CUTTING OUT ARITHMETIC TEICHMÜLLER CURVES IN GENUS TWO VIA THETA FUNCTIONS

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Abstract. We compute the class of arithmetic genus two Teichmüller curves in the Picard group of pseudo-Hilbert modular surfaces, distinguished according to their torsion order and spin invariant. As an application, we compute the number of genus two square-tiled surfaces with these invariants.

The main technical tool is the computation of divisor classes of Hilbert Jacobi forms on the universal abelian surface over the pseudo-Hilbert modular surface.

Contents

1. Introduction

The aim of this paper is to contribute to the classification of arithmetic Teichmüller curves and the computation of their basic invariants. The extension of the bundle of Jacobi forms to the universal family of abelian surfaces over pseudo-Hilbert modular surfaces and the computation of its class will be our main technical tool.

Arithmetic Teichmüller curves. Square-tiled surfaces are covers of the square torus, ramified over at most one point. Affinely deforming the squares into parallelograms yields a curve in the moduli space of curves, called arithmetic Teichmüller curve. Non-arithmetic Teichmüller curves, which are generated by flat surfaces that do not arise via branched coverings of the torus, have been classified in genus two ([McM05], [McM06]), and in higher genus there is a growing number of partial results. For Teichmüller curves generated by square-tiled surfaces, the classification problem is solved only for genus two surfaces with a single ramification point

Date: April 4, 2015.
The authors are partially supported by the ERC-StG 257137.

([HL06] for prime degree coverings and [McM05] in general). They are classified by two invariants, the number of squares and the spin.

Genus two, two ramification points. Genus two square-tiled covers with two ramification points come with three obvious invariants. One is the spin invariant, the number of integral Weierstraß points. The other two are the torsion order of the two branch points in a minimal intermediate torus covering and the degree of this covering (see Section 2). It is conjectured (and well-supported by computer experiments of Delecroix and Lelièvre) that these are the only invariants, i.e. that the set $T_{d,M,\varepsilon}$ of genus two degree d covers of the torus with torsion order M and spin ε is irreducible. For one ramification point, both [HL06] and [McM05] solved the irreducibility question combinatorially by exhibiting prototypes for the flat surfaces and connecting any two of the same invariants by a change of direction. This approach might work for two ramification points as well, but the combinatorial complexity is challenging.

This paper does not contain any picture of a flat surface. Instead we propose to tackle the classification problem by first computing the class of $T_{d,M,\varepsilon}$ in the (rational) Picard group of a pseudo-Hilbert modular surface and in the second step to argue that this class is not too divisible and that potential summands cannot be Teichmüller curves.

Counting square-tiled surfaces. In this paper, we complete the first step in this program for odd d. As a result, we can solve the following counting problem. For $M = 1$ this has been conjectured by Zmiaikou ([Zmi11, p. 67]).

Theorem 1.1. *The number $t_{d,M,\varepsilon}$ of reduced square-tiled surfaces of genus two, two ramification points, odd degree d, torsion order M and spin invariant ε is given as follows.*
If $M > 1$ is odd, then

$$t_{d,M,\varepsilon=3} \;=\; \frac{1}{24}(d-1)\Delta_d\frac{\Delta_M}{M} \qquad\qquad t_{d,M,\varepsilon=1} \;=\; \frac{1}{8}(d-1)\Delta_d\frac{\Delta_M}{M}.$$

If M is even, then there is no spin invariant and

$$t_{d,M} \;=\; \frac{1}{6}(d-1)\Delta_d\frac{\Delta_M}{M}.$$

If $M = 1$, then

$$t_{d,M=1,\varepsilon=3} \;=\; \frac{1}{24}(d-3)(d-5)\frac{1}{d}\Delta_d \quad \text{and} \quad t_{d,M=1,\varepsilon=1} \;=\; \frac{1}{8}(d-1)(d-3)\frac{1}{d}\Delta_d.$$

Remark 1.2. *In principle, the same program can be carried out for even d, but it requires performing similar computations as we present them for covering surfaces with an extra level of two (see Section 9.4). The conjectural values for the counting problem are as follows. For $M > 1$ and d even we have*

$$t_{d,M,\varepsilon=0} \;=\; \frac{1}{24}(d-1)\Delta_d\frac{\Delta_M}{M} \qquad\qquad t_{d,M,\varepsilon=2} \;=\; \frac{1}{8}(d-1)\Delta_d\frac{\Delta_M}{M},$$

and for $M = 1$ and d is even the values are

$$t_{d,M=1,\varepsilon=0} \;=\; \frac{1}{24}(d-2)\Delta_d \quad \text{and} \quad t_{d,M=1,\varepsilon=2} \;=\; \frac{1}{8}(d-2)(d-4)\frac{1}{d}\Delta_d\,.$$

The *sum* of contributions of the two spin structures appeared in [Kan06, Theorem 3] and in [EMS03], see Proposition 9.9 for the conversion of the two methods of counting.

Classes in the Picard group. The above counting result is a consequence of the following statement that gives the class of the (union of) Teichmüller curves $T_{d,M,\varepsilon}$ generated by the square-tiled surfaces of degree d, with torsion order M and spin ε on the compactified pseudo-Hilbert modular surface X_{d^2}, whose open part $X^\circ_{d^2}$ parametrizes abelian surfaces with multiplication by a pseudo-quadratic order. See Section 3 for the definition of X_{d^2} and the Hodge bundles λ_i.

Theorem 1.3. *Let d be odd. The class of $T_{d,M,\varepsilon}$ in $\mathrm{Pic}_{\mathbb{Q}}(X_{d^2})$ is given as follows. If $M > 1$ is odd, then*

$$\begin{aligned} [T_{d,M,\varepsilon=3}] &= \tfrac{1}{2}\, d\,\frac{\Delta_M}{M}\Big((1-\tfrac{1}{d})\lambda_1 + (2-\tfrac{2}{d})\lambda_2\Big), \\ [T_{d,M,\varepsilon=1}] &= \tfrac{3}{2}\, d\,\frac{\Delta_M}{M}\Big((1-\tfrac{1}{d})\lambda_1 + (2-\tfrac{2}{d})\lambda_2\Big). \end{aligned} \tag{1}$$

If M is even then

$$[T_{d,M}] = 2d\frac{\Delta_M}{M}\Big((1-\tfrac{1}{d})\lambda_1 + (2-\tfrac{2}{d})\lambda_2\Big). \tag{2}$$

If $M = 1$, then

$$\begin{aligned} [T_{d,M=1,\varepsilon=3}] &= \tfrac{1}{d}\Big(\tfrac{1}{2}(d-3)(d-5))\lambda_1 + (d-3)(d-5)\lambda_2\Big), \\ [T_{d,M=1,\varepsilon=1}] &= \tfrac{3}{d}\Big((\tfrac{1}{2}(d-1)(d-3))\lambda_1 + (d-1)(d-3)\lambda_2\Big). \end{aligned} \tag{3}$$

Remark 1.4. *The conjectural classes for the case d even are given as follows. If $M > 1$ is odd, then*

$$\begin{aligned} [T_{d,M,\varepsilon=0}] &= \tfrac{1}{2}d\frac{\Delta_M}{M}\Big((1-\tfrac{1}{d})\lambda_1 + (2-\tfrac{2}{d})\lambda_2\Big), \\ [T_{d,M,\varepsilon=2}] &= \tfrac{3}{2}d\frac{\Delta_M}{M}\Big((1-\tfrac{1}{d})\lambda_1 + (2-\tfrac{2}{d})\lambda_2\Big). \end{aligned} \tag{4}$$

If M is even then

$$[T_{d,M}] = 2d\frac{\Delta_M}{M}\Big((1-\tfrac{1}{d})\lambda_1 + (2-\tfrac{2}{d})\lambda_2\Big). \tag{5}$$

If $M = 1$, then

$$\begin{aligned} [T_{d,M=1,\varepsilon=0}] &= \Big(\tfrac{1}{2}(d-2)\lambda_1 + (d-2)\lambda_2\Big), \\ [T_{d,M=1,\varepsilon=2}] &= \tfrac{3}{d}\Big(\tfrac{1}{2}(d-2)(d-4)\lambda_1 + (d-2)(d-4)\lambda_2\Big). \end{aligned} \tag{6}$$

Strategy of the proof. Instead of locating a Teichmüller curve inside X_{d^2}, we locate the branch points of the covering map from the flat surface to the torus inside the universal family $A^\circ_{d^2}$ of abelian surfaces over the open subset $X^\circ_{d^2}$. The points that we want to single out lie on image of the flat surface in its Jacobian (i.e. on the theta divisor), they are branch points (i.e. the derivative of the theta function vanishes in some direction), and they have the property that their image in a certain intermediate elliptic curve is M-torsion. Theorem 8.3 expresses that the image of this intersection of three divisorial conditions in X_{d^2} is the Teichmüller curve. The basic idea to use theta functions builds on that in [Möl14], but there one could work entirely in the two-dimensional base, while most of the difficulties here come from performing the triple intersection in the four-dimensional total space. Of course, for intersection theory calculations, we need to work on a reasonable (normal, at most quotient singularities) compactification A_{d^2} of $A^\circ_{d^2}$. We recall the background on toroidal compactifications and construct A_{d^2} in Section 5. The family A_{d^2} comes with some obvious divisors (boundary components, Hodge bundle, zero sections), whose intersection product is readily computed. The goal is hence to express the ingredients of the triple intersection in these terms.

Jacobi forms for pseudo-Hilbert modular surfaces. Hilbert Jacobi forms are functions on the universal covering $\mathbb{H}^2 \times \mathbb{C}^2$ of $A^\circ_{d^2}$ whose transformation law combines the elliptic behavior on $\mathbb{C}^2$ and the modular behavior on $\mathbb{H}^2$ in the usual way as for elliptic Jacobi forms. The precise definitions are given in Section 6.3. The basic example of a Jacobi form is the theta function, both in the elliptic and in the pseudo-Hilbert modular case. We would like to express the divisor class of a Jacobi form on A_{d^2} in terms of the natural divisors mentioned above. We stress that, however, this question is *not even well-defined*. Only after making some artificial choice at the boundary (our choice is (30) in Section 6.3) we can determine the class of a Jacobi form in Theorem 6.1.

At the end of the day, we are only interested in the class of a divisor (the Teichmüller curve) generically lying in $X^\circ_{d^2}$. Consequently, we have to determine and subtract in Section 9.2 the spurious boundary components, thereby compensating the arbitrariness in the boundary extension of Jacobi forms.

Finally, in the case of $M = 1$, the analogous statement of Theorem 8.3 is Theorem 8.4 and there two other spurious summands occur. One contribution is from the reducible locus in X_{d^2}, whose class we determine in Section 7. The other contribution stems from square-tiled surfaces with only one branch point. The classes of the corresponding Teichmüller curves have been determined in [Bai07].

