathbb{E}\mathrm{nd}(\mathbb{V})^{(0,0)}$ says that $n\varphi$ preserves the graded pieces of the Hodge filtration. The universal covering of an abelian variety $A$ can be identified with the tangent space at zero or with $H^1(A, \mathcal{O}_A)$. Consequently, the map on the graded pieces associated with $n\varphi$ defines a family of $\mathbb{C}$-linear maps between the family of universal coverings.

In the first case the map $\mathrm{id}_{\mathbb{V}_1}$ is certainly a global section of $\mathbb{E}\mathrm{nd}(\mathbb{V}_\mathbb{Q})$ that lies in $\mathbb{E}\mathrm{nd}(\mathbb{V})^{(0,0)}$. In the second case we map $a \in F$ to $\sum_{\sigma \in \mathrm{Gal}(L/\mathbb{Q})/\mathrm{Gal}(L/F)} \sigma(a) \cdot \mathrm{id}_{\mathbb{L}^\sigma}$. This endomorphism certainly lies in $\mathbb{E}\mathrm{nd}(\mathbb{V})^{(0,0)}$. It also lies in $\mathrm{End}(\mathbb{V}_L)$ and the action of $\mathrm{Gal}(L/\mathbb{Q})$ just renumbers the summands by the hypothesis on the $\mathbb{L}^\sigma$. Consequently, this endomorphism lies in $\mathrm{End}(\mathbb{V}_\mathbb{Q})$. □

### 4.2. Examples

We present some examples of families of curves whose VHS decomposes. We will see later that the examples are related to Teichmüller curves.

**Cyclic coverings.** Consider the family of curves $f : \mathcal{X} \to C$ with

$$\mathcal{X} : y^N = x^{a_1}(x-1)^{a_2}(x-t)^{a_3}$$

over $C = \mathbb{P}^1$ with coordinate $t$. Define $0 < a_4 < N$ such that $a_1 + a_2 + a_3 + a_4 \equiv 0 \mod N$. This is a family of covers of the projective line ramified over 4 points $0, 1, t, \infty$. The automorphism $\varphi(x, y) = (x, \zeta_N y)$ generates the Deck group of this covering. It acts on $H^1(X, \mathbb{C})$, $H^0(X, \Omega^1_X)$ and $H^1(X, \mathcal{O}_X)$ for every fiber $X$ of $\mathcal{X}$ and the decomposition of $H^1(X, \mathbb{C})$ commutes with parallel transport. Consequently, the VHS $\mathbb{V}$ decomposes over $\mathbb{C}$ (see the remark after Theorem 4.1) into the eigenspaces of $\varphi$. The quotient of $\mathcal{X}$ by the group generated by $\varphi$ is a family of projective spaces $\mathbb{P}^1_x$ with coordinate $x$. Only when considered as family of projective spaces with 4 marked points $0, 1, t, \infty$, this is a non-trivial family over $\mathbb{P}^1_t$.

More concretely, one can explicitly write down a basis of holomorphic one-forms in the eigenspace with eigenvalue $\zeta_N^i$ as

$$\omega_j^i = \frac{x^{j-1} y^i dx}{x^{b_{1,i}}(x-1)^{b_{2,i}}(x-t)^{b_{3,i}}}$$

for $j = 1, \ldots, 2 - b(i)$, where $b_{k,i} = \lfloor i a_k / N \rfloor$ and where $b(i) = \sum_{k=1}^4 \langle i a_k / N \rangle - 1$. Here $\langle \cdot \rangle$ denotes the fractional part. Similarly, one can represent elements in $H^1(X, \mathcal{O}_X)$ e.g. in Czech cocycles explicitly.

As a result, one obtains that for all $i$ the eigenspaces $\mathbb{L}_i$ corresponding to the eigenvalue $\zeta_N^i$ is of rank two. The details appear in many places in the literature, e.g. [**Bou01**], [**BM10b**], [**EKZ10**].

For special values of the covering parameters the curve $\mathcal{X}$ has additional automorphisms. E.g. take $m, n$ odd and coprime, let $N = 2mn$ and $a_i = mn \pm m \pm n$. Then the $(\mathbb{Z}/2\mathbb{Z})^2$-action on $\mathbb{P}^1_x$ acting as double transpositions of the 4 branch points lifts to a $(\mathbb{Z}/2\mathbb{Z})^2$-action on $\mathcal{X}$. The VHS of the quotient family decomposes over the *reals* into rank two VHS $\mathbb{L}_i$, where the $\mathbb{L}_i$ are isomorphic to the $\mathbb{L}_i$ appearing in the VHS associated with $f : \mathcal{X} \to C$, but only subset of indices $i$ of $\{1, \ldots, N\}$ appears. Consequently, by Theorem 4.3 the quotient family has RM (on some part of its Jacobian). Details can be found in [**BM10b**].

## 5. Teichmüller curves

A *Teichmüller curve* $C \to \mathcal{M}_g$ is an algebraic curve in the moduli space of curves that is totally geodesic with respect to the Teichmüller metric.

We do not recall the definition of this metric since we only use two consequences. First, the Teichmüller metric on Teichmüller space $T_g$ is the same as the Kobayashi metric explained below. Moreover, if $C \to \mathcal{M}_g$ is a Teichmüller curve the universal covering map is a map $\mathbb{H} \to T_g$ to Teichmüller space. This map is also totally geodesic with respect to the Teichmüller metric and those totally geodesic maps $\mathbb{H} \to T_g$ are called *Teichmüller discs*. The second property of the Teichmüller metric we use is that every Teichmüller disc is the $\mathrm{SL}_2(\mathbb{R})$-orbit of a flat surface $(X, \omega)$ or a half-translation surface $(X, q)$ (see e.g. [**Hub06**, Chapter 5.3]). We say that the $(X, \omega)$ or $(X, q)$ *generates* the Teichmüller curve. In the sequel we exclusively consider Teichmüller curves generated by flat surfaces $(X, \omega)$. For Teichmüller curves generated by a half-translation surface $(X, q)$ one can pass to the canonical double covering and then apply the results below for some information about their trace fields, slopes etc.

Since $C$ is an algebraic curve and $C \cong \mathbb{H}/\mathrm{SL}(X,\omega)$ ([**McM03a**]), this implies that $\mathrm{SL}(X,\omega)$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$. A surface $(X,\omega)$ with this property is called a lattice surface or, more frequently, a *Veech surface.*

Consequently, a Teichmüller curve is the image of the projection of a closed $\mathrm{SL}_2(\mathbb{R})$-orbit from $\Omega\mathcal{M}_g$ to $\mathcal{M}_g$. Also the converse holds:

**Theorem 5.1** (Smillie, Weiss, [**Vee95**], [**SW04**]). *If the* $\mathrm{SL}_2(\mathbb{R})$*-orbit of* $(X,\omega)$ *is closed then* $(X,\omega)$ *generates a Teichmüller curve.*

We emphasize that Teichmüller curves are closed in $\mathcal{M}_g$, but never closed in $\overline{\mathcal{M}}_g$, i.e. they are never compact curves, as we have seen in Proposition 2.4.

**Motivation: Veech dichotomy.** A flat surface satisfies *Veech dichotomy* or is *dynamically optimal* if the following property of the straight line flow $\phi_t^\theta$ holds. For every fixed direction $\theta$ either $\phi_t^\theta$ is uniquely ergodic or all trajectories of $\phi_t^\theta$ are closed. (Here saddle connections also count as closed trajectories.)

The fact that Veech surfaces are dynamically optimal is one of the key observations of [**Vee89**], but the proof has to be fixed concerning the unique ergodicity statement. See [**MT02**] for a complete proof. The property Veech dichotomy is not quite characterizing Veech surfaces: this holds for $g = 2$ ([**McM05b**]), but counterexamples exist for higher genus. The most frequently used consequence of Veech dichotomy is that on a Veech surface every direction that contains a saddle connection is indeed periodic.

We give examples of Teichmüller curves before we turn to their algebraic characterization.

### 5.1. Square-tiled surfaces and primitivity

A *square-tiled surface* is a flat surface $(X,\omega)$, where $X$ is obtained as a covering of a torus ramified over one point only and $\omega$ is the pullback of the holomorphic one-form on the torus. ('Parallelogram-tiled' would be slightly more accurate, but square-tiled has become standard terminology.) The affine group of the torus is $\mathrm{SL}_2(\mathbb{Z})$ and such coverings change the affine group only by a finite amount. For a more precise statement we introduce the following notions. Two subgroups $\Gamma_1$ and $\Gamma_2$ of $\mathrm{SL}_2(\mathbb{R})$ are called *commensurable*, if there is a subgroup $\Gamma$ that has finite index both in $\Gamma_1$ and in $\Gamma_2$.

A *translation covering* $\pi : (X,\omega) \to (Y,\eta)$ is a covering $\pi : X \to Y$ of Riemann surfaces such that $\omega = \pi^*\eta$.

**Theorem 5.2** ([**GJ00**]). *Let* $\pi : (X,\omega) \to (Y,\eta)$ *be a translation covering. If* $\pi$ *is branched only over* $Z(\eta)$ *or if* $g(Y) = 1$ *and* $\pi$ *is branched over at most one point, then* $\mathrm{SL}(X,\omega)$ *and* $\mathrm{SL}(Y,\eta)$ *are commensurable.*

In view of Proposition 2.5 we restate that the trace field of a square-tiled surface is $\mathbb{Q}$. Square-tiled surfaces are a rich source of examples of Teichmüller curves. More precisely:

**Proposition 5.3.** *For every* $g$ *and each connected component of every stratum of* $\Omega\mathcal{M}_g$*, the set of Teichmüller curves generated by square-tiled surfaces is dense (for the analytic topology).*

Proof. If a flat surface $(X,\omega)$ has all its period coordinates in $\frac{1}{N}\mathbb{Z}[i]$ for some $N$, then $X$ admits a branched cover to the square torus branched over one point only. In fact, for any point $P \in X$ choose a path $\gamma$ joining a zero of $\omega$ to $X$ and map $P$ to $N \cdot \int_\gamma \omega$. This map is well-defined by hypothesis.

Since period coordinates are indeed coordinates on every stratum, we may find a point with period coordinates in $\frac{1}{N}\mathbb{Z}[i]$ (for appropriately large $N$) in a neighborhood of every point of that stratum. □

There is a convenient way to present square-tiled surfaces. In order to specify such a surface, with say $d$ squares, it suffices to specify the monodromy of the covering. The fundamental group of a once-punctured torus is a free group on two generators. This monodromy is thus given by two permutations $(\pi_u, \pi_r)$ on $d$ letters, corresponding to going up and going to the right, respectively. The square-tiled surfaces drawn in Figure 2 is given in permutation representation by

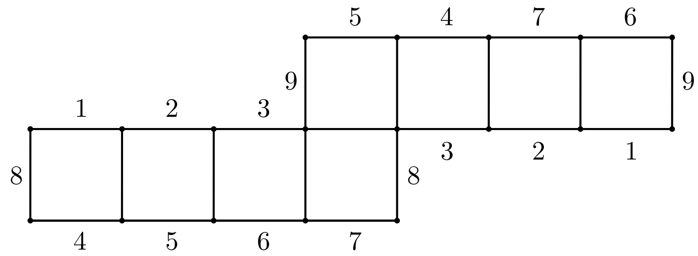


FIGURE 2. The 'eierlegende Wollmilchsau'

$(\pi_r = (1234)(5678), \pi_u = (1836)(2745))$. The name will be explained later.

**Algebraic and geometric primitivity.** From the point of view of classification of Teichmüller curves, square-tiled surfaces can be thought of as just one example plus a large amount of combinatorial decoration. One thus wants to classify Teichmüller curves generated from flat surfaces that do not stem from coverings. More precisely, we call a flat surface $(X, \omega)$ *geometrically primitive* if there is no translation covering $\pi : (X, \omega) \to (Y, \eta)$ with $g(Y) < g(X)$. A technical notion that is easy to check and a criterion of geometric primitivity is the following. Recall first, that the trace field of the affine group is unchanged under translation coverings. This is proven in [**McM03b**]. The proof combines Theorem 5.2 with the ideas in the proof of Proposition 2.5. We say that a Veech surface $(X, \omega)$ is *algebraically primitive*, if $g(X) = [K : \mathbb{Q}]$, where $K$ is the trace field of $\mathrm{SL}(X, \omega)$. Examples of algebraically primitive Veech surfaces are the $n$-gons where $n$ is an odd prime ([**Vee89**]).