2. Origamis, Square-tiled surfaces and their spin structure

Let $\Omega\mathcal{M}_g$ be the moduli space of flat surfaces (X, ω) and for any partition κ of $2g-2$, let $\Omega\mathcal{M}_g(\kappa)$ be the stratum, where the divisor of ω has type κ. In this paper (X, ω) will always be an arithmetic Veech surface of genus $g > 1$. This is equivalent to requiring the existence of an *origami map*, a covering $p : X \to E$ to an elliptic curve E such that p is branched over only one point and $\omega = p^*\omega_E$. The map p is unique only up to isogeny and translation on E (the latter can be dispensed with by translating the unique branch point to the origin). We call p *reduced*, if it does not factor over an origami map $p' : X \to E'$ that has strictly smaller degree. Equivalently, p is reduced, if and only if the lattice of generated by relative periods

$$\mathrm{Per}(\omega) = \Big\{ \int_\gamma \omega \mid \gamma \in H_1(X, Z(\omega), \mathbb{Z}) \Big\} \subset \mathbb{C}$$

is equal to $\mathrm{Per}(\omega_E) = \{\int_\gamma \omega_E : \gamma \in H_1(E, \mathbb{Z})\}$.

If E is the particular elliptic curve with $j(E) = 1728$, then X is called *square-tiled surface*. In this case, $\mathrm{Per}(\omega) \subset \mathbb{Z} \oplus i\mathbb{Z}$.

A covering $q : X \to E'$ to an elliptic curve E' is called *minimal* or *optimal*, if it does not factor over an isogeny of degree > 1. A covering is minimal, iff the induced map q_* on the first absolute homology is surjective.

Let $E'[2] = \{P_0, P_1, P_2, P_3\}$ denote the set of 2-torsion points, where $P_0 = 0$, and let $\sigma \in \mathrm{Aut}(X, \omega)$ denote the hyperelliptic involution. Let W_X denote the Weierstraß divisor on X. From now on we restrict to the case of genus two surfaces.

Proposition 2.1. *For any arithmetic Veech surface of genus* 2*, there is a reduced origami map* $p : X \to E$ *and a decomposition* $p = \iota \circ q$ *into a minimal covering* $q : X \to E'$ *of degree* d *and an isogeny* $\iota : E' \to E$ *of degree* $M \geq 1$.

The map q*, and a fortiori* p*, is uniquely determined by the requirement that*

$$q_* W_X = \begin{cases} 2(P_1 + P_2 + P_3), & \text{if } d \equiv 0 \bmod 2 \\ 3P_0 + P_1 + P_2 + P_3, & \text{if } d \equiv 1 \bmod 2\,. \end{cases}$$

We call the origami map p with a factorization and location of branch points as in this proposition *normalized.*

Proof. By [Kan03, Proposition 2.2], there is a uniquely determined minimal, normalized covering $q : X \to E'$. Moreover, this covering satisfies

$$[-1] \circ q = q \circ \sigma$$

and since the ramification points of q are not fixed by σ, their images P, Q satisfy $[-1]Q = P$. Let $\iota : E' \to E$ be an isogeny with $\iota(P) = \iota(Q) = [-1]\iota(P)$, or equivalently $\iota([2]P) = 0$. Such an isogeny exists since (X, ω) is a Veech surface, and hence $P - Q$ is of finite order. The minimal such is given by the quotient map $E' \to E'/T$, where T is the subgroup generated by $[2]P$. □

It is possible that $M = 1$. In this case, the branching divisor is non-reduced, i.e. $P = Q \in E'[2]$. The integers d and M are uniquely determined by the Veech surface. We call $d = d(X, \omega)$ the *degree* and $M = M(X, \omega)$ the *torsion order* of (X, ω).

2.1. **Spin structure.** Let $(X, \omega) \in \Omega\mathcal{M}_2$ be an arithmetic Veech surface with reduced, normalized covering $p : X \to E$. A Weierstraß point $\widetilde{P}$ is called *integral*, if $p(\widetilde{P})$ is equal to the branch point of p. The number of integral Weierstraß points is an invariant of the $\mathrm{SL}_2(\mathbb{R})$-orbit of (X, ω), called the *spin invariant* $\varepsilon(X, \omega)$. Depending on the parity of d and M, we determine when it distinguishes orbits.

Let $p : X \to E$ factorize as $p = \iota \circ q$ with a minimal, normalized covering q and an isogeny ι of degree $M \geq 1$. Let P denote one of the branch points of q. Then $\iota(P) \in E[2]$. If $M \equiv 1 \bmod 2$, then the induced map $\iota[2]$ on the 2-torsion points is an isomorphism. Thus, if $d \equiv 1 \bmod 2$

$$\varepsilon(X, \omega) = \begin{cases} 3, & \text{if } \iota(P) = 0 \\ 1, & \text{if } \iota(P) \neq 0 \end{cases}$$

and if $d \equiv 0 \bmod 2$, then

$$\varepsilon(X, \omega) = \begin{cases} 0, & \text{if } \iota(P) = 0 \\ 2, & \text{if } \iota(P) \neq 0 \end{cases}$$

If on the other hand, $M \equiv 0 \bmod 2$, then P is a primitive $2M$-torsion point, and the fiber of $\iota(P)$ does not contain a 2-torsion point, since the equation $P + [2k]P = -P - [2k]P$ has no solution $k \in \mathbb{Z}$. Thus in this case

$$\varepsilon(X, \omega) = 0.$$

Note that the preceding discussion applies both to arithmetic Veech surfaces in $\Omega\mathcal{M}_2(1, 1)$ and to arithmetic Veech surfaces in $\Omega\mathcal{M}_2(2)$. In the second case $M = 1$ of course.

Next we consider the case that X is a *reducible genus two surface* but with compact Jacobian, i.e. $X = E_1 \cup E_2$ is the union of two elliptic curves joined at a node S. In this case an *origami map* $p : X \to E$ is simply defined to be a map that is non-constant on both factors, or equivalently $\omega = p^*\omega$ is non-zero on both components. This implies that E_1 and E_2 (and E) are isogenous. If $d_i = \deg(p|_{E_i})$ then obviously $d = \deg(p) = d_1 + d_2$. We call Weierstraß divisor W_X on X the set of fixed points different from S of the elliptic involutions on E_1 and E_2 with respect to the zero S. Obviously $|W_X| = 6$ as in the smooth case. This notion is justified since one easily checks that for any family of flat surfaces (X_t, ω_t) degenerating to (X, ω), the Weierstraß divisor W_{X_t} converges to W_X. Again we let $\varepsilon(X, \omega)$ be the number of integral Weierstraß points, i.e. the number of points in W_X with image equal to $p(S)$.

There are no integral Weierstraß points on a component E_i iff d_i is odd. If d_i is even, there is three or one Weierstraß point, depending on whether $p|_{E_i}$ factorizes through multiplication by two or not. The latter can happen only if d_i is divisible by four. For $d \equiv 1 \bmod 2$ consequently

$$\varepsilon(X,\omega) \in \{1,3\},$$

since precisely one of the d_i is odd. If d is even, then both d_i might be odd, resulting in no integral Weierstraß points. If both d_i are even and one of the maps p_i factors through multiplication by two, then p factors through a two-isogeny. Consequently, if p is a reduced origami map and $d \equiv 1 \bmod 2$, then

$$\varepsilon(X,\omega) \in \{0,2\}.$$

where $\varepsilon(X,\omega) = 0$ corresponds to both d_i odd.

3. Pseudo-Hilbert modular surfaces

In this section we introduce the surfaces containing the Teichmüller curves we are interested in. These are moduli spaces for Abelian surfaces with multiplication by pseudo-quadratic orders that we call *pseudo-Hilbert modular surfaces* X_{d^2}. They admit a finite cover, which is a product of two modular curves. Consequently, many line bundles on X_{d^2} arise from line bundles on the modular curves and we summarize the main properties. Next, we introduce the Teichmüller curves on X_{d^2} and fix notation for all the divisors on X_{d^2} we need. See also [Bai07], [Her91] or [McM07] for basic properties of pseudo-Hilbert modular surfaces.

3.1. **Modular curves and modular forms.** We let $\Gamma(d) \subset \mathrm{SL}_2(\mathbb{Z})$ be the principal congruence group of level $d \in \mathbb{N}$ and $X(d)^\circ = \mathbb{H}/\Gamma(d)$ be the (open) *modular curve*. Its smooth compactification is denoted by $X(d)$. If $d \geq 3$, the curve $X(d)$ has $\nu_{\infty,d} = \frac{[\Gamma(1):\Gamma(d)]}{2d}$ cusps $R_{d,j}$ and genus $g(X(d)) = 1 + \frac{d-6}{24d}\,|\mathrm{SL}_2(\mathbb{Z}/d\mathbb{Z})|$.

We record that $X(d) \to X(1)$ is a covering of degree

$$\Delta_d := |\mathrm{SL}_2(\mathbb{Z}/d\mathbb{Z})| = [\Gamma(1):\Gamma(d)] = d^3 \prod_{p|d}(1-p^{-2})$$

if we consider these curves as quotient stacks. (In terms of coarse moduli spaces, if we let $\overline{\Gamma}(d)$ denote the image of $\Gamma(d)$ in $\overline{\Gamma}(1) = \mathrm{PSL}_2(\mathbb{Z})$, the covering is of degree $[\overline{\Gamma}(1):\overline{\Gamma}(d)]$, which is half the degree above for $d \geq 3$.)

The *Hodge bundle* on $X(d)$ is $\lambda = \varpi_*(\omega_{E(d)/X(d)})$, where $\varpi : E(d) \to X(d)$ is the (compactified) universal family (see Section 5). We also write $\lambda_{X(d)}$ if we want to emphasized the level. Global sections of $\lambda_{X(d)}^{\otimes k}$ are modular forms of weight k for $\Gamma(d)$. Moreover, $\lambda_{X(d)}^{\otimes 2} = K_{X(d)}(R_d)$, where R_d is the divisor of cusps and $K_{X(d)}$ is the canonical bundle.

The discriminant f_Δ is a modular form of weight 12 for $\Gamma(1)$. It is non-zero on $X(d)^\circ$ and vanishes to the order d at each cusp $R_{d,j}$ $(j = 1,\ldots,\nu_{\infty,d})$ of $X(d)$. Thus

$$12\lambda_{X(d)} = d \cdot R_d. \tag{7}$$

The principal congruence group of level d is conjugate to another congruence group

$$\Gamma(d)_d = \mathrm{diag}(d,1)\cdot\Gamma(d)\cdot\mathrm{diag}(d^{-1},1).$$

Consequently, the action of $\Gamma(d)_d$ and $\Gamma(d)$ on $\mathbb{H}$ are equivariant with respect to the multiplication map by d on $\mathbb{H}$ and there is an isomorphism

$$X(d)^\circ = \mathbb{H}/\Gamma(d) \cong \mathbb{H}/\Gamma(d)_d =: X(d)_d^\circ.$$

The two-fold product of the groups $\Gamma(d)_d$ appears naturally as subgroup of pseudo-Hilbert modular groups, as we will see next.

3.2. **Pseudo-Hilbert Modular surfaces.** Let $d \in \mathbb{N}$ and $D = d^2$. Following the conventions for Hilbert modular surfaces, we let $K = \mathbb{Q} \oplus \mathbb{Q}$, whose subring

$$\mathfrak{o}_{d^2} = \{x = (x', x'') \in \mathbb{Z} \oplus \mathbb{Z} : x' \equiv x'' \bmod d\} \subset K$$

will be called a pseudo-quadratic order of discriminant D. Let $\mathfrak{o}_{d^2}^\vee = \frac{1}{(d,-d)}\mathfrak{o}_{d^2}$ be the inverse different. The pseudo-Hilbert modular group is

$$\Gamma_{d^2} \;=\; \mathrm{SL}(\mathfrak{o}_{d^2} \oplus \mathfrak{o}_{d^2}^\vee)$$

and *pseudo-Hilbert modular surface* is the quotient[1]

$$X_{d^2}^\circ = \mathbb{H}^2/\Gamma_{d^2}.$$

It is the moduli space parameterizing abelian surfaces with multiplication by the pseudo-quadratic order of discriminant d^2 as we will see in Section 4. Since

$$\Gamma(d)_d^2 \subset \Gamma_{d^2} \subset \Gamma(1)_d^2,$$

both inclusions being of degree $|\mathrm{SL}_2(\mathbb{Z}/d\mathbb{Z})|$, the pseudo-Hilbert Modular surface admits a useful covering given by

$$\tau : (X(d)_d^\circ)^2 \to X_{d^2}^\circ$$

and a quotient given by

$$\beta : X_{d^2}^\circ \to (X(1)_d^\circ)^2.$$

The factor group $\Gamma(1)_d^2/\Gamma(d)_d^2$, and thus a fortiori $\Gamma_{d^2}/\Gamma(d)_d^2$, acts on the smooth compactification $X(d)_d^2$ of $(X(d)_d^\circ)^2$ and the quotient maps τ and β extend to quotient maps

$$\tau : X(d)_d^2 \to X_{d^2}, \qquad \text{and} \qquad \beta : X_{d^2} \to X(1)_d^2.$$

We will work with this normal (but not smooth) compactification X_{d^2} of $X_{d^2}^\circ$. In fact X_{d^2} is the Baily-Borel compactification of $X_{d^2}^\circ$. We now list the divisors on X_{d^2} that will be important in the sequel.

Boundary divisors. The image of $(\overline{\mathbb{H}} \setminus \mathbb{H}) \times \mathbb{H}$ is a curve $R^{(1),\circ} \subset X_{d^2}$ and the image of $\mathbb{H} \times (\overline{\mathbb{H}} \setminus \mathbb{H})$ is a curve $R^{(2),\circ} \subset X_{d^2}$. Their closures are denoted by $R^{(i)}$. The curves $R^{(i),\circ}$ are irreducible and isomorphic to $\mathbb{H}/\Gamma_1(d)^\pm$ ([Bai07, Proposition 2.4]).[2]

The Hodge bundles. We let $\lambda_\square^{(i)} = pr_i^*\lambda_{X(d)}$ be the pullback of the Hodge bundle to the product. The next important divisor classes on X_{d^2} are the Hodge bundles

$$\lambda_i = (pr_i \circ \beta)^* \lambda_{X(1)}.$$

By definition $\tau^*\lambda_i = \lambda_\square^{(i)}$.

In the same way, we define $R_\square^{(i)} = pr_i^* R_d$ as the pullback of the boundary divisors to $X(d)_d^2$. They consist of $\nu_{\infty,d}$ irreducible components $R_{\square,j}^{(i)}$, $j = 1, \ldots, \nu_{\infty,d}$.

Pulling back (7) to the product $X(d)_d^2$ and then taking its τ-pushforward we obtain the important relation

$$R^{(i)} = \frac{12}{d}\lambda_i. \tag{8}$$

in $\mathrm{Pic}(X_{d^2})$.

[1] topologically, but not as a quotient stack, see Section 3.3

[2] There are different indexing conventions for the boundary divisors in [Bai07] and in [Her91]. As mnemonic for our convention, keep in mind that $R^{(i)}$ and λ_i are pulled back via pr_i.

The product locus. We denote by $P^\circ_{d^2}$ the *product locus*, the locus of abelian surfaces that split as a polarized surface. We will determine the class of this locus in Section 7. The complement $X^\circ_{d^2} \setminus P^\circ_{d^2}$ consists of principally polarized abelian surfaces that are Jacobians of genus two curves.

The Teichmüller curves. The projection of an $\mathrm{SL}_2(\mathbb{R})$-orbit of a square-tiled surface (X,ω) is a Teichmüller curve C in $\mathcal{M}_2$. If $q : X \to E$ is a minimal torus covering of degree d, then the kernel of $\mathrm{Jac}(q) : \mathrm{Jac}(X) \to E$ is a connected abelian subvariety of exponent d (cf. (11)) by [BL04, Lemma 12.3.1, Corollary 12.1.5 and Proposition 12.1.9]. Consequently, by Proposition 4.1 below, a square-tiled surface that factorizes through such a map q defines a point in X_{d^2} and the corresponding Teichmüller curve C is a curve in X_{d^2}.

We let W_D $(D = d^2)$ be the union of Teichmüller curves generated by reduced square-tiled surfaces of degree d where ω has a double zero. By the results in the preceding section, W_D decomposes into spin components W^ε_D. The topology of W_D is completely determined by the work of [McM05], [Bai07], and [Muk14]. In particular the spin components are irreducible.

We let $T_{d,M}$ be the union of Teichmüller curves generated by reduced square-tiled surfaces of degree d such that ω has two simple zeros and (X,ω) has torsion order M. By the preceding section, $T_{d,M}$ decomposes into its spin components $T_{d,M,\varepsilon}$.

3.3. **On quotient stacks.** Since we suppose $d \geq 3$ throughout, the stack discussion on $X(1)$ in the beginning of this section was inessential. The group Γ_{d^2} however contains for all d an element of finite order that acts trivially on $\mathbb{H}^2$, namely $-I$ embedded diagonally. We want the main object of our studies, the pseudo-Hilbert modular surface X_{d^2} to be a variety, rather than a stack with global non-trival isotropy group of order two. For this purpose we consider X_{d^2} as the quotient stack $\mathbb{H}^2/\mathbb{P}\Gamma_{d^2}$. As a set, $X^\circ_{d^2} = \mathbb{H}^2/\Gamma_{d^2}$, as introduced above, but the morphism τ is of degree $|\mathrm{PSL}_2(\mathbb{Z}/d\mathbb{Z})| = \Delta_d/2$ throughout this paper. In particular, it is also possible to define the Hodge bundles 'from above' without invoking the orbifold bundles on $X(1)$ by the relation $\lambda_i = \frac{2}{\Delta_d}\tau_*\lambda^{(i)}_\square$. The equation (8) holds with this convention (and with the reduced scheme structure on $R^{(i)}$).

The reason for this discussion is that the diagonally embedded $-I$ does no longer act trivially when considering the universal family, see (13) in the next section. So there is no choice but to let the universal family $A^\circ_{d^2}$ and its compactification be really the quotient stack by the group $\widetilde{\Gamma}_{d^2}$. In particular, the map $\widetilde{\tau}$ is of degree $\Delta_d d^2$. This has the irritating consequence that the map of the universal family $\pi^\circ : A^\circ_{d^2} \to X^\circ_{d^2}$ is the composition of the forgetful map $\mathbb{H}^2 \times \mathbb{C}^2/\widetilde{\Gamma}_{d^2} \to \mathbb{H}^2/\Gamma_{d^2}$ composed with a (pointwise identity) map $\mathbb{H}^2/\Gamma_{d^2} \to X_{d^2}$ of degree $\frac{1}{2}$. This factor has to be taken into account in push-forwards, see Section 8.

4. Abelian surfaces with multiplication by pseudo-quadratic orders and modular embeddings

Here, we sketch how $X^\circ_{d^2}$ parametrizes abelian surfaces with multiplication by $\mathfrak{o}_{d^2}$ and describe the universal family

$$\pi^\circ : A^\circ_{d^2} \;=\; \mathbb{H}^2 \times \mathbb{C}^2/\widetilde{\Gamma}_{d^2} \to X^\circ_{d^2} \tag{9}$$

where

$$\widetilde{\Gamma}_{d^2} \;=\; \mathrm{SL}(\mathfrak{o}_{d^2} \oplus \mathfrak{o}^\vee_{d^2}) \ltimes (\mathfrak{o}^\vee_{d^2} \oplus \mathfrak{o}_{d^2}) \quad \subset \mathrm{SL}_2(K) \ltimes K^2. \tag{10}$$

One should be aware that $A^\circ_{d^2} \to X^\circ_{d^2}$ is the universal family only when considered as a quotient stack. The fibers of the underlying variety are Kummer surfaces, and in particular singular. Nevertheless, the open family and its compactification, introduced in Section 5, are both quotients of smooth varieties by finite groups and thus smooth when considered as stacks.

It will be convenient to compare this family to the universal family of all principally polarized abelian surfaces via a map $\widetilde{\psi} : \mathbb{H}^2 \times \mathbb{C}^2 \to \mathbb{H}_2 \times \mathbb{C}^2$ that is equivariant with respect to a group inclusion $\widetilde{\Psi} : \widetilde{\Gamma}_{d^2} \to \mathrm{Sp}(4,\mathbb{Z}) \ltimes \mathbb{Z}^4$. Such a pair $(\widetilde{\psi}, \widetilde{\Psi})$ is sometimes called *modular embedding* and it will be used in the next section to pull back theta functions.

Recall that the *exponent* $e(Y)$ of an abelian subvariety Y of dimension r in a principally polarized abelian variety (A, Θ) is defined as

$$e(Y) = d_r, \quad \text{if } \Theta|_Y \text{ has type } (d_1, \dots, d_r), \tag{11}$$

see [BL04, Section 1.2 and 12.1].

Proposition 4.1. *The pseudo-Hilbert modular surface surface $X^\circ_{d^2}$ is the moduli space of all pairs (A, ρ), where A is a principally polarized abelian surface and $\rho : \mathfrak{o}_{d^2} \to \mathrm{End}(A)$ is a choice of multiplication by $\mathfrak{o}_{d^2}$.*

Equivalently, $X^\circ_{d^2}$ is the moduli space of all pairs consisting of a principally polarized abelian surface A together with a projection $q : A \to E$ to an elliptic curve E such that $\ker(q)$ *is a connected abelian subvariety of exponent d.*

For the convenience of the reader and to fix notations, we provide a sketch of the proof the first statement, following [Bai07, Theorem 2.2]. The second statement follows from [BL04, Proposition 12.1.1 and Proposition 12.1.9] after unwinding the definitions.