If $(X, \omega)$ is algebraically primitive, then $(X, \omega)$ is geometrically primitive by Proposition 2.5. The converse does not hold in general. An infinite series of examples for this situation are given in [**McM06a**]. The surface in Figure 1 is an example of this situation if we perform the Thurston-Veech construction with multiplicities $d_n = (n, 1, n, 1, n, 1)$. We say that a Teichmüller curve is (algebraically) primitive, if a generating Veech surface is.

**Proposition 5.4.** *The equivalence class of a flat surface $(X, \omega)$ given by the relation 'translation covering' contains an elliptic curve or a unique 'minimal' element, i.e. there is a flat surfaces $(Y, \eta)$ such that all translation surfaces in the class of $(X, \omega)$ admit a translation covering to $(Y, \eta)$.*

If the equivalence class contains an elliptic curve, i.e. if $(X, \omega)$ is square-tiled, the uniqueness of the minimal element does not hold, since we may always postcompose $\pi$ by isogenies.

The major classification goal for Teichmüller curves seeks to classify those curves generated by the minimal representatives. The proof of this proposition is one of the corollaries in the next section.

### 5.2. The VHS of Teichmüller curves

The reader should now recall the conventions for curves and flat surfaces that we set up in Section 3.1. Let $L/\mathbb{Q}$ be a Galois closure of $K/\mathbb{Q}$.

**Theorem 5.5** ([**Möl06b**, Prop. 2.4])**.** *Let $B$ be a finite unramified cover of a Teichmüller curve generated by a flat surface $(X, \omega)$ and let $f : \mathcal{X} \to B$ be the universal family. Then the $\mathbb{R}$-variation of Hodge structures defined by $B$ decomposes into sub-VHS*

$$R^1 f_* \mathbb{R} = (\oplus_{\sigma \in \mathrm{Gal}(L/\mathbb{Q})/\mathrm{Gal}(L/K)} \mathbb{L}^\sigma) \oplus \mathbb{M},$$

*where $\mathbb{L}$ is the VHS with the standard 'affine group' representation of $\mathrm{SL}(X, \omega) \subset \mathrm{SL}_2(\mathbb{R})$, where $\mathbb{L}^\sigma$ are the Galois conjugates and where $\mathbb{M}$ is just some representation. Moreover, we have*

$$2 \deg(\mathbb{L}^{(1,0)}) = 2g(B) - 2 + |\Delta_B|.$$

There is also a converse of this, i.e. a characterization of Teichmüller curves.

**Theorem 5.6** ([**Möl06b**, Theorem 2.13])**.** *Let $f : \overline{\mathcal{X}} \to \overline{B}$ be a fibered surface and suppose that the VHS has a rank two sub-VHS $\mathbb{L}$, such that*

$$2 \deg(\mathbb{L}^{(1,0)}) = 2g(B) - 2 + |\Delta_B|.$$

*Then $B \to \mathcal{M}_g$ is a finite unramified covering of a Teichmüller curve $C \to \mathcal{M}_g$ generated by a flat surface $(X, \omega)$.*

A sub-VHS $\mathbb{L}$ as in the statement of the theorem was called 'maximal Higgs' in [**Möl06b**]. Very few is known on which representations occur for $\mathbb{M}$ or what their numerical data (see the subsequent chapters) are. Note that for square-tiled surfaces $\mathbb{M}$ is almost the whole VHS!

We give two applications before we turn to the proof of these theorems.

Proof of Proposition 5.4. Suppose that $f : \mathcal{X} \to B$ and $g : \mathcal{Y} \to B$ are two families of curves over a (finite covering of a) Teichmüller curve that arise from a translation covering $\pi : (X, \omega) \to (Y, \eta)$ of generating flat surfaces. Then the VHS associated with both $f$ and $g$ contains the generating local system $\mathbb{L}$ and hence also its Galois conjugates. Note that the one-form $\omega$ on the family of Jacobians is zero on any complement to the abelian variety associated with $\oplus_\sigma \mathbb{L}^\sigma$.

Consequently, given $(X, \omega)$ we take the limit over all (ordered by inclusion) abelian subvarieties $A \subset \mathrm{Jac}(X)$ such that $\omega|_A = 0$ of the normalization of the image $Z$ of $X$ in $\mathrm{Jac}(X)/A$. Obviously, we can provide $Z$ with a one-form $\omega_Z$, such that $(X, \omega)$ is a translation covering of $(Z, \omega_Z)$. Moreover any $(Y, \eta)$ covers $(Z, \omega_Z)$ since we may take $A = \mathrm{Ker}(\mathrm{Jac}(X) \to \mathrm{Jac}(Y))$, where the map comes from viewing Jac as Albanese variety. □

**Corollary 5.7.** *The family of Jacobians associated with the fibered surface $f : \mathcal{X} \to B$ over a Teichmüller curve decomposes up to isogeny into two families of abelian* *varieties $g_1 : \mathcal{A}_1 \to B$ and $g_2 : \mathcal{A}_2 \to B$ of dimension $r = [K : \mathbb{Q}]$ and $g - r$ respectively. Moreover, $g_1 : \mathcal{A}_1 \to B$ has real multiplication by $K$.*

Proof. Given the decomposition in Theorem 5.5 and the criterion Theorem 4.3 it suffices to verify the condition in the second statement of that criterion. Moreover, once we verify this condition for $\mathbb{L}$ it follows for the other $\mathbb{L}^\sigma$ by Galois conjugation.

If $\mathbb{L} \cong \mathbb{L}^\sigma$ then the traces of the underlying representation have to be fixed by $\sigma$, hence $\sigma$ has to fix $K$. Conversely, suppose that $\mathbb{L} \not\cong \mathbb{L}^\sigma$ for some $\sigma$ fixing $K$. Then both $\mathbb{L}$ and $\mathbb{L}^\sigma$ appear in the VHS decomposition of $\mathbb{V}$. Let $\varphi$ be some pseudo-Anosov element in $\mathrm{SL}(X,\omega)$ with dilatation $\lambda$ and $t(\lambda) = \mathrm{tr}(D\varphi) = \lambda + \lambda^{-1} > 2$. Both the fibers of $\mathbb{L}$ and of $\mathbb{L}^\sigma$ in $H^1(X,L)$ are in the kernel of $\varphi^* + (\varphi^*)^{-1} - t \cdot \mathrm{id} \in \mathrm{End}(H^1(X,L))$. Since the function $t(\lambda)$ is monotone for $\lambda \in (1,\infty)$, this implies that $\lambda$ is a multiple eigenvalue of the action of $\varphi$ on $H^1(X,L)$. But $\lambda$ is known to be the largest eigenvalue of this action and simple by Perron-Frobenius (see e.g. [**McM03a**, Theorem 5.3]). This contradiction concludes the proof. □

**Real multiplication and the classification problem.** We give some indication of why real multiplication might be useful to classify Teichmüller curves and a warning why one should not be too optimistic. See Section 5.5 for more details on the classification problem.

Suppose for simplicity that we are interested in algebraically primitive Teichmüller curves only. The locus of real multiplication $\mathcal{R}A_{\mathfrak{o}}$ in $\mathcal{A}_g$ is the image of Hilbert modular varieties $X(M)$. They are of dimension $g$ while $\dim \mathcal{A}_g = g(g+1)/2$. Since $\dim \mathcal{M}_g = 3g-3$, for large enough $g$ the expected dimension of the intersection $X(M) \cap \mathcal{M}_g$ considered in $\mathcal{A}_g$ via the map defined in Section 4.1 resp. via the Torelli map is zero. Hence there should be no algebraically primitive Teichmüller curves for large genus – which is known to be false as is shown in the initial paper [**Vee89**] already.

On the other hand the union of all Hilbert modular varieties is Zariski dense in $\mathcal{A}_g$ by the Borel density theorem, so this explains why one should be cautious.

There is an analogous conjecture of Coleman stating that there are no Shimura curves (see Section 6.5) in $\mathcal{A}_g$ whose generic point lies in $\mathcal{M}_g$ for $g$ large enough. Again, being a Shimura curve gives additional endomorphisms or at least additional Hodge cycles, so one can make a similar dimension heuristics. But there has not been much progress in the last decade on this question.

### 5.3. Proof the VHS decomposition and real multiplication

The proof of Theorem 5.5 was given in [**Möl06b**] using $\mathbb{C}$-VHS and the corresponding $\mathbb{C}$-version of the Semisimplicity Theorem 4.1. Moreover, the fact that trace fields are totally real was deduced from that proof. We give here the proof for algebraically primitive Teichmüller curves, reducing the statement to Theorem 4.2. Theorem 2.8 allows us to stay entirely within the more well-known context of $\mathbb{R}$-VHS.

Proof (Algebraically primitive case): Let $\mathbb{V} = R^1 f_* \mathbb{R}$ be the $\mathbb{R}$-local system underlying the weight one VHS of the given family of curves. Since we deal with a Teichmüller curve, the family is generated as the $\mathrm{SL}_2(\mathbb{R})$-orbit of some flat surface $(X,\omega)$. That is, $X$ is the fiber of $f$ over some point $b \in B$ and in that fiber $\Re(\omega)$ and $\Im(\omega)$ generate a rank two real sub-vector space $L$ of $\mathbb{V}_b = H^1(X,\mathbb{R})$.

All the other fibers of $f$ are the image of $(X, \omega)$ for some matrix $A \in \mathrm{SL}_2(\mathbb{R})$. All elements in $\gamma \in \pi_1(B)$ can be represented in the universal covering of $B$ by a path from $(X, \omega)$ to $A_\gamma \cdot (X, \omega)$ for some matrix $A_\gamma \in \mathrm{SL}(X, \omega) \subset \mathrm{SL}_2(\mathbb{R})$. The matrix $A_\gamma$ acts, by definition of the action of $\mathrm{SL}_2(\mathbb{R})$ by the standard representation on $L$. In particular it preserves the subspace $L$ and we have found a local subsystem $\mathbb{L}$.

Now we use the simplifying additional hypothesis that $r = g$. Consequently, there is a collection of $\gamma_i \in \mathrm{SL}(X, \omega)$ such that the traces of $\gamma_i$ generate a field of degree $r$ over $\mathbb{Q}$. We denote by $\iota_1, \ldots, \iota_r$ the different embeddings of $K$ into $\mathbb{R}$ and choose automorphisms $\sigma_j$ of the Galois closure of $K/\mathbb{Q}$ such that $\sigma_j \circ \iota_1 = \iota_j$. Since $\mathbb{V}$ is defined over $\mathbb{Q}$ the $\sigma_j$-images of $\mathbb{L}$ are also $K$-local subsystems of $\mathbb{V}$. Since the standard two-dimensional representation of the lattice $\mathrm{SL}(X, \omega)$ is irreducible, so are the Galois conjugates and consequently, $\mathbb{L}^{\sigma_j} \cap \mathbb{L}^{\sigma_k} = 0$ for $j \neq k$.

To sum up, we know that

$$R^1 f_* \mathbb{R} = (\oplus_{j=1}^r \mathbb{L}^{\sigma_j})$$

as local systems. Said differently, we have a decomposition $V = H^1(X, \mathbb{R}) = (\oplus_{j=1}^r L_j)$ into $\pi_1(B)$-stable sub-vector spaces and we have to show that the Hodge filtration

$$0 \subset F^1(\mathcal{V}) \subset \mathcal{V} = (R^1 f_* \mathbb{R}) \otimes_{\mathbb{R}} \mathcal{O}_B$$

intersects each of the summands $\mathbb{L}_j \otimes_{\mathbb{R}} \mathcal{O}_B$ in a vector bundle. Dimensions of intersections of vector bundles are lower semicontinuous, and if the dimension is constant, such an intersection is again a vector bundle. Hence it suffices to show that at a general point (which we may suppose $b \in B$ to be)

$$L_j = (L_j \cap V^{(1,0)}) \oplus (L_j \cap V^{(0,1)}) \tag{7}$$

for all $j$, since then a dimension jump in a special fiber would lead to a dimension contradiction.

Consider the element $P_j \in \mathrm{Hom}(V, V)$ that consists of projection to the subspace $L_j$ composed with the inclusion. Since $L_j$ is $\pi_1(B)$-invariant, $P_j$ is a flat global section of the local system $\mathrm{Hom}(\mathbb{V}, \mathbb{V})$.

On the other hand, $\mathrm{Hom}(\mathbb{V}, \mathbb{V})$ carries a Hodge structure of weight zero and we may decompose $P_j$ into its Hodge components

$$P_j = P_j^{(1,-1)} \oplus P_j^{(0,0)} \oplus P_j^{(-1,1)},$$

where the upper index indicates the shift in bidegree, if we consider $P_j$ as endomorphism. Now we apply Theorem 4.2 to conclude that all the components, in particular $P_j^{(1,-1)}$ and $P_j^{(-1,1)}$ are also flat. The only flat global sections of $\oplus_{j=1}^r L_j$ are $\mathbb{C}^g$ acting diagonally, since the $L_j$ (as $\pi_1(B)$-representations) are irreducible. In particular, no power of a non-zero global section vanishes. Since $(P_j^{(1,-1)})^2 = 0 = (P_j^{(-1,1)})^2$, these sections have to be zero. Hence $P_j = P_j^{(0,0)}$ and this implies (7), which we needed to show. □

Instead of fully proving the characterization we highlight one of the main arguments of the proof, the use of the Kobayashi metric in the next proposition. A fully self-contained proof will appear soon in a paper of A. Wright.

**Proposition 5.8.** *Let $f : \overline{\mathcal{X}} \to \overline{B}$ be a fibered surface and suppose that the VHS has a rank two sub-VHS $\mathbb{L}$, such that the monodromy representation underlying $\mathbb{L}$ is the Fuchsian representation of $\pi_1(B) = \mathbb{H}/\Gamma$ in $\mathrm{SL}_2(\mathbb{R})$. Then $B \to \mathcal{M}_g$ is a finite unramified covering of a Teichmüller curve $C \to \mathcal{M}_g$.*

This is the only place in this text, where the Teichmüller metric actually appears. We use that the Teichmüller metric on Teichmüller space is indeed equivalent to the Kobayashi metric $k_W$, which is defined for any complex space $W$ as follows.

We denote by $d_\Delta$ the Poincaré metric on the unit disc $\Delta$ and for all $x, y \in W$ we define a *chain* from $x$ to $y$ by points $x_0, x_1, \ldots, x_n \in \Delta$ together with maps holomorphic $f_i : \Delta \to W$ such that

$$f_1(x_0) = x, \quad f_j(x_j) = f_{j+1}(x_j),\ j = 1, \ldots, n-1, \quad f_n(x_n) = y.$$

Then

$$k_W(x, y) = \inf \sum_{i=1}^{n} d_\Delta(x_{i-1}, x_i),$$

where the infimum is over all chains from $x$ to $y$.

From the definition already one can deduce that all holomorphic maps are distance non-increasing for the Kobayashi metric and that a composition of two holomorphic maps is a Kobayashi isometry only if the first map is a Kobayashi isometry, too.

Proof. The VHS $\mathbb{L}$ gives rise to a period map $p$ from the universal cover of $B$ to the period domain of $\mathbb{L}$, i.e. $p : \mathbb{H} \to \mathbb{H}$. By definition, this map is equivariant with respect to the action of $\pi_1(B)$ on the domain and by the monodromy representation on its range. Since the image of the monodromy representation is Fuchsian, we may pass to the quotient map $\overline{p}$. Since the two quotient surfaces are isomorphic to $B$ and since we may assume that $g(B) \geq 2$, Riemann-Hurwitz implies that $\overline{p}$ is an isomorphism. This implies that $p$ is a Mobius transformation. On the other hand, $p$ factors as

$$p : \mathbb{H} \to T_g \to \mathbb{H}_g \to \mathbb{H},$$

where the first two maps are the universal covering maps associated with the maps $C \to \mathcal{M}_g$ and with the Torelli map $\mathcal{M}_g \to \mathcal{A}_g$. The composition $\mathbb{H} \to \mathbb{H}_g$ is the period map for the full VHS $\mathbb{V}$, and since $\mathbb{L}$ is a factor of $\mathbb{V}$, we may represent the period map of $\mathbb{V}$ as a composition of this map and the projection onto one factor.

Finally, $p$ is a composition of holomorphic maps and an isomorphism, hence an isometry for the Kobayashi metric. Consequently, the first map $\mathbb{H} \to T_g$ is an isometry for the Kobayashi metric, too. This implies that $B \to \mathcal{M}_g$ is a finite cover of a Teichmüller curve. $\square$

### 5.4. Cusps and sections of Teichmüller curves

A Teichmüller curve $C$ is obtained as $\mathbb{H}/\mathrm{SL}(X, \omega)$, so the set of cusps $\Delta$ corresponds to the set of $\mathrm{SL}(X, \omega)$-conjugacy classes of maximal parabolic elements. In this section we first describe the stable curves associated with $\Delta_C \subset \overline{\mathcal{M}}_g$ and also neighborhoods of these points to perform intersection theory calculations. We then construct sections of the family of curves over $C$ using the singularities $Z(\omega)$ and calculate their intersection number.

Let $\theta$ be a fixed direction of some parabolic element of $\mathrm{SL}(X, \omega)$. By conjugation we may suppose that $\theta$ is the horizontal direction. Then the geodesic $g_{-t}(X, \omega)$ runs into the cusp. We now describe how to obtain the surfaces along this geodesic. The horizontal direction is parabolic, in particular decomposes into maximal cylinders $C_i$ of heights $h_i$ and widths $w_i$ for $i \in I$. Cut the surface $X$ open along the core geodesics (at height $h_i/2$) of the cylinders. On the cut-open surface $X^0$ the original cylinders are decomposed into their top and bottom part $C_i^\top, C_i^\perp \subset X^0$. We define strips $U_i^\top \subset C_i^\top$ as the lower half (of height $h_i/4$) of the top part and $U_i^\perp \subset C_i^\perp$ as the top half (of height $h_i/4$) of the bottom part. For each $i$ take a cylinder $Z_i$ of height $(e^{2t} - 1/2)h_i$ and width $w_i$. Glue the top strip of height $h_i/4$ of $Z_i$ to $U_i^\top$ and the bottom strip of height $h_i/4$ of $Z_i$ to $U_i^\perp$ to form a flat surface $(Y_t, \eta_t)$.

Obviously, the holomorphic map $z \mapsto e^{-t}z$ on each cylinder extends to a biholomorphism between the two Riemann surfaces $g_{-t}(X, \omega)$ and $(Y_t, \eta_t)$.

Finally we recall that the topology from the complex structure of $\overline{\mathcal{M}}_g$ can be phrased in quasi-conformal language as follows. A basis of neighborhoods of the stable surface $X_\infty$ consists of stable curves $X$ together with ('smaller and smaller') compact sets $V_X$ and $V_\infty$ around the cusps of $X$ and $X_\infty$ such that there is a $K$-quasiconformal map $\varphi : X \setminus V_X \to V_\infty$ (with $K$ close to one). This together with the previous description implies that $g_{-t}(X, \omega)$ converges to the stable curve where we replace each cylinder $C_i$ of $(X, \omega)$ by two half-infinite cylinders. We summarize:

**Proposition 5.9.** *The stable curves corresponding to the boundary points $\Delta_C$ of a Teichmüller curve $C \to \mathcal{M}_g$ are obtained by choosing a parabolic direction of a generating flat surface $(X, \omega)$ and replacing each cylinder by a pair of half-infinite cylinders whose points at $i\infty$ resp. at $-i\infty$ are identified.*

Since half-infinite cylinders are conformal to punctured discs, we may equivalently replace each cylinder by two punctured discs to obtain the (punctured) normalization of the stable curve. The stable curve itself is then obtained by adding the zeros to the punctured discs and identifying the corresponding pairs.

We defined a Teichmüller curve to be a curve in $\mathcal{M}_g$ and saw that Teichmüller curves stem from $\mathrm{SL}_2(\mathbb{R})$-orbits of flat surfaces in $\Omega\mathcal{M}_g$. If we quotient by scalar (of absolute value one) or equivalently by the $\mathrm{SO}_2(\mathbb{R})$-action we obtain a curve in $\mathbb{P}\Omega\overline{\mathcal{M}}_g$, more precisely in $\mathbb{P}\Omega\overline{\mathcal{M}}_g(\mu)$, where $\mu$ is the signature of a generating surface. We call this curve the *canonical lift* of the Teichmüller curve.

**Proposition 5.10.** *Suppose that $C$ is a Teichmüller curve generated by an abelian differential $(X, \omega)$ in $\Omega\mathcal{M}_g(\mu)$ and let $\mu'$ be a degeneration of the signature $\mu$. Then the canonical lift of $C$ to $\mathbb{P}\Omega\overline{\mathcal{M}}_g(\mu)$ is disjoint from $\mathbb{P}\Omega\overline{\mathcal{M}}_g(\mu')$.*

*The holomorphic one-form given as the $\mathrm{SL}_2(\mathbb{R})$-image over each smooth fiber over a Teichmüller curve extends to a section $\omega_\infty$ of the dualizing sheaf (i.e. a stable one-form) for each singular fiber $X^\infty$ over the closure of a Teichmüller curve.*

Proof. The claim is obvious over the interior of the moduli space. We only need to check the disjointness over the boundary. We may approach the boundary along a geodesic ray. In the construction of the limiting surface the open subset

$$H = X^0 \setminus \bigcup_{i \in I} (U_i^\top \cup U_i^\perp)$$

is never touched as explained at the beginning of this section. This subset $H$ contains the zeros of $\omega$. Since the multiplicity of a zero is a local property, this implies the claim. □

**Corollary 5.11.** *Let $X^\infty$ be a stable curve corresponding to a boundary point of a Teichmüller curve. Then $X^\infty$ does not contain separating nodes. In particular $C \cdot \delta_i = 0$ for $i \geq 1$.*

*For each irreducible component $Y$ of $X^\infty$ the number of zeros of $\omega_\infty$ is equal to $2g(Y) - 2 + n$, where $g(Y)$ is the arithmetic genus of $Y$ and $n = |Y \cap \overline{X^\infty \setminus Y}|$. In particular each irreducible component $Y$ of $X^\infty$ contains at least one zero of $\omega_\infty$.*

PROOF. If a node was separating, then by the description of the degeneration a core curve $\gamma_i$ of a cylinder $C_i$ was separating. If this was true, view $X \setminus C_i = X^\top \cup X^\perp$ both as gluing of cylinders. For a translation structure saddle connections on the top of a cylinders have to be glued to the bottom of a cylinder. But the total length of top sides of cylinders of $X^\top$ exceeds the total length of bottom sides of cylinders in $X^\top$. It is thus impossible to form a closed surface $X^\top$ with boundary $\partial C_i$ with the gluing rules just described. This contradiction proves the first statement.

The second statement is an immediate consequence of the degeneration description and the Gauss-Bonnet formula. □

Note that for a quadratic differential it is permitted to glue a saddle connection on the top of a cylinder to another one on the top of a cylinder. Consequently, the above proof does not apply to Teichmüller curves generated by quadratic differentials. They may indeed have degenerate fibers that are stable curves with separating nodes.

**Sections defined by singularities.** Let $C$ be a Teichmüller curve generated by $(X, \omega) \in \Omega\mathcal{M}_g(m_1, \ldots, m_k)$. The $\mathrm{SL}_2(\mathbb{R})$-orbit of each singularity $Z$ defines a section $S(Z)$ over the Teichmüller disc $\mathbb{H} = \mathrm{SO}_2(\mathbb{R})\backslash\mathrm{SL}_2(\mathbb{R})$. The affine group permutes the singularities. Since there are only finitely many of them, there is a finite index subgroup of $\Gamma$ of $\mathrm{SL}(X, \omega)$ that fixes each of the singularities. We may take $\Gamma$ even smaller, but still of finite index, and suppose that all the conditions of Section 3.1 are met for $\Gamma$. Let $B = \mathbb{H}/\Gamma \to C$ be a finite unramified cover of the Teichmüller curves such that the zero $Z_i$ of order $m_i$ defines a section $\sigma_i$ (not only a multi-section) with image $S_i$ of the pullback family $f : \overline{\mathcal{X}} \to \overline{B}$. Such a section can be used to define a lift of $\overline{B}$ to $\overline{\mathcal{M}}_{g,1}$.