We want to provide $\mathfrak{o}^\vee_{d^2} \oplus \mathfrak{o}_{d^2}$ with a polarization. For this purpose we define the 'Galois conjugation' on $\mathfrak{o}_{d^2}$ by $(x', x'')^\sigma = (x'', x')$. With the usual definition of trace the pairing

$$\big\langle (x_1, y_1), (x_2, y_2) \big\rangle = \mathrm{Tr}(x_1 y_2 - x_2 y_1).$$

on $\mathfrak{o}^\vee_{d^2} \oplus \mathfrak{o}_{d^2}$ is unimodular, alternating and $\mathbb{Z}$-valued, hence a polarization. Moreover, we let $\sqrt{D} = (d, -d) \in K$. Then, a symplectic basis of $\mathfrak{o}^\vee_{d^2} \oplus \mathfrak{o}_{d^2}$ is

$$a_1 = (\tfrac{1}{\sqrt{D}} \eta_2^\sigma, 0), \quad a_2 = (-\tfrac{1}{\sqrt{D}} \eta_1^\sigma, 0), \quad b_1 = (0, \eta_1), \quad b_2 = (0, \eta_2),$$

where η_1, η_2 is an arbitrary basis of $\mathfrak{o}_{d^2}$. For $z = (z_1, z_2) \in \mathbb{H}^2$, define the embedding

$$\mathfrak{o}^\vee_{d^2} \oplus \mathfrak{o}_{d^2} \to \mathbb{C}^2, \qquad (x, y) \mapsto \left(\begin{smallmatrix} x' z_1 + y' \\ x'' z_2 + y'' \end{smallmatrix} \right).$$

The image is a lattice in $\mathbb{C}^2$ spanned by the columns of

$$\Pi_z = \begin{pmatrix} \frac{1}{d}\eta_2'' z_1 & -\frac{1}{d}\eta_1'' z_1 & \eta_1' & \eta_2' \\ -\frac{1}{d}\eta_2' z_2 & \frac{1}{d}\eta_1' z_2 & \eta_1'' & \eta_2'' \end{pmatrix} = (z^* \cdot A^T, B)$$

where $z^* = \left(\begin{smallmatrix} z_1 & 0 \\ 0 & z_2 \end{smallmatrix} \right)$ and where $B = \left(\begin{smallmatrix} \eta_1' & \eta_2' \\ \eta_1'' & \eta_2'' \end{smallmatrix} \right)$ and $A = B^{-1}$. We will work throughout with the choice

$$B = \left(\begin{smallmatrix} 1 & 0 \\ 1 & d \end{smallmatrix} \right), \quad \text{hence} \quad A = \left(\begin{smallmatrix} 1 & 0 \\ -\frac{1}{d} & \frac{1}{d} \end{smallmatrix} \right).$$

The quotient $A_{d^2,z} = \mathbb{C}^2 / \Pi_z \mathbb{Z}^4$ is a principally polarized abelian surface (ppas), polarized by the hermitian form with matrix $\mathrm{Im}\,(z^*)^{-1}$ and the columns of Π_z are a symplectic basis for the pairing with matrix $\left(\begin{smallmatrix} 0 & I_2 \\ -I_2 & 0 \end{smallmatrix} \right)$. The associated point in $\mathbb{H}_2$ is $Z = A \cdot z^* \cdot A^T$, with the convention that $Z \in \mathbb{H}_2$ corresponds to the ppas with lattice spanned by the columns of (Z, I_2). It admits multiplication by $\mathfrak{o}_{d^2}$ via the diagonal action on the embedding $\mathfrak{o}^\vee_{d^2} \oplus \mathfrak{o}_{d^2} \to \mathbb{C}^2$. This justifies the claims made in Section 3.2.

Since both eigenspaces of multiplication by K are defined over $\mathbb{Q}$, the abelian surface is isogenous to a product of elliptic curves with an isogeny of degree d^2. We give an explicit basis of the sublattice corresponding to the product decomposition. It is generated by the columns of

$$\Pi_z \cdot \begin{pmatrix} B^T & 0 \\ 0 & d \cdot A \end{pmatrix} = \begin{pmatrix} z_1 & 0 & d & 0 \\ 0 & z_2 & 0 & d \end{pmatrix}$$

For an $\mathbb{R}$-basis (w_1, w_2) of $\mathbb{C}$, define the elliptic curve $E_{w_1,w_2} = \mathbb{C}/(w_1\mathbb{Z} + w_2\mathbb{Z})$. Then the isogeny between abelian varieties

$$E_{z_1,d} \times E_{z_2,d} \longrightarrow A_{d^2,z}$$

is induced by the identity on the universal cover. The coordinate projections $p_i : \mathbb{C}^2 \to \mathbb{C}$, $i = 1, 2$ induce the dual isogeny

$$A_{d^2,z} \longrightarrow E_{z_1/d,1} \times E_{z_2/d,1}$$

which after composition with the isomorphism covered by $\mathbb{C}^2 \to \mathbb{C}^2$, $z \mapsto d \cdot z$ becomes multiplication by d on $E_{z_1,d} \times E_{z_2,d}$.

This completes the sketch of the proof of Proposition 4.1.

Modular embeddings. The universal family is now easily obtained by pullback of the universal family of principally polarized abelian surfaces over $\mathbb{H}_2$ via a modular embedding.

Lemma 4.2. *The embedding*

$$\widetilde{\psi} : \mathbb{H}^2 \times \mathbb{C}^2 \to \mathbb{H}_2 \times \mathbb{C}^2, \quad (z, u) \mapsto (Az^*A^T, Au)$$

is equivariant with respect to

$$\begin{array}{rcl} \widetilde{\Psi}: \widetilde{\Gamma}_{d^2} & \to & \mathrm{Sp}(4,\mathbb{Z}) \ltimes \mathbb{Z}^4, \\ (M, r) & \mapsto & S \cdot (M^*, r) \cdot S^{-1} = (\begin{pmatrix} Aa^*B & Ab^*A^T \\ B^Tc^*B & B^Te^*A^T \end{pmatrix}, (r_1B, r_2A^T)) \end{array}$$

where $M^* = \left(\begin{smallmatrix} a^* & b^* \\ c^* & e^* \end{smallmatrix}\right)$, $r = (r_1, r_2)$ *and* $S = (\mathrm{diag}(A, B^T), 0) \in \mathrm{Sp}(4,\mathbb{Q}) \ltimes \mathbb{Q}^4$.

Note that the induced map $X^\circ_{d^2} \to \mathcal{A}_2$ does not depend on the choice of the matrix B. If B' is another basis, $A' = B'^{-1}$, and $(\widetilde{\psi}', \widetilde{\Psi}')$ is the embedding associated with B', then

$$\widetilde{\psi}' = g \circ \widetilde{\psi} \quad \text{and} \quad \widetilde{\Psi}' = g \cdot \widetilde{\Psi} \cdot g^{-1} \qquad \text{where } g = \mathrm{diag}(A'B, B'^TA^T) \in \mathrm{Sp}(4,\mathbb{Z})$$

The proof of Lemma 4.2 is a straightforward calculation, once one fixes the precise definition of the group actions on source and target. We define the semidirect products $\mathrm{Sp}(2g,\mathbb{R}) \ltimes \mathbb{R}^{2g}$ by the rule

$$(M_1, r_1) \cdot (M_2, r_2) := (M_1M_2, r_1M_2 + r_2).$$

This semidirect product acts on the product $\mathbb{H}_g \times \mathbb{C}^g$ by

$$(Z, v) \mapsto (M(Z), ((CZ + E)^T)^{-1}(v + (Z, I_g)r^T)) \tag{12}$$

where $M = \left(\begin{smallmatrix} A & B \\ C & E \end{smallmatrix}\right)$ and $r \in \mathbb{Z}^{2g}$, and $M(Z) = (AZ + B)(CZ + E)^{-1}$. The action is compatible with the projection on the first factor and standard action of $\mathrm{Sp}(2g,\mathbb{R})$ on $\mathbb{H}_g$.

Next, we explicitly write out the action of $\widetilde{\Gamma}_{d^2}$ on $\mathbb{H}^2 \times \mathbb{C}^2$, or more generally of $\mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^4$ on $\mathbb{H}^2 \times \mathbb{C}^2$, which is implicitly already given by (12) and the modular embedding. For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{C}^2$, set $\alpha^* = \left(\begin{smallmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{smallmatrix}\right)$. Then $(M, r) \in \mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^4$ acts via

$$(z, u) \mapsto (M(z), (c^*z^* + e^*)^{-1}(u + (z^*, I_2)r^T)) \tag{13}$$

where $M = \left(\begin{smallmatrix} a & b \\ c & e \end{smallmatrix}\right)$, $r = (r_1, r_2)$, and $r^T = (r_1', r_1'', r_2', r_2'')^T$ and where

$$M(z) = (az + b)(cz + e)^{-1} = \left(\frac{a'z_1 + b'}{c'z_1 + e'}, \frac{a''z_2 + b''}{c''z_2 + e''}\right).$$

5. Compactifying the universal family over X_{d^2}

We will compute the classes of the curves $T_{d,M,\varepsilon}$ as the image of a locus cut out in the universal family of abelian surfaces over the pseudo-Hilbert modular surface. Over the open pseudo-Hilbert modular surfaces, this family is described as the quotient (see Section 4)

$$\pi^\circ : A^\circ_{d^2} \;=\; \mathbb{H}^2 \times \mathbb{C}^2/\widetilde{\Gamma}_{d^2} \to X^\circ_{d^2}$$

To perform intersection calculations, we need to work on a compact space and the aim of this section is to describe explicitly such a compactification of $A^\circ_{d^2}$. Our strategy is as follows. The universal family over the modular curve has a simple compactification, by adding a 'm-gon' of rational curves at every cusp, the simplest instance of a toroidal compactification. In order to reduce from $A^\circ_{d^2}$ to such a situation, we have to pass from $X^\circ_{d^2}$ to a finite cover where this surface is a product, as explained in the previous section, and then to pass fiberwise to an isogenous abelian variety.

The aim of this section is to exhibit a compactification of $A^\circ_{d^2}$ by describing the action of the 2-step covering group on the product of two compactified universal elliptic curves. We thus present a compactification of $A^\circ_{d^2}$ as a quotient of a smooth compact variety by a finite group action. Along with this, we introduce local coordinates at the boundary that will be used to define bundle extensions in the next section.

For this purpose we note that $\widetilde{\Gamma}_{d^2}$ has a normal subgroup that is equal to a product $\widetilde{\Gamma}(d)^2_d$, where

$$\widetilde{\Gamma}(d)_d = \mathrm{diag}(d,1)\cdot(\Gamma(d)\ltimes d\mathbb{Z}^2)\cdot \mathrm{diag}(d^{-1},1).$$

The quotient $\mathbb{H}^2\times\mathbb{C}^2/\widetilde{\Gamma}(d)^2_d$ is a product family

$$\varpi^\circ\times\varpi^\circ : (E(d)^\circ_d)^2 \to (X(d)^\circ_d)^2,$$

in fact of two copies on a universal family of elliptic curves.

As a general guide to the notation in the sequel, groups Γ act on $\mathbb{H}$ or $\mathbb{H}^2$, while groups with a tilde are semidirect products acting on $\mathbb{H}\times\mathbb{C}$ or $(\mathbb{H}\times\mathbb{C})^2$.

Theorem 5.1. *There exists a proper, smooth 4-dimensional stack A_{d^2} containing $A^\circ_{d^2}$ as a Zariski open subset such that*

a) The canonical projection π° extends to a flat, proper morphism

$$\pi : A_{d^2} \to X_{d^2}.$$

b) The map $\widetilde{\tau}^\circ : (E(d)^\circ_d)^2 \to A^\circ_{d^2}$ induced by the inclusion $(\widetilde{\Gamma}(d)_d)^2 \subset \widetilde{\Gamma}_{d^2}$ extends to a finite morphism of degree $\Delta_d d^2$

$$\widetilde{\tau} : (E(d)_d)^2 \to A_{d^2}$$

over $\tau : (X(d)_d)^2 \to X_{d^2}$

c) The scheme underlying the stack A_{d^2} has at most quotient singularities.