**Proposition 5.12.** *The section $S_i$ has self-intersection number*

$$S_i^2 = \frac{-\chi}{2(m_i + 1)},$$

*where $\chi = 2g(\overline{B}) - 2 + |\Delta|$ and $\Delta$ is the set of cusps in $\overline{B}$. In particular the intersection number of $\overline{B}$ with $\omega_{i,\mathrm{rel}}$, which is by definition equal to $-S_i^2$, is given by*

$$\overline{B} \cdot \omega_{i,\mathrm{rel}} = \frac{\overline{B} \cdot \lambda - (\overline{B} \cdot \delta)/12}{(m_i + 1)\kappa_\mu},$$

*where $\kappa_\mu = \frac{1}{12} \sum_{j=1}^k \frac{m_j(m_j+2)}{m_j+1}$.*

PROOF. Let $\mathcal{L} \subset f_* \omega_{\widetilde{\mathcal{X}}/\overline{B}}$ be the ('maximal Higgs', see [**Möl06b**]) line bundle whose fiber over the point corresponding to $[X]$ is $\mathbb{C} \cdot \omega$, the generating differential of the Teichmüller curve. The property 'maximal Higgs' says by definition that

$$\deg(\mathcal{L}) = \chi/2. \tag{8}$$

Let $S$ be the union of the sections $S_1, \ldots, S_k$. Pulling back the above inclusion to $\mathcal{X}$ gives an exact sequence

$$0 \to f^*\mathcal{L} \to \omega_{\widetilde{\mathcal{X}}/\overline{B}} \to \mathcal{O}_S\left(\sum_{j=1}^k m_j S_j\right) \to 0,$$

since the multiplicities of the vanishing locus of the generating differential of the Teichmüller curve are constant along the whole compactified Teichmüller curve. This implies that $\omega_{\widetilde{\mathcal{X}}/\overline{B}}$ is numerically equal to $f^*\mathcal{L}+\sum_{j=1}^k m_jS_j$. By the adjunction formula we get

$$S_i^2 = -\omega_{\mathcal{X}/\overline{B}} \cdot S_i = -m_iS_i^2 - \deg(\mathcal{L}),$$

since the intersection product of two fibers of $f$ is zero. Together with (8) we thus obtain the desired self-intersection formula.

We write the Noether formula (5) as

$$12(\overline{B}\cdot\lambda) - (\overline{B}\cdot\delta) = \omega^2_{\widetilde{\mathcal{X}}/\overline{B}}$$

and calculate

$$\begin{aligned}\omega^2_{\widetilde{\mathcal{X}}/\overline{B}} &= (f^*\mathcal{L} + \sum_{j=1}^k m_jS_j)^2 = \sum_{j=1}^k m_j^2S_j^2 + 2\deg(\mathcal{L})\sum_{j=1}^k m_j \\ &= \frac{\chi}{2}\sum_{j=1}^k \frac{-m_j}{m_j+1} + 2m_j = \frac{\chi}{2}\sum_{j=1}^k \frac{m_j(m_j+2)}{m_j+1} \\ &= -12(m_i+1)\kappa_\mu S_i^2.\end{aligned}$$

Solving for $-S_i^2$ gives the claimed formula. □

In Proposition 2.3 we saw that in the interior of $\mathcal{M}_g$ a Teichmüller curve generated by a hyperelliptic curve always stays inside the hyperelliptic locus. By contraposition the argument implies that a Teichmüller curve generated by a non-hyperelliptic curve does not meet the hyperelliptic locus in $\mathcal{M}_g$. In boundary points the behavior is much more subtle. Sometimes the dichotomy can still be proved, which will be important for the non-varying results in Section 6.4.

**Proposition 5.13.** *Let $C$ be a Teichmüller curve generated by $(X,\omega)$ in $\Omega\mathcal{M}_g(\mu)$. Suppose that an irreducible degenerate fiber $X^\infty$ over a cusp of $C$ is hyperelliptic. Then $X$ is hyperelliptic, hence the whole Teichmüller curve lies in the locus of hyperelliptic flat surfaces.*

*Moreover, if $\mu \in \{(4),(3,1),(6),(5,1),(3,3),(3,2,1),(8),(5,3)\}$ and $(X,\omega)$ is not hyperelliptic, then no degenerate fiber of the Teichmüller curve is hyperelliptic.*

The last conclusion does not hold for all strata. For instance, Teichmüller curves generated by a non-hyperelliptic flat surface in the stratum $\Omega\mathcal{M}_3(2,1,1)$ always intersect the hyperelliptic locus at the boundary. The proof is an intersection number argument similar to the non-varying results for Lyapunov exponents below. It can be found in [**CM11**].

Proof. Suppose that the stable model $X^\infty$ of the degenerate fiber is irreducible of geometric genus $h$ with $(g-h)$ pairs of points $(p_i,q_i)$ identified. A semi-stable model of $X^\infty$ admits a degree two admissible cover of the projective line if and only if the normalization $X_n$ of $X^\infty$ is branched at $2h+2$ branch points over a main component $\mathbb{P}^1$ with covering group generated by an involution $\phi$ and, moreover, for each of the $2(g-h)$ nodes there is a projective line intersecting $X_n$ in $p_i$ and $q_i = \phi(p_i)$ with two branch points.

In the flat coordinates of $X_n$ given by $\omega$, the surface consists of a compact surface $X_0$ with boundary of genus $h$ and $2(g-h)$ half-infinite cylinders (corresponding to the nodes) attached to the boundary of $X_0$. We may define $X_0$ canonically by sweeping out the half-infinite cylinder at $p_i$ (or $q_i$) with lines of slope equal to the residue (considered as element in $\mathbb{R}^2$) of $\omega$ at $p_i$ until such a line hits a zero of $\omega$, i.e. a singularity of the flat structure.

With this normalization, the above discussion shows that for irreducible stable curves the hyperelliptic involution exchanges the half-infinite cylinders corresponding to $p_i$ and $q_i$ and it defines an involution $\phi$ of $X_0$. As in the smooth case, $\phi$ acts as $-\mathrm{Id}$ on $X_0$.

To obtain smooth fibers over the Teichmüller curve (in a neighborhood of $X^\infty$) one has to glue cylinders of finite (large) height in place of the half-infinite cylinders of appropriate ratios of moduli. The hypothesis on $\phi$ acting on $X_0$ and on the half-infinite cylinders implies that $\phi$ is a well-defined involution on the smooth curves. Moreover, $\phi$ has two fixed points in each of the finite cylinders and $2h + 2$ fixed points on $X_0$, making $2g + 2$ fixed points in total. This shows that the smooth fibers of the Teichmüller curve are hyperelliptic.

To complete the proof we have to consider the two-component degenerations for $\mu \in \{(3, 1), (5, 1), (5, 3)\}$. In both cases, the hyperelliptic involutions can neither exchange the components (since the zeros are of different order) nor fix the components (since the zeros are of odd order).

For $\mu = (3, 3)$ a hyperelliptic involution $\phi$ cannot fix the component, since 3 is odd. It cannot exchange the two components and exchange a pair of half-infinite cylinders that belong to different nodes, since $\phi$ could then be used to define a non-trivial involution for each component. This involution fixes the zeros and this contradicts that 3 is odd. If $\phi$ exchanges all pairs of half-infinite cylinders that belong to the same node, $\phi$ has two fixed points in each cylinder on the smooth 'opened up' surface. Now we can apply the same argument as in the irreducible case to conclude that the 'opened up' flat surfaces are hyperelliptic as well.

For $\mu = (3, 2, 1)$ a hyperelliptic involution can neither fix the component with the (unique) zero of order three, since 3 is odd, nor map it elsewhere, since the zeros are of different order. □

### 5.5. The classification problem of Teichmüller curves: state of the art

One of the main questions of the theory is the classification of Teichmüller curves. In this section we summarize what is known today and what the open problems are. It also explains to which extent the VHS decomposition and real multiplication have so far been useful for solving the classification problem.

**Genus two.** In genus two the notion of primitive and algebraically primitive coincide. In the stratum $\Omega\mathcal{M}_2(2)$ an infinite series of primitive Teichmüller curves was found independently in [**Cal04**] and in [**McM03a**]. This was shown to be the complete list of primitive Teichmüller curves in this stratum ([**McM05a**]). This is the only stratum where a complete classification is known, since square-tiled surfaces in this stratum have been classified in [**HL06b**] and also in [**McM05a**]. In the stratum $\Omega\mathcal{M}_2(1, 1)$ primitive Teichmüller curves have been classified: There is only one example, the regular decagon with opposite sides identified. The proof in [**McM06b**] relies on a 'torsion' characterization of periodic points ([**Möl06a**]) on those Teichmüller curves, that in turn relies on the VHS decomposition in Theorem 5.5.

**Infinite series for fixed genus.** The only genera that are known to have an infinite number of primitive Teichmüller curves are $g = 2, 3, 4$. They are not algebraically primitive for $g = 3$ and $g = 4$. The construction of these examples is in [**McM06a**]. A classification of Teichmüller curves with quadratic trace field in these strata has recently been announced by Lanneau and D.-M. Nguyen.

**Infinite series of (algebraically primitive) Teichmüller curves.** Besides the examples in low genus $g \leq 4$, the only known primitive Teichmüller curves have a triangle group as affine group. These are the original examples of Veech ([**Vee89**]) and Ward, as well as a series of examples of primitive Teichmüller curves realizing all possible triangle groups as affine groups in [**BM10b**].

**Finiteness results.** In the hyperelliptic component of $\Omega\mathcal{M}_g(g-1, g-1)$ there are (for given $g$) only finitely many algebraically primitive Teichmüller curves. This is again a consequence of the 'torsion condition' ([**Möl08**]). These components are particular, since they have more than one zero to apply the criterion, but not too many zeros and moreover the hyperelliptic involution in order to reduce combinatorial complexity of the problem.

A more conceptual approach in order to exploit real multiplication for the classification of algebraically primitive Teichmüller curves was taken in [**BM09**]. It combines the boundary behavior of the real multiplication locus in $\mathcal{M}_g$ with the 'torsion condition' to give finiteness in the stratum $\Omega\mathcal{M}_3(3, 1)$. Moreover, the real multiplication condition gives an algorithm and quite convincing numerical evidence for finiteness in the stratum $\Omega\mathcal{M}_3(4)^{\text{hyp}}$.

**Open problems.** Finiteness for the number of algebraically primitive Teichmüller curves in the strata with many zeros seems likely to hold, following the torsion point approach in [**Möl06a**], but the combinatorics might be so difficult that essential new ideas are needed even for strata in low genus. On the other hand, the approach in [**BM09**] should be extended to give finiteness at least for the strata $\Omega\mathcal{M}_g(2g-2)$.

If we consider square-tiled surfaces instead of algebraically primitive curves, the smallest open strata are $\Omega\mathcal{M}_2(1, 1)$ and $\Omega\mathcal{M}_3(4)$. The combinatorial approach of [**HL06b**] goes as follows. Take the 'obvious' combinatorial invariants and show that one can connect any pair of cusps by passing from one direction on a given flat surface to another. This approach might work also in both cases just mentioned, but the reader should be warned of the combinatorial complexity.

## 6. Lyapunov exponents

### 6.1. Motivation: Asymptotic cycles, deviations and the wind-tree model

Fix a generic surface $(X, \omega)$ in some stratum $\Omega\mathcal{M}_g(m_1, \ldots, m_k)$ and a generic point $P$ on that surface. Consider a vertical straight line starting at $P$ and close it up along a small horizontal slit $I$ once this slit is hit for the first time as in Figure 3. We thus obtain a closed cycle $c_1 \in H_1(X, \mathbb{R})$.