The following diagram gives an overview of the spaces and maps involved.

$$\begin{tikzcd}
(E(d)_d)^2 \arrow[r, "\widetilde{\tau}"] \arrow[d, "\varpi\times\varpi"'] & A_{d^2} \arrow[d, "\pi"] \\
(X(d)_d)^2 \arrow[r, "\tau"'] & X_{d^2}
\end{tikzcd} \tag{14}$$

In order to prove this theorem, we employ the usual toroidal compactification of a family of elliptic curves. For $\ell \in \mathbb{N}$ we define more the twisted level subgroup $\Gamma(\ell)_d = \mathrm{diag}(d,1)\cdot\Gamma(\ell)\cdot\mathrm{diag}(d^{-1},1)$. We let

$$\widetilde{\Gamma}(\ell)_d = \mathrm{diag}(d,1)\cdot(\Gamma(\ell)\ltimes \ell\mathbb{Z}^2)\cdot\mathrm{diag}(d^{-1},1).$$

The quotient $X(d)_d^\circ = \mathbb{H}/\Gamma(d)_d$ is the moduli space of d-polarized elliptic curves with a level d-structure and

$$\varpi^\circ : E(d)_d^\circ = \mathbb{H}\times\mathbb{C}/\widetilde{\Gamma}(d)_d \to X(d)_d^\circ \tag{15}$$

is the universal family over it if $d \geq 3$. (Here and everywhere in the sequel we do not discuss the supplementary stack issues arising when $d = 2$.) In particular, $E(d)_d^\circ$ and $X(d)_d^\circ$ is smooth.

The following statement is the point of departure for the compactification. It is well-known (see e.g. [HKW93, Section I.2]), but we give its proof below since we need the coordinates introduced there later on.

Proposition 5.2. *There exists a compactification of $E(d)_d^\circ$ to a smooth, projective surface $E(d)_d$ with the following properties.*

a) The projection ϖ° has an extension to a flat, proper morphism

$$\varpi : E(d)_d \to X(d)_d.$$

b) The boundary $\partial E(d)_d$ consists of $d\cdot\nu_{\infty,d}$ rational curves $D_{C,k}$, where C is a cusp of $\Gamma(d)_d$ and $k \in \mathbb{Z}/d\mathbb{Z}$. We have

$$D_{C_i,k}.D_{C_j,l} = \begin{cases} -2, & i=j, k=l \\ 1, & i=j, k=l\pm 1 \\ 0, & \textit{else} \end{cases}$$

c) There is an action of $\widetilde{\Gamma}(1)_d/\widetilde{\Gamma}(d)_d \cong \mathrm{SL}_2(\mathbb{Z}/(d))\ltimes(\mathbb{Z}/(d))^2$ on $E(d)_d$ extending the action on $E(d)_d^\circ$.

Proof of Theorem 5.1. Thanks to the last item, we can define quotients of $E(d)_d^2$ by all subgroups of $\big(\widetilde{\Gamma}(1)_d/\widetilde{\Gamma}(d)_d\big)^2$. Therefore, setting

$$A_{d^2} = (E(d)_d)^2 \;/\; (\widetilde{\Gamma}_{d^2}/\widetilde{\Gamma}(d)_d^2)$$

immediately yields the claims of Theorem 5.1. $\square$

We also obtain a description of the boundaries of the compactification A_{d^2}. As for X_{d^2} there are boundaries components where the first resp. the second elliptic curve degenerates. While for each of them there is a d-gon over every cusp in the $E(d)_d\times E(d)_d$, there are only two boundary components $D^{(i)}$ for $i\in\{1,2\}$ on A_{d^2}.

More precisely, let S be the set of equivalence classes of cusps of $\Gamma(d)_d$. For $C \in S$, $k = 0,\dots,d-1$, we define the following divisors

$$D_{C,k}^{(1)} = D_{C,k}\times E(d)_d \qquad\qquad D_{C,k}^{(2)} = E(d)_d\times D_{C,k}$$

in $E(d)_d\times E(d)_d$. Then the boundary components are as follows.

Corollary 5.3. *The boundary of A_{d^2} consists of the two irreducible components of codimension one*

$$D^{(i)} = \widetilde{\tau}(D_{C,k}^{(i)}) \qquad\qquad i = 1,2$$

where $C\in S$, $k\in\mathbb{Z}/d\mathbb{Z}$ are arbitrary.

5.1. **Toroidal compactification of families of elliptic curves.** Here, we describe the compactification of the universal family $E(d)_d$ of elliptic curves, and thereby prove Proposition 5.2.

Let $T = (\mathbb{C}^*)^2$ with coordinates ζ and q. For each integer k we define an inclusion $T \to T_{\sigma_k} \cong \mathbb{C}^2$, given by

$$(\zeta, q) \mapsto (\zeta_k, q_k) = (\zeta q^{-k}, \zeta^{-1} q^{k+1}). \tag{16}$$

Inside each T_{σ_k} we define the open set $T_{\xi_{k+1}} = \{q_k \neq 0\} = D(q_k)$ and we consider this as an open subset of $T_{\sigma_{k+1}}$ via

$$T_{\xi_{k+1}} \to T_{\sigma_{k+1}}, \qquad (\zeta_k, q_k) \mapsto (\zeta_{k+1}, q_{k+1}) = (q_k^{-1}, \zeta_k q_k^2)\,.$$

Gluing T_{σ_k} to $T_{\sigma_{k+1}}$ along the open set $T_{\xi_{k+1}}$ gives an infinite chain of rational lines D_{k+1}.

The line D_k is covered by two affine charts. It is given by

$$V(\zeta_{k-1}) \subset T_{\sigma_{k-1}} \quad \text{and} \quad V(q_k) \subset T_{\sigma_k}$$

which are glued along $D(q_{k-1}) \leftrightarrow D(\zeta_k)$ by $q_{k-1} = \zeta_k^{-1}$. As $\zeta_{k-1} = q_{k-1}^{-2} q_k$, this is indeed well-defined, and moreover D_k has self-intersection -2. (In fact, we described a partial toroidal compactification of T, using the collection $\sigma = \{\sigma_k\}_{k\in\mathbb{Z}}$ of rational polyhedral cones in $\mathbb{R}^2$ defined by

$$\sigma_k = \mathbb{R}_{\geq 0} \cdot (k, 1) + \mathbb{R}_{\geq 0} \cdot (k+1, 1), \quad k \in \mathbb{Z},$$

but we will not need this viewpoint. See [HKW93] for details.)

We now compactify $E(d)_d^\circ$ by adding suitable d-gons over the cusps of $\Gamma(d)_d$. We can carry this out for one cusp at a time, and in fact, it suffices to describe a compactification for the cusps ∞, since $\Gamma(d)_d$ is normal in $\Gamma(1)_d$, which has only one cusp.

Compactification over ∞. We carry out the standard construction of a toroidal compactification. It will be convenient to represent elements of the semidirect product $\mathrm{Sp}_{2g}(\mathbb{R}) \ltimes \mathbb{R}^{2g}$ in matrix form via

$$(M, r) \mapsto \begin{pmatrix} 1 & r \\ 0 & M \end{pmatrix}\,.$$

The stabilizer $P = P_\infty(d)_d$ of a small neighborhood in $E(d)_d^\circ$ of the preimage of the cusp ∞ will have a normal subgroup $P^n = P_\infty^n(d)_d$ such that the quotient map by P^n is given by a suitable coordinatewise exponential map and such that the image is isomorphic to T. On the partial compactification of T defined above the factor group $P^q = P/P^n$ acts, e.g. on the boundary curves by a shift of indices. For each of the cusps these quotients are glued to the family over the open curve to obtain a compact space. In the sequel we need the precise form of the coordinates, in particular (17) and (18).

More precisely, let $N = \{\mathrm{Im}\, z \gg 1\}$ be a neighborhood of $\infty \in \mathbb{H}$ not fixed by any element outside the stabilizer of ∞ in $\Gamma(d)_d$. The preimage $P_\infty(d)_d$ of the stabilizer of N in $\widetilde{\Gamma}(d)_d$ is equal to

$$P = P_\infty(d)_d = \{ \begin{pmatrix} 1 & \mathbb{Z} & d\mathbb{Z} \\ 0 & 1 & d^2\mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix} \}.$$

It contains the normal subgroup

$$P^n = P_\infty^n(d)_d = \{ \begin{pmatrix} 1 & 0 & d\mathbb{Z} \\ 0 & 1 & d^2\mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix} \}.$$

that acts on the ϖ-preimage of N, which is isomorphic to $N \times \mathbb{C}$. The quotient map $N \times \mathbb{C} \to N \times \mathbb{C}/P^n$ is given by

$$(z,u) \mapsto (\zeta_\infty, q_\infty), \quad \text{with } \zeta_\infty = \mathbf{e}(\tfrac{1}{d}u), \quad q_\infty = \mathbf{e}(\tfrac{1}{d^2}z), \tag{17}$$

(where $\mathbf{e}(\cdot) = \exp(2\pi i \cdot)$) and identifies $N \times \mathbb{C}/P^n$ with an open set X_∞ in T. We compactify T as above and take $X_{\infty,\Sigma}$ to be the interior of the closure of X_∞. The boundary

$$\partial X_{\infty,\Sigma} = X_{\infty,\Sigma} \setminus X_\infty$$

is an infinite chain of rational curves $D_{\infty,k}$.

The group P acts on X_∞ through the factor group P^q and the compactification is compatible with this action. In fact, the bigger group $P_\infty(1)_d$, the preimage of the stabilizer of N in $\widetilde{\Gamma}(1)_d$, acts on T, and thus on $X_{\infty,\Sigma}$, as the following lemma shows. Its proof is a straight-forward calculation. Let $\eta_d = e(1/d)$.