We let $c_n \in H_1(X, \mathbb{R})$ be the cycle obtained by closing up the $n$-th hit on $I$. The limit

$$c = \lim_{n \to \infty} \frac{c_n}{||c_n||}$$

exists and is called asymptotic cycle. We are interested on the *deviation from this first order approximation.* For that purpose we provide $H_1(X, \mathbb{R})$ with the Hodge norm and let $p_2$ be the projection onto the orthogonal complement of $V_1 = \langle c \rangle$.

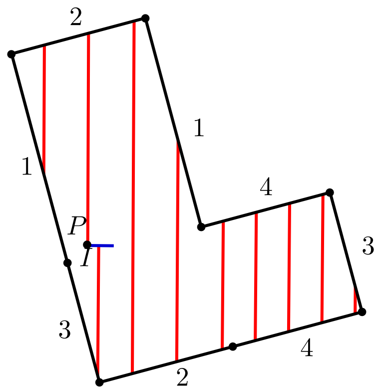


FIGURE 3. Straight line converging to the asymptotic cycle

Then the limit

$$\lambda_2 = \lim_{n\to\infty} \frac{\log ||p_2(c_n)||}{\log n}$$

exists. In fact, the projections $p_2(c_n)$ normalized to Hodge length one converge and we let $V_2$ be the subspace generated by the limit. Then we define $p_3$ to be the projection onto the orthogonal complement of $V_1$ and $V_2$ and let

$$\lambda_3 = \lim_{n\to\infty} \frac{\log ||p_3(c_n)||}{\log n}.$$

This process can be repeated to produce a full flag on $H_1(X, \mathbb{R})$.

For another example we consider the *wind-tree model* for the diffusion of gas molecules. In this model a particle drifts in billiard paths in the plane or the space and is reflected at randomly placed scatterers. Mathematically even the simplified model of scatterers places in a regular pattern at fixed positions is interesting. We restrict even further to the regular pattern being the lattice $\mathbb{Z}^2$ in the plane and the scatterers being boxes of side lengths $(a, b)$ with $a, b \in (0, 1)$ centered at the lattice points as shown in Figure 4. For some (e.g. rational) values of parameters $(a, b)$ there is a dense subset of $S^1$ such that trajectories in that given direction are periodic. Clearly, periodicity is a rare phenomenon. But still, for almost every direction the directional flow is recurrent.

On the other hand, most trajectories make excursions further and further out. Let $\phi_t^\theta$ denote the flow in the direction $\theta$ for time $t$. In fact, for every $(a, b)$ there is some $\lambda_2$ such that for almost every direction $\theta$ and almost every starting point we have

$$\lim_{t\to\infty} \sup \frac{\log d(\phi_t^\theta(x), x)}{\log t} = \lambda_2.$$

The (non-)dependence of $\lambda_2$ on the parameters $(a, b)$ is one of the main motivations of the notion 'non-varying' that we introduce below.

**References:** In both cases the answer to the problem is related to Lyapunov exponents as defined in the following section. The strata of $\Omega\mathcal{M}_g$ carry a finite invariant measure $\mu_{\rm gen}$ ([**Mas82**], [**Vee82**]) with support equal to the whole stratum and the $\lambda_i$ that stem from the deviations are the Lyapunov exponents for the Teichmüller geodesic flow acting on the Hodge bundle over $\Omega\mathcal{M}_g$ with respect to $\mu_{\rm gen}$.

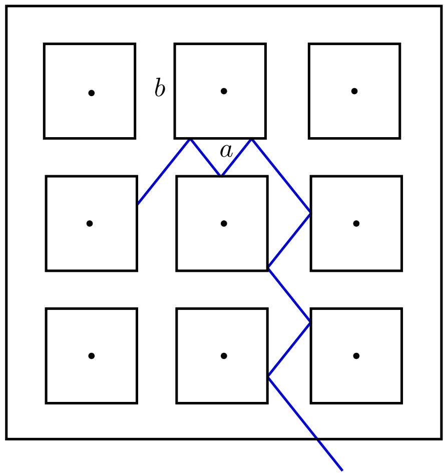


FIGURE 4. Wind-tree model

For the wind-tree model the $\lambda_2$ turns out to be the second Lyapunov exponent for the Haar measure supported on a Teichmüller curve in some cases, e.g. when $a$ and $b$ are rational, and for the Masur-Veech measure in the remaining cases.

The proof of these results will not be discussed here. It can be found in [**Zor94**] and [**Zor99**] (see also [**Zor06**]) for asymptotic cycles and in [**HLT09**] and [**DHL11**] for the wind-tree model.

### 6.2. Lyapunov exponents

We first state Oseledec's general theorem for the existence of Lyapunov exponents and then explain the instances we want to apply this theorem to.

**Theorem 6.1** (Oseledec). *Let $g_t : (M, \mu) \to (M, \mu)$ be a flow that acts ergodically on a space $M$ with finite measure $\mu$. Suppose that the action of $t \in \mathbb{R}^+$ lifts equivariantly to a flow also denoted by $g_t$ on some measurable real vector bundle $V$ on $M$. Suppose there exists a norm $|| \cdot ||$ on $V$ (of course not supposed to be $g_t$-equivariant) such that for all $t \in \mathbb{R}^+$*

$$\int_M \log(1 + ||g_t||(m))\mu(m) < \infty,$$

*where $||g_t||(m)$ denotes the operator norm at the point $m$ induced by the map $g_t$ and the norm on $V$.*

*Then there exist real constants $\widetilde{\lambda_1} > \cdots > \widetilde{\lambda_k}$ and a filtration*

$$V = V_1 \supsetneq \cdots \supsetneq V_k \supset 0$$

*by measurable vector subbundles such that, for almost all $m \in M$ and all $v \in V_m \setminus \{0\}$, one has*

$$||g_t(v)|| = \exp(\widetilde{\lambda}_i t + o(t)),$$

*where $i$ is the maximal value such that $v \in (V_i)_m$.*

*The $\widetilde{\lambda}_i$ and the $V_i$ do not change if $|| \cdot ||$ is replaced by another norm of 'comparable' size (e.g. if one is a scalar multiple of the other).*

Note that the $\widetilde{\lambda_i}$ and $V_i$ are unchanged if we replace the support of $\mu$ by a finite unramified covering with a lift of the flow and the pullback of $V$.

From now on we adopt the convention to repeat the exponents $\widetilde{\lambda_i}$ according to the rank of $V_i/V_{i+1}$ such that we will always have $2g$ of them, possibly some of them equal. The resulting sequence of numbers

$$\lambda_1 = \widetilde{\lambda_1} \geq \lambda_2 \geq \cdots \geq \lambda_{2g} = \widetilde{\lambda_k}$$

is called the spectrum of *Lyapunov exponents of* $g_t$. If $V$ is a symplectic vector bundle, the spectrum is symmetric, i.e. $\lambda_{g+k} = -\lambda_{g-k+1}$.

We will use measures $\mu$ on $\Omega\mathcal{M}_g$ that will even be $\mathrm{SL}_2(\mathbb{R})$-invariant and such that the Teichmüller geodesic flow $g_t$ acts ergodically. Let $V$ be the restriction of the real Hodge bundle (i.e. the bundle with fibers $H^1(X,\mathbb{R})$) to the support $M$ of $\mu$. Let $g_t$ be the lift of the geodesic flow to $V$ via the Gauss-Manin connection. The norm on $V$ will be the Hodge norm, the norm associated with the bilinear form $Q$ defined in Section 4.

To sum up, the Lyapunov exponents for the Teichmüller geodesic flow on $\Omega\mathcal{M}_g$ measure the logarithm of the growth rate of the Hodge norm of cohomology classes during parallel transport along the geodesic flow. The reader may consult [**For06**] or [**Zor06**] for a more detailed introduction to this subject.

Most of our results will be about the *sum* of the top half of the Lyapunov exponents defined as

$$L = \sum_{i=1}^{g} \lambda_i.$$

**Two sorts of measures.** We will apply Oseledec's theorem in two instances. The first are Masur-Veech measures $\mu_{\mathrm{gen}}$ ([**Mas82**], [**Vee82**]) with support equal to the whole hypersurface of flat surfaces of area one in a connected component. These measures are constructed using period coordinates, giving the unit cube $(\mathbb{Z}[i])^N$ volume one. Since coordinate changes are in the symplectic group with integral coefficients, this is a well-defined normalization. The sum of Lyapunov exponents for these measures can be calculated by first calculating Siegel-Veech constants using [**EMZ03**] and then transferring the information using [**EKZ11**]. We will denote by $L_{\Omega\mathcal{M}_g(m_1,\ldots,m_k)}$ the sum of Lyapunov exponents for the measure $\mu_{\mathrm{gen}}$ supported on the stratum $\Omega\mathcal{M}_g(m_1,\ldots,m_k)$. It is a combinatorially very involved procedure to actually compute these values.

When talking about Lyapunov exponents for Teichmüller curves we take $\mu$ to be the measure on the unit tangent bundle $T^1B$ to a Teichmüller curve that stems from the Poincaré metric $g_{\mathrm{hyp}}$ on $\mathbb{H}$ with scalar curvature $-4$. This normalization is equivalent to require that our choice of scaling of the geodesic flow $g_t = \mathrm{diag}(e^t, e^{-t})$ has unit speed. It implies that the first Lyapunov exponent (corresponding to the subbundle $\Re\omega$ where $(X,\omega)$ is a generating flat surface of the Teichmüller curve) equals one. This normalization is consistent with the normalization of $\mu_{\mathrm{gen}}$ where also the first Lyapunov exponent is one. It is thus meaningful to compare Lyapunov exponents for $\mu$ and for the $\mu_{\mathrm{gen}}$ of the stratum the generating flat surface lies in.

In both cases, the integrability condition of Oseledec's theorem has to be verified. For $\mu_{\mathrm{gen}}$ this is done using a discretization and the language of matrix-valued cocycles in [**Zor99**], see also [**Zor06**]. For $\mu$ on a Teichmüller curve this can be deduced in continuous time from [**For02**, Lemma 2.1 and Corollary 2.2]. It holds in much greater generality as we shall soon see, since Lemma 6.10 below can also be used here.

### 6.3. Lyapunov exponents for Teichmüller curves

The bridge between the 'dynamical' definition of Lyapunov exponents and the 'algebraic' method applied in the sequel is given by the following result. Note that if the VHS splits into direct summands one can apply Oseledec's theorem to the summands individually. The full set of Lyapunov exponents is the union (with multiplicity) of the Lyapunov exponents of the summands. Note that both sides of the equations of the following theorem are invariant when passing to finite unramified covers. We will thus use a convenient model of a fibered surface as explained in Section 3.1 instead of the Teichmüller curve itself. If one wants to evaluate the right hand side on the Teichmüller curve, one has to take into account orbifold degrees of line bundles.

**Theorem 6.2** ([**Kon97**], [**KZ97**], [**BM10b**])**.** *If the VHS over the Teichmüller curve contains a sub-VHS $\mathbb{W}$ of rank $2k$, then the sum of the corresponding $k$ non-negative Lyapunov exponents equals*

$$\sum_{i=1}^{k} \lambda_i^{\mathbb{W}} = \frac{2\deg \mathbb{W}^{(1,0)}}{2g(\overline{B}) - 2 + |\Delta|},$$

*where $\mathbb{W}^{(1,0)}$ is the $(1,0)$-part of the Hodge-filtration of the vector bundle associated with $\mathbb{W}$. In particular, we have*

$$\sum_{i=1}^{g} \lambda_i = \frac{2\deg f_*\omega_{\overline{\mathcal{X}}/\overline{B}}}{2g(\overline{B}) - 2 + |\Delta|}.$$

*In particular, the first Lyapunov exponent of a Teichmüller curve is one by our normalization convention.*

The following proof is from [**EKZ11**]. This proof has three main ingredients. Instead of averaging over a $g_t$-orbit as stated in the definition of Lyapunov exponents one averages also over $\mathrm{SO}_2(\mathbb{R})$-orbits first and thus over the whole disc. This does not change the limit appearing in the definition of Lyapunov exponents, since the set where some other limit occurs has measure zero. Second, instead of taking the limit for any special vector we can (to determine the top Lyapunov exponent) average over all vectors or any subset whose intersection with the next filtration step has measure zero. Third, we rewrite a $k$-fold wedge-product of flat sections into a $k$-fold wedge-product of holomorphic sections plus contributions that are killed when taking $\partial\overline{\partial}$ in order to compute a curvature form. This last step works for the middle wedge power only and this is why this method only determines the sum of Lyapunov exponents. All the arguments are perfectly valid in the setting of Section 6.5 and justify Proposition 6.11.

We now give the details of this outline. Let

$$\Omega = \wedge^k(Q) : \wedge^{2k}\mathbb{W} \to \mathbb{C}$$

be the volume form on $\mathbb{W}$ induced by the Hodge inner product with values in the constant local system $\mathbb{C}$ on $M$, i.e.

$$\Omega(w_1 \wedge \ldots \wedge w_k) = \sum_{\sigma \in S_{2k}} \mathrm{sign}(\sigma) Q(w_{\sigma(1)}, w_{\sigma(2)}) \cdots Q(w_{\sigma(2k-1)}, w_{\sigma(2k)}).$$

The volume form $\Omega$ can be extended $\mathcal{O}_M$-linearly to a volume form on the vector bundle $\wedge^{2k}\mathcal{W}$. For any section $L = v_1 \wedge \cdots \wedge v_k$ of $\Lambda^k(\mathbb{W})$ given by a decomposable vector we have a norm induced by the Hodge norm, that can be written in terms of the Hodge-* as

$$||L||^2 = \Omega(v_1 \wedge \cdots \wedge v_k \wedge *v_1 \wedge \cdots \wedge *v_k).$$

One checks that for any basis of holomorphic sections $\omega_i$ of $\mathbb{W}^{(1,0)}$ over some open subset $U$ of $B$ we have

$$(9) \quad ||L||^2 = \frac{|\Omega(\omega_1 \wedge \cdots \wedge \omega_k \wedge v_1 \wedge \cdots \wedge v_k)| \cdot |\Omega(v_1 \wedge \cdots \wedge v_k \wedge \bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_k)|}{|\Omega(\omega_1 \wedge \cdots \wedge \omega_k \wedge \bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_k)|}.$$

The denominator can also be written as

$$|\Omega(\omega_1 \wedge \cdots \wedge \omega_k \wedge \bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_k)| = \det(Q(\omega_i, \omega_j)_{i,j=1}^k).$$

**Lemma 6.3.** *Given a section $L$ of $\Lambda^k(\mathbb{W})$ as above over some open subset $U$ of $B$, we have on $U$ the equality of functions*

$$\Phi_k := \Delta_{\text{hyp}}(\log(||L||)) = -\frac{1}{2}\Delta_{\text{hyp}} \log |\det(Q(\omega_i, \omega_j)_{i,j=1}^k)|,$$

*where $\Delta_{\text{hyp}}$ is the Laplacian for the hyperbolic metric along the Teichmüller disc.*

Proof. We apply $\Delta_{\text{hyp}}$ to the defining equation (9), calculate $||L||$ and obtain

$$2\Phi_k = \Delta_{\text{hyp}} \log |\Omega(\omega_1 \wedge \cdots \wedge \omega_k \wedge v_1 \wedge \cdots \wedge v_k)|$$
$$+ \Delta_{\text{hyp}} \log |\Omega(v_1 \wedge \cdots \wedge v_k \wedge \bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_k)| - \Delta_{\text{hyp}} \log |\det(Q(\omega_i, \omega_j)_{i,j=1}^k)|.$$

Since the $v_i$ are flat sections, the first summand on the right hand side is a holomorphic function and the second summand is an antiholomorphic function of the parameter on the Teichmüller disc. Hence they are harmonic and vanish after applying $\Delta_{\text{hyp}}$. □

Let $\mathrm{Gr}_k(B)$ be the bundle whose fiber over $b$ consists of the $k$-dimensional $\mathbb{R}$-subspaces of $(\mathbb{W})_b$. The Grassmanian carries a Haar measure which we denote by $\gamma$. We pull back this Grassmanian bundle to the unit tangent bundle $T_1B = \mathrm{SL}_2(\mathbb{R})/\Gamma$ of $B$. Within the Grassmanian bundle there is the subset of decomposable vectors $\mathrm{Gr}_k^{\mathrm{dec}}(T_1B)$ and this will be the set we are averaging over.

Proof of Theorem 6.2. We first rewrite the sum of Lyapunov exponents as an integral over $\Phi_k$ in the following way. First, for almost every $L$ in $\mathrm{Gr}_k^{\mathrm{dec}}(T_1B)$ we have

$$\sum_{i=1}^k \lambda_i = \lim_{T\to\infty} \frac{1}{T} \log ||g_T(L)||.$$

Together with the main theorem of calculus and additional averaging where the loci that give smaller contributions are of measure zero, we obtain the first line of (10). In the second line we do yet another circle averaging, in order to write the inner integral over a disc $\Delta_t$ of radius $t$. To pass to the third line we use a result from harmonic analysis stating that for any smooth rotation invariant function $L$ we have

$$\frac{1}{2\pi}\frac{\partial}{\partial t}\int_0^{2\pi} L(t,\theta)d\theta = \frac{1}{2}\tanh(t)\frac{1}{\mathrm{vol}(\Delta_t)}\int_{\Delta_t} \Delta_{\text{hyp}} L d\mu.$$

We are interpreting points $c$ in $\Delta_t$ as elements in $\mathrm{SL}_2(\mathbb{R})$ by writing $c = g_t(r_\theta(b))$. We may thus let $c$ act on points in $\mathbb{H}$ and vector bundles on $\mathbb{H}$ using parallel transport. We indicate these actions by $c_*$. Passing to the forth line we perform a change of variables and use the $\mathrm{SL}_2(\mathbb{R})$-invariance of $\mu$. Passing to the fifth line we use the preceding lemma and get rid of the additional Grassmannian averaging. The sixth line needs a global bound ([**For06**]) to justify the change of integration order. Then we may take the limit of the integral over the tanh first.

(10)

$$\begin{aligned}
\mathrm{vol}(B)\sum_{i=1}^{k}\lambda_i &= \int_{\mathrm{Gr}_k^{\mathrm{dec}}(T_1B)} \lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{d}{dt}\log\|g_t(L,b)\| dt d\mu(b) d\gamma(L)\\
&= \int_{\mathrm{Gr}_k^{\mathrm{dec}}(T_1B)} \lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{1}{2\pi}\int_0^{2\pi}\frac{d}{dt}\log\|g_t(r_\theta L,b)\| d\sigma(\theta) dt d\mu(b) d\gamma(L)\\
&= \int_{\mathrm{Gr}_k^{\mathrm{dec}}(T_1B)} \lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{\tanh(t)}{2\,\mathrm{vol}(\Delta_t)}\int_{\Delta_t}\Delta_{\mathrm{hyp}}\log\|c_*(L,b)\| d\mu(c) dt d\mu(b) d\gamma(L)\\
&= \int_{T_1B} \lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{\tanh(t)}{2\,\mathrm{vol}(\Delta_t)}\int_{\Delta_t}-\frac{1}{2}\Delta_{\mathrm{hyp}}\log|\det Q(\omega_i,\omega_j)_{i,j=1}^k|(c_*(b)) d\mu(c) dt d\mu(b)\\
&= \int_{T_1B} \lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{\tanh(t)}{2\,\mathrm{vol}(\Delta_t)}\int_{\Delta_t}-\frac{1}{2}\Delta_{\mathrm{hyp}}\log|\det Q(\omega_i,\omega_j)_{i,j=1}^k|(b') d\mu(c) dt d\mu(b')\\
&= \int_{T_1B} \Phi_k(b')\left(\lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{\tanh(t)}{2\,\mathrm{vol}(\Delta_t)}\int_{\Delta_t} 1\, d\mu(c)\right) d\mu(b')\\
&= \int_B \Phi_k(b) d\overline{\mu}(b),
\end{aligned}$$

where $\overline{\mu}$ denotes the direct image of $\mu$ under the projection $T_1B \to B$. Now, if $\Theta(\det(\mathbb{W}^{(1,0)}))$ denotes the curvature form of the line bundle $\det(\mathbb{W}^{(1,0)})$, this curvature form is represented by the differential form $-2\partial\overline{\partial}\log|\det(Q(\omega_i,\omega_j)_{i,j=1}^k)|$. Since $\mathrm{vol}(B) = \frac{\pi}{2}(2g(\overline{B}) - 2 + |\Delta|)$ we obtain

$$\begin{aligned}
\int_B \Phi_k(b) d\overline{\mu}(b) &= -\frac{1}{4}\int_B \Delta_{\mathrm{hyp}}\log|\det(Q(\omega_i,\omega_j)_{i,j=1}^k)|(b)\, d\overline{\mu}(b)\\
&= -\frac{1}{4}\int_B 4\partial\overline{\partial}\log|\det(Q(\omega_i,\omega_j)_{i,j=1}^k)|\,\frac{i}{2}dz\wedge d\overline{z} \qquad (11)\\
&= \frac{i}{2}\int_B [\Theta(\det(\mathbb{W}^{(1,0)}))] = \pi\deg(\mathbb{W}^{(1,0)}).
\end{aligned}$$

□

In the remainder of this section we give the bridge between the above formula for the sum of Lyapunov exponents and the slope for Teichmüller curves.

Recall that we denote the signature of a stratum of $\Omega\mathcal{M}_g$ by the tuple $\mu = (m_1,\ldots,m_k)$ where $\sum m_i = 2g-2$. Let $\kappa_\mu$ be the constant

$$\kappa_\mu = \frac{1}{12}\sum_{i=1}^{k}\frac{m_i(m_i+2)}{m_i+1}.$$

**Proposition 6.4.** *Let $C \to \mathcal{M}_g$ be a Teichmüller curve generated by a flat surface in $\Omega\mathcal{M}_g(m_1,\ldots,m_k)$. Then knowing the slope is equivalent to knowing the sum of*

*Lyapunov exponents, since the two quantities are related by the formula*

$$s(C) = 12 - \frac{12\kappa_\mu}{L(C)}. \tag{12}$$

This is yet another consequence of the Noether formula and the self-intersection number of the sections $S_i$. Versions of the formula appear also in [**Che10b**, Thm. 1.8] and in [**EKZ11**].

Proof. As in the proof of Proposition 5.12 we deduce that

$$\frac{12(\overline{C}\cdot\lambda) - (\overline{C}\cdot\delta)}{12\kappa_\mu} = \frac{\chi}{2} \tag{13}$$

and Theorem 6.2 states that $L = 2(\overline{C}\cdot\lambda)/\chi$. Hence the relation (12) follows immediately. □

The preceding proposition and Xiao's bound (3.4) gives immediately the following upper bound.

**Corollary 6.5.** *Let $C \to \mathcal{M}_g$ be a Teichmüller curve generated by a flat surface in $\Omega\mathcal{M}_g(m_1, \ldots, m_k)$. Then*

$$L(C) \le \frac{g}{4(g-1)} \sum_{i=1}^{k} \frac{m_i(m_i+2)}{m_i+1}.$$

**An explicit formula.** For any given Teichmüller curve at a time the sum of Lyapunov exponents can be calculated, provided the Veech group can be calculated. This is a non-trivial condition, since at the time of writing there is still no deterministic algorithm to determine the Veech groups of the primitive Teichmüller curves in genus two (see Section 5.5), if the order $D$ is large.

**Proposition 6.6.** *Suppose the Teichmüller curve $C = \mathbb{H}/\mathrm{SL}(X,\omega)$ generated by $(X,\omega)$ has orbifold Euler characteristic $\chi$ and let $\Delta$ be the the set of cusps of $C$. For each $i \in \Delta$ let $C_{ij}$ for $j \in J_i$ be the set of maximal cylinders of $(X,\omega)$ in the direction corresponding to the cusp. Suppose that the generator of the cusp stabilizer in $\mathrm{SL}(X,\omega)$ acts on $C_{ij}$ as $k_{ij}$-fold Dehn twist. Then the sum of Lyapunov exponents can be calculated as follows:*

$$L(C) = \kappa_\mu + \frac{1}{6}\frac{\sum_{i\in\Delta}\sum_{j\in J_i} k_{ij}}{\chi}. \tag{14}$$

Note that the fraction is invariant under passing to a finite cover unramified outside the cusps. Hence working with the orbifold Euler characteristic and the true Teichmüller curve or the finite cover introduced for technical reasons in Section 3.1 does not change the result.