Lemma 5.4. *For $b \in \mathbb{Z}$, $s_i \in \mathbb{Z}$ and $\varepsilon \in \{\pm 1\}$, let*

$$\widetilde{g} = \widetilde{g}(s_1, s_2, \varepsilon, b) = \begin{pmatrix} 1 & \frac{1}{d}s_1 & s_2 \\ 0 & \varepsilon & bd \\ 0 & 0 & \varepsilon \end{pmatrix} \in P_\infty(1)_d = \left\{ \begin{pmatrix} 1 & \frac{1}{d}\mathbb{Z} & \mathbb{Z} \\ 0 & \pm 1 & d\mathbb{Z} \\ 0 & 0 & \pm 1 \end{pmatrix} \right\}, \quad .$$

The element

a) $\widetilde{g}$ acts on the coordinates $(\zeta, q) = (\zeta_\infty, q_\infty)$ by

$$\begin{aligned} \zeta &\mapsto \zeta^\varepsilon \cdot q^{\varepsilon s_1} \cdot \eta_d^{\varepsilon s_2} \\ q &\mapsto q \cdot \eta_d^{\varepsilon b} \end{aligned} \tag{18}$$

b) $\widetilde{g}$ acts on the coordinates $(\zeta_k, q_k) = (\zeta_{\infty,k}, q_{\infty,k})$ by

$$\zeta_k \mapsto \begin{cases} \zeta_{k-s_1} \cdot \eta_d^{s_2 - bk}, & \varepsilon = 1 \\ q_{-s_1-k-1} \cdot \eta_d^{bk - s_2}, & \varepsilon = -1 \end{cases}, \quad q_k \quad \mapsto \begin{cases} q_{k-s_1} \cdot \eta_d^{(k+1)b - s_2}, & \varepsilon = 1 \\ \zeta_{-s_1-k-1} \cdot \eta_d^{s_2 - (k+1)b}, & \varepsilon = -1 \end{cases}$$

c) $\widetilde{g}$ acts on set of rational curves $D_{\infty,k}$ ($k \in \mathbb{Z}$) by

$$D_{\infty,k} \mapsto D_{\infty,\varepsilon(k+s_1)}.$$

In particular, the action of $P_\infty(1)_d$ on $\{D_{\infty,k}\}$ is transitive.

The action of P^q on $X_{\infty,\Sigma}$ is properly discontinuous and free. Let $Y_{\infty,\Sigma} = X_{\infty,\Sigma}/P^q$ be the quotient. The action of P^q identifies $D_{\infty,k}$ with $D_{\infty,k+dr}$, $r \in \mathbb{Z}$, whence the boundary of the quotient $Y_{\infty,\Sigma}$ consists of a d-gon of rational curves, also called $D_{\infty,k}$ ($k \in \mathbb{Z}/d\mathbb{Z}$).

Compactification over an arbitrary cusp. Let S be a system of representatives of the cusps of $\Gamma(d)_d$. For $C \in S$, choose an element

$$M_C = \left(\begin{smallmatrix} \alpha_C & \beta_C \\ \gamma_C & \delta_C \end{smallmatrix}\right) \in \Gamma(1)_d \qquad \text{such that } \Gamma(d)_d M_C(\infty) = C.$$

The neighborhood $N_C = M_C(N)$ of C in $\mathbb{H}$ is not fixed by an element outside the stabilizer of C. We define $P_C(d)_d$ as the preimage of the stabilizer of N_C, and $P_C(d)_d$ its unipotent radical. The coordinates on the quotient $N_C \times \mathbb{C}/P'_C(d)_d$ are

$$\zeta_C = \mathbf{e}((-\gamma_C z + \alpha_C)^{-1}\tfrac{u}{d}), \qquad q_C = \mathbf{e}(\tfrac{M_C^{-1}z}{d^2}). \tag{19}$$

As before, the image of $N_C \times \mathbb{C}$ is an open set X_C in the torus $T = \operatorname{Spec}\mathbb{C}[\zeta_C^\pm, q_C^\pm]$ and, using the same torus embedding as above, we compactify it by taking $X_{C,\Sigma}$ to be the interior of the closure of X_C in T_Σ. Again let $Y_{C,\Sigma} = X_{C,\Sigma}/P_C(d)_d$ be the quotient. Let Y_C be the image of X_C in $Y_{C,\Sigma}$. Then the map $i_C : Y_C \to E(d)_d^\circ$ that sends an orbit of $P_C(d)_d$ to its $\widetilde{\Gamma}(d)_d$-orbit is an embedding.

The space $E(d)_d$ is now obtained by taking the disjoint union

$$E(d)_d^\circ \,\dot\cup\, \dot{\bigcup}_{C\in S} Y_{C,\Sigma}$$

and dividing out the equivalence relation generated by identifying $x \in E(d)_d^\circ$ with $y \in Y_C$ if $i_C(y) = x$. This completes the proof of Proposition 5.2.

5.2. **Description of the boundaries of A_{d^2}.** In this section, we analyze the action of the quotient group $H_{d^2} = \widetilde{\Gamma}_{d^2}/(\Gamma(d)_d)^2$ on the set of boundary components of $E(d)_d^2$, showing the claims of Corollary 5.3. Secondly, we determine local coordinates of a neighborhood of $D^{(i)}$ by showing that the isotropy group of a generic point is trivial.

Recall the group isomorphisms

$$\overline{red}^{(i)} : H_{d^2} \to \mathrm{SL}_2(\mathbb{Z}/d\mathbb{Z}) \ltimes (\mathbb{Z}/d\mathbb{Z})^2, \quad i = 1,2$$

induced by

$$red^{(i)} : \widetilde{\Gamma}_{d^2} \to \mathrm{SL}_2(\mathbb{Z}/d\mathbb{Z}) \ltimes (\mathbb{Z}/d\mathbb{Z})^2, \quad (A,s) \mapsto \overline{\mathrm{diag}(d^{-1},1)\cdot(A^{(i)},s^{(i)})\cdot\mathrm{diag}(d,1)}.$$

where $\bar{\cdot}$ denotes the reduction modulo d.

Lemma 5.5. *The group H_{d^2} acts transitively on $\{D^{(i)}_{C,k} \mid C \in S, k \in \mathbb{Z}/d\mathbb{Z}\}$ for each $i = 1,2$. The stabilizer of $D^{(i)}_{\infty,0}$ is given by*

$$\overline{red}^{(i)}(\mathrm{Stab}_{H_{d^2}}(D^{(i)}_{\infty,0})) = \{[\left(\begin{smallmatrix} \pm1 & * \\ 0 & \pm1 \end{smallmatrix}\right), (0,*)]\} \subset \mathrm{SL}_2(\mathbb{Z}/d\mathbb{Z}) \ltimes (\mathbb{Z}/d\mathbb{Z})^2$$

and is of order $2d^2$. Moreover the pointwise stabilizer

$$\mathrm{Stab}_{H_{d^2}}(D^{(i)}_{\infty,0})$$

is trivial.

Proof. By symmetry, we may focus on $i = 1$. The group Γ_{d^2} acts transitively on the set $\{C \times X(d)_d \mid C \in S\}$, so it suffices to show that $\widetilde{\Gamma}_{d^2} \cap (P_\infty(1)_d \times \widetilde{\Gamma}(1)_d)$ acts transitively on $\{D^{(1)}_{\infty,k} \mid k \in \mathbb{Z}/d\}$. For $s_1 \in \mathbb{R}^2$, let $h(s_1) = [I,(s_1,0)] \in \widetilde{G}$. We have

$$h(\tfrac{1}{\sqrt{D}}) = [I,((\tfrac{1}{d},-\tfrac{1}{d}),0)] \in \widetilde{\Gamma}_{d^2} \cap (P_\infty(1)_d \times \widetilde{\Gamma}(1)_d),$$

which maps $D^{(1)}_{\infty,k}$ to $D^{(1)}_{\infty,k+1}$.

Concerning the stabilizer group of $D^{(1)}_{\infty,0}$, we have

$$\overline{red}^{(i)}\big(\mathrm{Stab}_{H_{d^2}}(D^{(1)}_{\infty,k})\big) = red^{(i)}\Big(\widetilde{\Gamma}_{d^2} \cap \big(\mathrm{Stab}_{P_\infty(1)_d}(D_{\infty,k}) \times \widetilde{\Gamma}(1)_d\big)\Big).$$

Using this observation and Lemma 5.4, one can easily determine the stabilizer and the pointwise stabilizer. □

Local coordinates at the boundaries. We describe local coordinates in the neighborhood of a point $x \in D^{(i)}$ $(i = 1,2)$. These will be used to extend the line bundles in the next section.

For $i = 1,2$ and $k \in \mathbb{Z}$, we introduce, following (16) and (17), the notations

$$\zeta_i = \mathbf{e}(\tfrac{1}{d}u_i) \qquad q_i = \mathbf{e}(\tfrac{1}{d^2}z_i) \tag{20}$$

$$\zeta_{i,k} = \zeta_i q_i^k \qquad q_{i,k} = \zeta_i^{-1} q_i^{k+1} \tag{21}$$

It will be helpful to keep in mind the relations

$$\zeta_i = \zeta_{i,k}^{k+1} q_{i,k}^k, \qquad q_i = \zeta_{i,k} q_{i,k} \tag{22}$$

Note also that we work throughout over the cusps ∞, but we suppress this from the notation.

Lemma 5.6. *Let $x \in D^{(i)}$ be a generic point and let $\widetilde{x}$ be a lift of x in $D_{\infty,k} \times \mathbb{H} \times \mathbb{C}$ in case $i = 1$, respectively in $\mathbb{H} \times \mathbb{C} \times D_{\infty,k}$ in case $i = 2$. Then*

$$(\zeta_{1,k},\ q_{1,k},\ z_2,\ u_2) \qquad i = 1$$
$$(z_1,\ u_1,\ \zeta_{2,k},\ q_{2,k}) \qquad i = 2$$

are local coordinates at x, in the sense that there exists an open neighborhood $\widetilde{U}$ of $\widetilde{x}$ such that the canonical projection $U \to A_{d^2}$ is a homeomorphism.

In particular, the generic point of $D^{(i)}$ is smooth.

Proof. By symmetry, we may restrict to the case $i = 1$. Since the action is properly discontinuous, it suffices to show that a generic $\widetilde{x}$ is not fixed by any element $g \in \widetilde{\Gamma}_{d^2} \setminus (P'_\infty(d)_d \times \{1\})$. Let us write $g = (M, r)$, $M = \left(\begin{smallmatrix} a & b \\ c & e \end{smallmatrix}\right)$, $r = (r_1, r_2)$ and suppose that it fixes $\widetilde{x}$. As x is generic, z_2 is not a fixed point of M'' and thus $M'' = \pm I$. For the same reason, u_2 is not a half-integral lattice point $\frac{1}{2d}\widetilde{z}_2\mathbb{Z} + \frac{1}{2}\mathbb{Z}$, and thus $u_2 \mapsto a''(u_2 + z_2 r_1'' + r_2''))$ does not fix a neighborhood of u_2 unless $a'' = 1$, $r_1'' = r_2'' = 0$. Since M' fixes a point in $X_{\infty,\Sigma}$, it is of the form

$$M' = \begin{pmatrix} 1 & r_1' & r_2' \\ 0 & \varepsilon & b' \\ 0 & 0 & \varepsilon \end{pmatrix}$$

The congruence condition together with $a'' = 1$ forces $\varepsilon = 1$. Since $b'' = 0$ and $r_2'' = 0$, we have $b' \in d^2\mathbb{Z}$ and $r_2' \in d\mathbb{Z}$. Moreover, M' has to fix the component $D_k \subset X_{\infty,\Sigma}$, which according to Lemma 5.4 entails $r_1' = 0$. Altogether, this shows $M \in P'_\infty(d)_d \times \{1\}$.