The $k_{ij}$ need not be integral. In fact in the square-tiled surface in Figure 2 the Veech group is equal to $\mathrm{SL}_2(\mathbb{Z})$. The generator of the stabilizer of the horizontal cusp produces a $1/4$ Dehn twist on both horizontal cylinders, i.e. its 4-th power produces a simple Dehn twist on both horizontal cylinders.

Proof. From the Noether formula in the version of (13) and $L(C) = 2(\overline{C}\cdot\lambda)/\chi$ we immediately obtain

$$L(C) = \kappa_\mu + \frac{1}{6}\frac{\overline{C}\cdot\delta}{\chi}.$$

We may suppose that we work over a base curve $B$ as in Section 3.1 and suppose moreover that all the $k_{ij}$ are integral by passing to yet another finite cover (if necessary) that we still denote by $B$. In the proof of Proposition 3.3 we checked that $\overline{B}\cdot\delta = \sum_{F \text{ sing.}} \Delta\chi_{\text{top}}(F)$ and that each singularity $xy = t^n$ of the stable model of $f : \mathcal{X} \to B$ contributes $n$ to this quantity. By Proposition 5.9 the degenerate fibers $X^\infty$ at the point $\infty \in \Delta$ of the Teichmüller curve is obtained by replacing, in a given direction, the core curves of the cylinders by half-infinite cylinders attached at their points at infinity to from a node. It thus suffices to check that if a simple loop around $\infty$ makes and $n$-fold Dehn twist around a cylinder, the stable model locally looks like $xy = t^n$. This is the classical computation of the Picard-Lefschetz monodromy of surface singularity. □

For any given square-tiled surface $(X, \omega)$ calculating the Veech group is no problem, since there is an explicit algorithm given in [**Sch04**]. In this case the formula specializes as follows. Let $X_i$ be the square-tiled surfaces in the $\mathrm{SL}_2(\mathbb{Z})$-orbit of $(X, \omega)$. For any square-tiled surface $X_i$ we decompose the horizontal direction into maximal cylinders $C_{ij}$ and denote by $m_{ij}$ their moduli. Then

$$L(C) = \kappa_\mu + \frac{1}{|\mathrm{SL}_2(\mathbb{Z}) \cdot (X,\omega)|} \sum_{X_i \in \mathrm{SL}_2(\mathbb{Z})\cdot(X,\omega)} \sum_{\substack{\text{all } C_{ij} \\ \text{cylinder of} X_i}} m_{ij}. \tag{15}$$

This formula is derived in [**EKZ11**] as a consequence of their main theorem relating the sum of Lyapunov exponents to Siegel-Veech constants (for *any* $\mathrm{SL}_2(\mathbb{R})$-invariant measure that satisfies a technical regularity condition). We provide here a proof that is algebraic, but it works for Teichmüller curves only.

Proof of Formula (15). We have $6\chi = [\mathrm{SL}_2(\mathbb{Z}) : \mathrm{SL}(X,\omega)] = |\mathrm{SL}_2(\mathbb{Z}) \cdot (X, \omega)|$ for a square-tiled surface $(X, \omega)$. To check that the double sums in (14) and (15) are equal, it suffices to group coset representatives of $\mathrm{SL}_2(\mathbb{Z})/\mathrm{SL}(X,\omega)$ according to the cusps and to observe that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ makes an $m_{ij}$-fold Dehn twist on a cylinder of modulus $m_{ij}$. □

This formula leaves open the need for a conceptual explanation which values occur for the sum of Lyapunov exponents. This is precisely the motivation for Section 6.4.

### 6.4. Non-varying properties for sums of Lyapunov exponents

We have seen that Teichmüller curves, in fact already those that are generated by square-tiled surfaces, are dense in each stratum. Indeed, on the level of sums of Lyapunov exponents we have the following convergence statement. Let $C_d$ denote the union of all Teichmüller curves in a fixed stratum generated by square-tiled surfaces with $d$ squares. We define the sum of Lyapunov exponents $L(C_d)$ as the average of the sums of Lyapunov exponents of the individual components weighted by the orbifold Euler characteristic (or hyperbolic volume) of the corresponding component.

**Proposition 6.7.** *For $d \to \infty$ the weighted sum of Lyapunov exponents $L(C_d)$ of square-tiled surfaces in a component of a stratum of $\Omega\mathcal{M}_g$ converges to the sum of*

*Lyapunov exponents $L_{\mu_{\rm gen}}$ for the measure $\mu_{\rm gen}$ with support on the whole component.*

The proof of this fact is due to Eskin and is published in [**Che10b**, App. A]. For the proof the statement is translated into the language of Siegel-Veech constants (that we will not define here). This translation works only for the sum of Lyapunov exponents, and thus the corresponding statement for individual exponents is still an open problem.

Even with such a limiting behavior there is no reason to expect that this sum should be the same for all Teichmüller curves. A reason to appreciate such a phenomenon – if it holds – is the wind-tree example given above. It implies the independence of the escape rate from the side lengths of the scatterers.

We say that a connected component of a stratum is *non-varying*, if for all Teichmüller curves $C$ in that component the sum of Lyapunov exponents $L(C)$ is the same. Such a non-varying phenomenon was observed numerically by Kontsevich and Zorich along with the initial observations on Lyapunov exponents for the Teichmüller geodesic flow ([**Kon97**]). Today, there are two types of non-varying results, one for low genus and one for hyperelliptic loci and two completely different methods of proof. One method uses a translation of the problem into algebraic geometry, in particular slope calculations, and the other relies on the correspondence to Siegel-Veech constants.

**Theorem 6.8** ([**CM11**])**.** *For all strata in genus $g = 3$ but the principal stratum the sum of Lyapunov exponents is non-varying.*

*For the strata with signature* $(6)^{\rm even}$, $(6)^{\rm odd}$, $(5,1)$, $(3,3)$, $(3,2,1)$ *and* $(2,2,2)^{\rm odd}$ *as well as for the hyperelliptic strata in genus $g = 4$ the sum of Lyapunov exponents is non-varying. For all the remaining strata, except maybe* $(4,2)^{\rm odd}$ *and* $(4,2)^{\rm even}$*, the sum of Lyapunov exponents is varying.*

*For the strata with signature* $(8)^{\rm even}$, $(8)^{\rm odd}$ *and* $(5,3)$ *as well as for the hyperelliptic strata in genus $g = 5$ the sum of Lyapunov exponents is non-varying. For all the other strata, except maybe* $(6,2)^{\rm odd}$*, the sum of Lyapunov exponents is varying.*

**Theorem 6.9** ([**EKZ11**])**.** *Hyperelliptic strata are non-varying. For a Teichmüller curve $C$ generated by $(X,\omega)$ we have*

$$
\begin{aligned}
L(C) &= \frac{g^2}{2g-1} \quad \text{and} \quad s(C) = 8 + \frac{4}{g} \quad \text{if} \quad (X,\omega) \in \Omega\mathcal{M}_g(2g-2)^{\rm hyp}, \\
L(C) &= \frac{g+1}{2} \quad \text{and} \quad s(C) = 8 + \frac{4}{g} \quad \text{if} \quad (X,\omega) \in \Omega\mathcal{M}_g(g-1,g-1)^{\rm hyp}.
\end{aligned} \tag{16}
$$

The result of [**EKZ11**] is a consequence of their main result relating sum of Lyapunov exponents to Siegel-Veech constants. It also gives non-varying statements in hyperelliptic loci. The case of genus two curves is a special case of this theorem and a proof appears in [**Bai07**] and in [**BM10a**].

In the left open cases of Theorem 6.8 computer experiments suggest that the strata should be non-varying, but the method of proof had no success so far, maybe due to our much limited knowledge of divisors on the moduli space of spin curves compared to the $\overline{\mathcal{M}}_g$. In most of the varying cases the method of proof of Theorem 6.8 gives interesting upper bounds for the sum of Lyapunov exponents. This can be translated e.g. into upper bounds on the escape rate of the 'wind-tree model' with other patterns of scatterers.

We now describe the idea of the proof of Theorem 6.8. Suppose that *all* Teichmüller curves $C$ in a stratum $\Omega\mathcal{M}_g(\mu)$ are disjoint from a divisor $D$ in $\overline{\mathcal{M}}_g$. Then $C \cdot D = 0$ for all these Teichmüller curves, hence $s(C) = s(D)$. Since the slopes are non-varying, so are the sums of Lyapunov exponents and the Siegel-Veech constants for these Teichmüller curves. In general, we also need to consider the moduli spaces of curves with marked points or spin structures. But the idea still relies on the non-intersection property of the Teichmüller curves with certain divisors on those moduli spaces.

We give the proof of the theorem in two instances. The first one is very simple and shows the general method. The second one is more involved. It shows that in general we have to work with the moduli space $\overline{\mathcal{M}}_{g,n}$ rather than $\overline{\mathcal{M}}_g$, since the slopes we expect for this stratum in $g = 4$ are smaller than the slope of any divisor in $\overline{\mathcal{M}}_4$

In the case $\Omega\mathcal{M}_3(4)^{\mathrm{odd}}$ the algorithm of [**EMZ03**] to calculate Siegel-Veech constants for components of strata can be translated using [**EKZ11**] into $L_{(4)^{\mathrm{odd}}} = 8/5$, hence if non-varying holds we expect by (12) any Teichmüller curve $C$ in that stratum to have $s(C) = 9$.

Proof of Theorem 6.8, Case $\Omega\mathcal{M}_3(4)^{\mathrm{odd}}$. For genus three, the connected components $\Omega\mathcal{M}_3(4)^{\mathrm{odd}}$ and $\Omega\mathcal{M}_3(4)^{\mathrm{hyp}}$ are not only disjoint in $\Omega\mathcal{M}_3$ they are also disjoint in $\Omega\overline{\mathcal{M}}_3$ since they are distinguished by the parity of spin structures, which is known to be deformation invariant over all of $\overline{\mathcal{M}}_g$ for any $g$.

We need this property only for boundary points of Teichmüller curves, which is shown in Proposition 5.13. In any case, Teichmüller curves in this stratum do not intersect the closure of the hyperelliptic locus $H$ in $\overline{\mathcal{M}}_3$. Recall the divisor class of $H$ in (2). From $s(H) = 9$ and $C \cdot H = 0$, we obtain that $s(C) = 9$, hence $L(C) = 8/5$ for all Teichmüller curves in this stratum using (12). $\square$

In the case $\Omega\mathcal{M}_4(3,3)^{\mathrm{non-hyp}}$ we can calculate as above $L_{(3,3)^{\mathrm{non-hyp}}} = 2$ hence if non-varying holds we expect for any Teichmüller curve $C$ in that stratum $s(C) = 33/4$. Note that $s(C)$ is smaller than the lower bound $17/2$ for slopes of effective divisors in $\overline{\mathcal{M}}_4$ ([**HM90**]).

Recall from Section 5.4 that we may define, after a finite unramified covering that does not change Lyapunov exponents, sections $\sigma_i$ for $i_1, \ldots, i_k$ of the family $f : \overline{X} \to \overline{B}$ over the Teichmüller curve corresponding to the singularities of the generating holomorphic one-from $\omega$. With the help of (a selection of) these sections we may lift the Teichmüller curve to a map $\overline{B} \to \overline{\mathcal{M}}_{g,n}$ for some $n \leq k$.

Proof of Theorem 6.8, Case $\Omega\mathcal{M}_4(3,3)^{\mathrm{non-hyp}}$. Let $C$ be a Teichmüller curve generated by a flat surface $(X, \omega)$ in the stratum $\Omega\mathcal{M}_4(3,3)^{\mathrm{non-hyp}}$, lifted to $\overline{B} \to \overline{\mathcal{M}}_{4,2}$. Recall that $\mathrm{Lin}_3^1 \subset \overline{\mathcal{M}}_{4,2}$ parametrizes pointed curves $(X, p, q)$ that admit a $g_3^1$ with a section vanishing at $p, q, r$ for some $r \in X$. We first want to show that $\overline{C}$ (or $\overline{B}$) does not intersect $\mathrm{Lin}_3^1$.