Alternatively, one can argue that $(q_{1,k}, \zeta_{1,k}, z_2, u_2)$ provide local coordinates about $D^{(1)}_{\infty,0}$ on $E(d)_d$, and that the pointwise stabilizer $\mathrm{Stab}_{H_{d^2}}(D^{(1)}_{\infty,0})$ is trivial. □

6. Divisors and line bundles on A_{d^2}

On the universal family over an (open) pseudo-Hilbert modular surface there is a natural collection of line bundles, the common generalization of the pullback of Hilbert modular forms and classical elliptic Jacobi forms. These are the called Hilbert Jacobi-forms. Theta functions will be the main instances of sections of these line bundles. Our aim is to express the classes of these line bundles in the rational Picard group $\mathrm{Pic}_{\mathbb{Q}}(A_{d^2})$ in terms of line bundles that are good for intersection theory calculations: the Hodge bundles, the boundary divisors and the pullbacks $N^{(i)}$ of the zero sections.

The main result of this section is the following. The notation will be explained in the rest of this section.

Theorem 6.1. *Let f be a Hilbert-Jacobi form of weight $\kappa \in (\frac{1}{2}\mathbb{Z})^2$, index $m \in \frac{1}{2}\mathfrak{o}_{d^2}$ and a multiplier of order ℓ for the group $\widetilde{\Gamma}_{d^2}$. Then the class of $\mathrm{div}(f)$ in $\mathrm{Pic}_{\mathbb{Q}}(A_{d^2})$ is*

$$(\kappa_1 + \tfrac{2m'}{d})\pi^*\lambda_1 + (\kappa_2 + \tfrac{2m''}{d})\pi^*\lambda_2 + \tfrac{2m'}{d}N^{(1)} + \tfrac{2m''}{d}N^{(2)}\,. \tag{23}$$

Note that it is almost meaningless to speak of the class of a line bundle defined by giving explicit automorphy factors on the open family. If $\mathcal{J}_{\kappa,m}$ is one extension to the compactification, any twist $\mathcal{J}_{\kappa,m}(nD^{(i)})$ for any integral n and a boundary component $D^{(i)}$ will also be an extension. The theorem becomes meaningful only together with the description of the behavior at the boundary (in terms of Laurent series in local coordinates) given in (30). For practical purposes, any other boundary conditions would work as well: we have to correct by the vanishing order at the boundary and the difference is independent of any choices, see Theorem 9.2 for our application.

6.1. **Divisors in the Picard group of the universal family: The boundary and torsion sections.** In this section we list some important divisor classes in the compactified universal family $\mathrm{Pic}_{\mathbb{Q}}(A_{d^2})$ over the pseudo-Hilbert modular surface. The classes of a Hilbert modular forms can be expressed in these bundles. For later use we also define the divisors corresponding to zero sections and compare it to the divisor of torsion sections.

Recall from Section 3.2 the definition of the Hodge bundles $\lambda_i = (pr_i \circ \beta)^* \lambda_{X(1)}$, where $\beta : X_{d^2} \to X(1)_d^2$ is the projection and λ is the Hodge class on $X(1)_d$. There, we also defined the boundary curves $R^{(i)}$, that obey the relation $R^{(i)} = \frac{12}{d}\lambda_i$.

In Corollary 5.3 we gave a description of the boundary with two components $D^{(i)}$, mapping surjectively to $R^{(i)}$ respectively for $i = 1, 2$. The discussion in Section 3.3 implies that $\pi^* R^{(i)} = D^{(i)}$. In particular, we have the relation

$$D^{(i)} = \pi^* R^{(i)} = \frac{12}{d}\pi^* \lambda_i \tag{24}$$

For $i = 1, 2$ let $N_{\square}^{(i)}$ be the pullback of the zero section $N_{X(d)}$ of the compactified universal family $E(d)_d$ of elliptic curves via the i-th projection to $E(d)_d^2$. We denote by

$$N^{(i)} = \widetilde{\tau}(N_{\square}^{(i)}) \tag{25}$$

the image of these zero sections in A_{d^2}. Note that $N^{(i)} = \frac{1}{\Delta_d}\widetilde{\tau}_*(N_{\square}^{(i)})$.

With the same letter and the additional subscript $\ell-$tor we denote the corresponding divisors of the multi-section of primitive ℓ-torsion points on the family over $X(d)$, over $X(d)^2$ and over X_{d^2} respectively. Their classes are related as follows.

Proposition 6.2. *In* $\mathrm{CH}^1(A_{d^2})$, *we have for* $\ell > 1$

$$N_{\ell-\mathrm{tor}}^{(i)} = \tfrac{\Delta_\ell}{\ell}(N^{(i)} + \pi^* \lambda_i).$$

Proof. All the quantities involved are pull backs from the universal family $E(1)$ (we calculate in $\mathrm{Pic}_{\mathbb{Q}}$ of a quotient stack) over $X(1)$ and we prove the relation there. The rational Picard group of an elliptic fibration is generated by the zero section N, the class F of a fiber and the components of the singular fibers, with the relation that the sum of all the components are equal to a smooth fiber. Since all the singular fibers are irreducible here we can disregard the singular fibers.

Consequently, we write $N_{\ell-\mathrm{tor}} = aN + bF$. Intersecting with another fiber shows that $a = \frac{\Delta_\ell}{\ell}$. Intersecting with N shows that $b = -aN^2 = a\deg(\lambda)$ ([Kod63, Eq. (12.6)]). Since the fiber classes are pulled back from $X(1)$, where any two points are linearly equivalent, we may write $bF = \frac{\Delta_\ell}{\ell}\bar{\varpi}^*\lambda$, where $\bar{\varpi} : E(1) \to X(1)$ is the map of the universal family. □

6.2. **Elliptic Jacobi forms.** In this section we recall the classical theory of elliptic Jacobi forms for $\widetilde{\Gamma}(1)$ (see e.g. [EZ85]), specify a bundle they are sections of and use this to determine the class of the divisor where the Jacobi form vanishes. Our method follows [Kra91], but we redo this case as preparation for the case of Hilbert Jacobi forms in the next section, to include non-integral weight and index as well as non-cusp forms, and clarify the imprecise statement in [Kra91, Proposition 2.4].

We start with the standard definition and explain the notations afterwards.

Definition 6.3. *An* elliptic Jacobi form *of weight* $\kappa \in \frac{1}{2}\mathbb{Z}$ *and index* $m \in \frac{1}{2d}\mathbb{Z}$ *for the group* $\widetilde{\Gamma}(d)_d = \Gamma(d)_d \ltimes (\mathbb{Z} \oplus d\mathbb{Z})$ *and the multiplier* χ *is a holomorphic function* $f : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ *such that* *(i)* $f\big|_{\kappa,m}[M,r](z,u) = \chi(M,r)f(z,u)$ *for all* $(M,r) \in \widetilde{\Gamma}(d)_d$.

(ii) For each cusp C *with* M_C, q_C *and* ζ_C *as defined in Section 5.1,* f *has a Fourier development*

$$f(z,u) \cdot j_{\kappa,m}(M_C^{-1}, z, u)^{-1} = \sum_{0 \le s \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} c_{C,s,t}\, q_C^s \zeta_C^t$$

for some $c_{C,s,t} \in \mathbb{C}$, *which vanish unless* $4sm - t^2 \ge 0$.

The divisor $\operatorname{div} f$ of a Jacobi form is well-defined as a subset of $E(d)_d^\circ$, since the exponential factors in the transformation rule (see (26)) do not change the vanishing order of the function. However, $\operatorname{div} f$ does not define a class in $\operatorname{Pic}(E(d)_d)$, since the boundary contribution is not well-defined. Later (compare Theorem 9.2) we are interested in the class of the topological closure $\overline{\operatorname{div} f}$ in $\operatorname{Pic}(E(d)_d)$. This class however is not determined by the parameters (weight, index, multiplier) of the Jacobi form, as one can easily see already for modular forms. We will talk about divisor classes once we introduced the bundle of Jacobi forms.

Note that condition ii) is for historical reasons only. It holds for the most important examples (theta functions introduced below, and also Fourier-Jacobi coefficients of Siegel modular forms) and guarantees the finite-dimensionality of the space of Jacobi forms for fixed parameters. However, many other (cone) conditions would do as well and fixing the bundle $\mathcal{J}_{\kappa,m}(E(d)_d)$ is independent of this choice.

The slash operator for $\widetilde{\Gamma}(d)_d$. In order to define the slash operator we let

$$j_{\kappa,m}(\gamma, z, u) = (cz+e)^{-\kappa} \cdot \mathbf{e}\left(-m\,\frac{c(u + r_1 z + r_2)^2}{cz+e}\right) \cdot \mathbf{e}(m(r_1^2 z + 2r_1 u)).$$

For κ integral, the function $j_{\kappa,m}$ is an *automorphy factor* for $\gamma \in \widetilde{\Gamma}(1)$ called classical automorphy factor, i.e.

$$j_{\kappa,m}(\gamma_1\gamma_2, z, u) = j_{\kappa,m}(\gamma_1, \gamma_2(z,u)) \cdot j_{\kappa,m}(\gamma_2, z, u).$$

and we define

$$f\big|_{\kappa,m}[\gamma](z,u) := f(\gamma(z,u)) \cdot j_{\kappa,m}(\gamma, z, u). \tag{26}$$

In this case $\chi : \widetilde{\Gamma}(d)_d \to \mathbb{C}^\times$ is just an abelian character. For general κ, the map χ is a *multiplier*, i.e. a map so that $j_{\kappa,m}(\cdot)\chi^{-1}(\cdot)$ is an automorphy factor for a fixed choice of the determination of $(cz+e)^{-\kappa}$. In any case, χ is supposed to be finite, i.e. $\chi^M = 1$ for some $M \in \mathbb{N}$.

Let $d \ge 3$, ℓ be integers. Recall that

$$\widetilde{\Gamma}(\ell)_d = \Gamma(\ell)_d \ltimes (\tfrac{\ell}{d}\mathbb{Z} \oplus \ell\mathbb{Z}) = \operatorname{diag}(d,1) \cdot \Gamma(\ell) \ltimes \ell\mathbb{Z}^2 \cdot \operatorname{diag}(d^{-1},1).$$

Lemma 6.4. *For* $\kappa \in \mathbb{Z}$, $m \in \mathbb{Z}$, *the function* $j_{\kappa,md/\ell^2}$ *is an automorphy factor for the twisted group* $\widetilde{\Gamma}(\ell)_d$.