Suppose that $(X, p, q)$ is contained in the intersection of $\overline{B}$ with $\mathrm{Lin}_3^1$. Since $\omega_X \sim \mathcal{O}_X(3p + 3q)$ and since being in $\mathrm{Lin}_3^1$ implies $h^0(\mathcal{O}_X(p + q + r)) \geq 2$, by the Riemann-Roch theorem we know that $h^0(\mathcal{O}_X(2p + 2q - r)) \geq 2$. If $r \neq p, q$, then $h^0(\mathcal{O}_X(2p + 2q)) \geq 3$, hence $2p + q$ and $2q + p$ both admit $g_3^1$. Since $X$ is not hyperelliptic the canonical image of $X$ is contained in a quadric in $\mathbb{P}^3$. This quadric has at most two rulings (only one if the quadric is singular) and each $g_3^1$ corresponds to a ruling of the quadric. Consequently, both $2p + q$ and $2q + p$ define the same

$g_3^1$, the one defined by the line connecting $p, q$. This implies $p \sim q$, a contradiction. If $r = p$ or $q$, again, $2p + q$ and $2q + p$ both admit $g_3^1$ and consequently $C$ is hyperelliptic. But this stratum is non-hyperelliptic, and Proposition 5.13 yields the desired contradiction.

From $\overline{C} \cdot \mathrm{Lin}_3^1 = 0$ together with Proposition 5.12 and $\kappa_{(3,3)} = 5/8$, the result follows immediately. □

The last calculation is the calculation for the second test curve in the proof of Proposition 3.2 read backwards.

### 6.5. Lyapunov exponents for general curves in $\mathcal{M}_g$ and in $\mathcal{A}_g$

The main point of this section is that all the formalism of Lyapunov exponents is also perfectly valid in this more general setting. For curves in $\mathcal{M}_g$ we show that calculating the sum is equivalent to the (interesting and partially understood) problem of computing slopes that we addressed in Section 3.4. Moreover, it raises the interesting question of identifying individual Lyapunov exponents for other curves besides Teichmüller curves.

Let $f : \mathcal{A} \to C$ be a non-constant family of abelian varieties. Then the universal covering of $C$ is the upper half plane and $C = \mathbb{H}/\Gamma$. We provide the unit tangent bundle $T^1C$ to $C$ with the metric $\mu$ that stems from Haar measure from $\mathrm{SL}_2(\mathbb{R})$ and with the geodesic flow $g_t$. In order to apply the calculation used in the proof of Theorem 6.2 we only need scalar curvature $-4$ for the correct relation between $\chi(C)$ and $\mathrm{vol}(C)$ and we needed $g_t$ to have geodesic unit speed, i.e. $g_t = \mathrm{diag}(e^t, e^{-t})$. We also use these conventions here.

The direct image $R^1f_*\mathbb{R}$ is a local system $\mathbb{V}$ on $C$ (with fiber $H^1(A, \mathbb{R})$ of dimension $2\dim A$). We pull this local system back to $T^1C$ and provide it with the Hodge metric. The key to get started is the following lemma.

**Lemma 6.10.** *The lift of $g_t$ to $\mathbb{V}$ with respect to $\mu$ and the Hodge norm satisfies the integrability hypothesis in Oseledec's theorem.*

Proof. It suffices to give a global bound (i.e. independent of the abelian variety $A$) on the operator norm of $g_t(\cdot)$ on the (Hodge) norm one-ball in $H^1(A, \mathbb{R})$. To do so, it suffices to bound the derivative of the Hodge norm of any element in the (Hodge) norm one-ball in $H^1(A, \mathbb{R})$ in the direction of some $v \in T^1\mathcal{A}_g$. If $p : \mathbb{H} \to \mathbb{H}_g$ denotes the period map associated with $f$, this can be rephrased by bounding $||dp||$, where $||dp||$ is the norm induced by the Bergmann-Siegel metric and the Poincaré metric on $\mathbb{H}_g$ and $\mathbb{H}$ respectively. A generalization of the Schwarz-Pick Lemma (e.g. [**Roy80**, Theorem 2]) implies that $||dp|| \le k/K$ where $k$ and $K$ are the curvatures of the metrics on domain and range of $p$ respectively. With our normalization we have $k = K = -4$ and thus $||dp|| \le 1$. □

Given the lemma, we may talk of the Lyapunov exponents $\lambda_i$ of the geodesic flow on $C$. From the proof we deduce that

$$\lambda_i \le 1 \tag{17}$$

for all $i$ with our curvature normalization, generalizing the expectation from the case of Teichmüller curves.

As before, we take a subgroup $\Gamma_1 \subset \Gamma$ without elements of finite order such that the monodromy around the cusps acts unipotently on the fibers. We let $B = \mathbb{H}/\Gamma_1$

and denote by $f : \mathcal{A} \to B$ the pullback of the family over $C$ to $B$. Again, such a pullback does not change the spectrum of Lyapunov exponents.

The following proposition has been justified along with Theorem 6.2.

**Proposition 6.11.** *Let $\mathbb{V}$ be the weight one VHS associated with the family of abelian varieties $f$ of dimension $g$. Then the sum of the $g$ non-negative Lyapunov exponents equals*

$$\sum_{i=1}^{g} \lambda_i^{\mathbb{V}} = \frac{2 \deg \mathbb{V}^{(1,0)}}{2g(\overline{B}) - 2 + |\Delta|},$$

*where $\mathbb{V}^{(1,0)}$ is the $(1,0)$-part of the Hodge-filtration of the vector bundle associated with $\mathbb{V}$.*

This proposition together with equation (17) implies the *Arakelov inequality*

$$\frac{2 \deg \mathbb{V}^{(1,0)}}{2g(\overline{B}) - 2 + |\Delta|} \leq g. \tag{18}$$

This inequality appears first in [**Fal83**]. See also [**VZ04**] for some background and references to other versions.

**Towards a characterization of Shimura curves.** A *Shimura curve* is a curve $C \to \mathcal{A}_g$ that is obtained as the quotient $C = K \backslash G_{\mathbb{R}}/\Gamma \to U(g)\backslash \mathrm{Sp}_{2g}(\mathbb{R})/\mathrm{Sp}_{2g}(\mathbb{Z})$ induced by some inclusion of $\mathbb{Q}$-algebraic groups $G \to \mathrm{Sp}_{2g}$, where $K$ is a maximal compact subgroup of $G_{\mathbb{R}}$ and where $\Gamma$ is some arithmetic lattice. Equivalently, a Shimura curve is a locus of abelian varieties admitting additional 'endomorphisms'. We put 'endomorphisms' in quotation marks, since some Shimura curves are defined by the existence of endomorphisms but in general the presence of Hodge classes is the appropriate condition. We refer to the recent survey [**MO10**] for details of the definition.

This is almost the same as requiring that $C \to \mathcal{A}_g$ is totally geodesic for the 'Bergmann-Siegel' Riemannian metric on $\mathcal{A}_g$, the unique (up to scalar) Riemannian metric on the Hermitian symmetric domain $\mathbb{H}_g$ that is invariant under the action of $\mathrm{Sp}_{2g}(\mathbb{R})$. (Note that for $g > 1$ the Bergmann-Siegel metric is not the Kobayashi metric on $\mathbb{H}_g$. The latter is just a Finsler metric, not Riemannian for $g > 1$.) In fact, such a totally geodesic curve is a Shimura curve if and only if it contains a CM point. This obstruction is very minor. The only thing that can happen to produce a totally geodesic curve that is not Shimura is to take a product of a family of abelian varieties over a Shimura curve times a constant family consisting of a non-CM abelian variety. See again [**MO10**] and the references therein for details.

We have the following partial characterization of Shimura curves.

**Proposition 6.12.** *If the curve $C \to \mathcal{A}_g$ is totally geodesic for the Riemannian metric on $\mathcal{A}_g$, then the Lyapunov spectrum contains only the values $\pm 1$ and zero.*

*Conversely, if the Lyapunov spectrum of $C \to \mathcal{A}_g$ contains only the values $\pm 1$, then $C$ is a Shimura curve.*

Proof. If $C \to \mathcal{A}_g$ is totally geodesic, then we may change $C$ into a Shimura curve without changing the Lyapunov spectrum. Consequently, we may apply [**Möl11**, Theorem 1.2]. This theorem states that the VHS decomposes into a unitary part (that gives rise to zero Lyapunov exponents) and the standard Fuchsian representation of $\Gamma$ (that gives rise to Lyapunov exponent $\pm 1$) tensored with some unitary representation (that accounts for the multiplicity of $\pm 1$).

Conversely, if the positive Lyapunov exponents are all one, then the VHS over $C$ attains the upper bound in the Arakelov equality (18). Now the fundamental result of Viehweg and Zuo ([**VZ04**, Theorem 0.5]) implies that $C$ is a Shimura curve (without unitary direct summand in the VHS). □

**Open problem.** Can one generalize Proposition 6.12 to characterize Shimura curves as those curves in $C \to \mathcal{A}_g$ with Lyapunov spectrum only zero and $\pm 1$ ? That is, do the Lyapunov subspaces for zero and those with $\pm 1$ span local subsystems?

### 6.6. Known results and open problems

There are many variants of the comparison problem of Lyapunov exponents for Teichmüller curves to those of the ambient strata. For the Masur-Veech measure it was shown by Avila and Viana that the Lyapunov spectrum is simple, thus solving the Kontsevich-Zorich conjecture ([**AV07**]). It is tempting to guess that for square-tiled surfaces with a large number of squares the Lyapunov spectrum of the corresponding Teichmüller curves (in a fixed connected component of a stratum) converges to the Lyapunov spectrum for the measure $\mu_{\mathrm{gen}}$ supported on the whole connected component. Besides Proposition 6.7 no such statement is presently known. Is the Lyapunov spectrum for all but finitely many Teichmüller curves in a given stratum simple?

The nature of individual Lyapunov exponents is a wide open question. Are they related to characteristic classes? It seems, on the contrary, that partial sums of (non-zero) Lyapunov exponents are rational only if the Oseledec's bundle of this partial sum is a summand in the decomposition of the VHS. At least, there are presently no counterexamples.

Lyapunov exponents are hard to determine even numerically. The presently known and implemented algorithms are based on the first motivation (Zorich, Kontsevich), but they are exponentially slow (in desired accuracy) due to the log in the definition of Lyapunov exponents. It would be interesting to have an alternative formulation that allows faster computation.

The stratum $\Omega\mathcal{M}_3(1,1,1,1)$ is one of smallest that does not have the non-varying property. With computer help one easily produces examples of Teichmüller curves in this stratum with

$$L(C) \in \{1, 3/2, 5/3, 7/4, 9/5, 11/6, 19/11, 33/19, 83/46, 544/297\}.$$

The value for the measure with support on the whole stratum is $L_{(1,1,1,1)} = 53/28$. What is the set of values that the sum of Lyapunov exponents for Teichmüller curves attains? At least, what is its set of accumulation points?

A Teichmüller curve has *maximally degenerate Lyapunov spectrum*, if $\lambda_1 = 1$ and $\lambda_{2g} = -1$ are the only two non-zero Lyapunov exponents. The 'eierlegende Wollmilchsau' in Figure 2 is one of the two known Teichmüller curves with maximally degenerate Lyapunov spectrum. Its name refers to the fact it has many different remarkable properties at the same time and serves ubiquitously as counterexamples to many naive conjectural properties of square-tiled surfaces. The eierlegende Wollmilchsau was discovered by [**HS08**] and [**For06**] independently. It is a cyclic cover

$$y^4 = x(x-1)(x-t), \quad \omega = \frac{dx}{y}$$

and also the other known Teichmüller curve with maximally degenerate Lyapunov spectrum ([**FM08**]) is a cyclic cover

$$y^6 = x(x-1)(x-t), \quad \omega = \frac{dx}{y}.$$

In fact, a Teichmüller curve has maximally degenerate Lyapunov spectrum if and only if it is one of the two families, possibly with exceptions in strata in $g = 5$ ([**Möl11**]). Does such an exception exist?

**References:** A good introduction to Lyapunov exponents with a lot of motivating examples is the survey by Zorich ([**Zor06**]). With (even) more emphasis on dynamics an introduction to Lyapunov exponents is given in the lecture notes of M. Viana ([**Via10**]).

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