Proof. Consider the map $\varphi : \mathbb{H} \times \mathbb{C} \to \mathbb{H} \times \mathbb{C}$, $(z,u) \mapsto (dz, \ell u)$. It is equivariant with respect to the map $\Phi : \Gamma(\ell) \ltimes \mathbb{Z}^2 \to \Gamma(\ell)_d \ltimes (\frac{\ell}{d}\mathbb{Z} \oplus \ell\mathbb{Z})$ given by

$$\left(\left(\begin{smallmatrix} a & b \\ c & e \end{smallmatrix}\right), (r_1, r_2)\right) \mapsto \left(\left(\begin{smallmatrix} a & bd \\ c/d & e \end{smallmatrix}\right), (\tfrac{\ell}{d}r_1, \ell r_2)\right).$$

Since by pullback

$$\begin{aligned} j_{\kappa,m} \circ (\Phi \times \varphi)^{-1}(\gamma, z, u) &= (cd\tfrac{z}{d} + e)^{-\kappa}\mathbf{e}\big(m(\tfrac{d^2}{\ell^2}r_1^2\tfrac{z}{d} + 2\tfrac{d}{\ell}r_1\tfrac{u}{\ell})\big) \\ &\quad \cdot \mathbf{e}\left(-m\frac{cd(\frac{u}{\ell} + \frac{d}{\ell}r_1\frac{z}{d} + \frac{1}{\ell}r_2)^2}{cd\frac{z}{d} + e}\right) \\ &= j_{\kappa,md/\ell^2}(\gamma, z, u)\,, \end{aligned}$$

the classical automorphy factor j restricted to $\Gamma(\ell) \ltimes \mathbb{Z}^2$ with $m \in \mathbb{Z}$ and $\kappa \in \mathbb{Z}$ is transformed into an automorphy factor for $\Gamma(\ell)_d \ltimes (\frac{\ell}{d}\mathbb{Z} \oplus \mathbb{Z})$. □

A bundle of elliptic Jacobi forms. It is well-known that an automorphy factor like $j_{\kappa,m}\chi^{-1}$ for a group like $\widetilde{\Gamma}(d)_d$ defines a line bundle $\mathcal{J}^{\chi}_{\kappa,m}(E(d)^{\circ}_d)$ on $\mathbb{H} \times \mathbb{C}/\widetilde{\Gamma}(d)_d = E(d)^{\circ}_d$. We specify an extension of $\mathcal{J}^{\chi}_{\kappa,m}(E(d)^{\circ}_d)$ to $E(d)_d$. For simplicity, let us first assume that $j_{\kappa,m}$ is already an automorphy factor. We consider the line bundle induced on the open set X_C introduced in Section 5.1; in fact, it suffices to work over the cusp ∞ and carry the arguments over to any other cusp C using the elements M_C. As the slash operator is trivial on $P^n_\infty(d)_d$, so is the line bundle induced by $j_{\kappa,m}$ on X_∞. We extend it to a line bundle on $X_{\infty,\Sigma}$ by declaring on T_{σ_k} the Laurent series

$$q_k^{-mk^2}\zeta_k^{-m(k+1)^2}\sum_{i,j\geq 0} c_{i,j}q_k^i\zeta_k^j$$

to be holomorphic. Since by Lemma 5.4, $f_k = q_k^{-mk^2}\zeta_k^{-m(k+1)^2}$ is mapped to $f_k\big|_{\kappa,m}[\widetilde{g}] = f_{k-s_1}\cdot\alpha$ for some d-th root of unity α by the element $\widetilde{g}(s_1,s_2,\varepsilon,b) \in P_\infty(1)_d$, it follows that this extension descends to a well-defined line bundle on $Y_{\infty,\Sigma}$. Performing this extension over all cusps, we obtain a well-defined line bundle $\mathcal{J}_{\kappa,m}(E(d)_d)$ on $E(d)_d$ that restricts to $\mathcal{J}_{\kappa,m}(E(d)^{\circ}_d)$ on the open family.

In the presence of a non-trivial multiplier χ, the line bundle induced on X_∞ may not be trivial. Still it is a local system, which means that the sections in two trivializations are transformed into each other by multiplication by a non-zero constant. This entails that we can use the same definition as above for the extension. Note also that the arguments show in fact that the extension $\mathcal{J}^{\chi}_{\kappa,m}(E(d)_d)$ is a $\widetilde{\Gamma}(d)_d/\widetilde{\Gamma}(1)_d$-equivariant bundle (as long as the automorphy factor $j_{\kappa,m}\chi^{-1}$ is well-defined on $\widetilde{\Gamma}(1)_d$).

In order to make the connection with Jacobi forms, we rewrite the Fourier expansion of a Jacobi form f at the cusp ∞ using

$$\zeta_\infty = \zeta_k^{k+1}q_k^k, \quad q_\infty = \zeta_k q_k$$

and obtain

$$f(z,u) = \sum_{\substack{s,t\in\mathbb{Z},s\geq 0,\\ 4sm-t^2\geq 0}} c_{s,t}\, q_\infty^s\zeta_\infty^t = \sum_{\substack{s,t\in\mathbb{Z},s\geq 0,\\ 4sm-t^2\geq 0}} c_{s,t}\, q_k^{s+kt}\zeta_k^{s+(k+1)t}.$$

It is easy to check that the smallest q_k-exponent appearing is

$$\min\{s+kt \mid s,t\in\mathbb{Z}, 4sm-t^2\geq 0, s\geq 0\} \geq -mk^2,$$

and that a similar statement holds for the smallest ζ_k-exponent. Thus, f is a holomorphic section of the bundle extension $\mathcal{J}_{\kappa,m}(E(d)_d)$.

With this choice of extension, the class of $\mathrm{div}(f)$ is well-defined and has been calculated in [Kra91, Proposition 2.4]. The result is not needed in the sequel, but we will follow his method in the next subsections very closely to prove Theorem 6.1.

6.3. **Hilbert Jacobi forms.** In this section, we define Jacobi forms for the pseudo-Hilbert modular surfaces analogously to the elliptic case by an automorphy factor and a condition on the Fourier development at the boundary. Then we describe an extension of the line bundle induced by the automorphy factor on $A^{\circ}_{d^2}$ to the compactification A_{d^2}, whose global sections will include all Hilbert Jacobi forms. Again, we first give the well-known definition and explain notation afterwards.

Definition 6.5. *A* Hilbert Jacobi form *of weight* $\kappa = (\kappa_1, \kappa_2) \in \frac{1}{2}\mathbb{Z}^2$ *and index* $m = (m', m'') \in \frac{1}{2}\mathfrak{o}_{d^2}$ *for the group* $\widetilde{\Gamma}_{d^2}$ *and multiplier* χ *is a holomorphic function* $f : \mathbb{H}^2 \times \mathbb{C}^2 \to \mathbb{C}$ *such that*

(i) $f((M,r)(z,u)) \cdot \widetilde{\jmath}_{\kappa,m}((M,r),z,u) = \chi(M,r)\, f(z,u)$ *for all* $(M,r) \in \widetilde{\Gamma}_{d^2}$.

(ii) f *has Fourier developments*

$$\begin{aligned} f(z,u) &= \sum_{s'\in\mathbb{Z}} \sum_{t'\in\mathbb{Z}} c_{s',t'}(z_2,u_2)\, q_1^{s'} \zeta_1^{t'} \\ &= \sum_{s''\in\mathbb{Z}} \sum_{t''\in\mathbb{Z}} c_{s'',t''}(z_1,u_1)\, q_2^{s''} \zeta_2^{t''} \end{aligned} \tag{27}$$

in the local coordinates

$$q_i = \mathbf{e}(\tfrac{z_i}{d^2}), \quad \zeta_i = \mathbf{e}(\tfrac{u_i}{d}),$$

where $c_{s',t'}$, $c_{s'',t''}$ *are holomorphic functions, which vanish unless*

$$4sm - t^2 \geq 0 \quad \textit{and} \quad s \geq 0.$$

In this definition,

$$\begin{aligned} \widetilde{\jmath}_{\kappa,m}(\gamma,z,u) \;=\; & \mathbf{e}(\mathrm{tr}_{K/\mathbb{Q}}(m(r_1^2 z + 2r_1 u))) \prod_{i=1}^{2} (c^{(i)} z_i + e^{(i)})^{-\kappa_i} \cdot \\ & \cdot \mathbf{e}(-\mathrm{tr}_{K/\mathbb{Q}}(m(cz+e)^{-1} c(u + z^* r_1^T + r_2^T)^2)) \end{aligned} \tag{28}$$

and one checks that for κ integral the function $(z,u) \mapsto \widetilde{\jmath}_{\kappa,m}(\gamma,z,u)$ is an automorphy factor for $\widetilde{\Gamma}_{d^2}$. In the general case, for κ not necessarily integral, a multiplier is defined to be a map $\chi : \widetilde{\Gamma}_{d^2} \to \mathbb{C}^\times$ such that for a fixed determination of $\widetilde{\jmath}_{\kappa,m}$ the product $\widetilde{\jmath}_{\kappa,m}\chi^{-1}$ is an automorphy factor for $\widetilde{\Gamma}_{d^2}$. We suppose throughout that $\chi(\gamma)$ has finite order for $\gamma \in \widetilde{\Gamma}_{d^2}$. We will not need more details, since the multipliers trivialize after taking tensor powers and so they do not effect a statement on the rational Picard group as Theorem 6.1.

Note also that

$$\widetilde{\jmath}_{\kappa,m} = j^{(1)}_{\kappa_1,m'} \cdot j^{(2)}_{\kappa_2,m''} \tag{29}$$

where $j^{(i)}_{\kappa_i,m^{(i)}}(\gamma,z,u) = j_{\kappa_i,m^{(i)}}(\gamma^{(i)}, z_i, u_i)$.

A bundle of Hilbert Jacobi forms. We denote by $\mathcal{J}^{\chi}_{\kappa,m}(A^{\circ}_{d^2})$ the line bundle defined by the automorphy factor $\widetilde{\jmath}_{\kappa,m}\chi^{-1}$ on the open variety $A^{\circ}_{d^2}$. In order to extend it, we proceed as in the elliptic case. We work local coordinates near a boundary divisor, say $D^{(1)}$ and suppose first that $\chi = 1$. The local coordinates are given by Lemma 5.6 by

$$\zeta_{1,k}, q_{1,k}, z_2, u_2,$$

and the line bundle induced by $\widetilde{\jmath}_{\kappa,m}$ is trivial. Again, we declare sections to be holomorphic if they are of the form

$$q_{1,k}^{-m'k^2} \zeta_{1,k}^{-m'(k+1)^2} \cdot f \tag{30}$$

for a holomorphic function $f = f(\zeta_{1,k}, q_{1,k}, z_2, u_2)$. For a non-trivial multiplier χ, we have to pass to local systems, but this definition still makes sense, since it is independent of the chosen trivialization of the local system.

Alternatively, we can construct the extension (for $\chi = 1$) by using (29), which translates into

$$\widetilde{\tau}^* \mathcal{J}_{\kappa,m}(A^{\circ}_{d^2}) \cong pr_1^* \mathcal{J}_{\kappa_1,m'}(E(d)^{\circ}_d) \otimes pr_2^* \mathcal{J}_{\kappa_2,m''}(E(d)^{\circ}_d).$$

and the fact that the latter bundle has an extension, which is in fact H_{d^2}-equivariant and thus induces a bundle on the quotient. (Note that for $m \in d\mathbb{Z}^2$, it is even

$\widetilde{\Gamma}(1)^2_d/\widetilde{\Gamma}(d)^2_d$-equivariant, but for general rational index m, j is not an automorphy factor for $\widetilde{\Gamma}(1)_d$.)

From the Fourier development (27) and the coordinate transformations (22) we deduce that a Hilbert Jacobi form has near the boundary divisors $D^{(1)}_{\infty,k}$ given by $q_{1,k} = 0$ a Fourier development

$$f(z,u) = \sum_{\substack{s',t':\\ 4s'm_1 - t'^2 \geq 0}} c_{s',t'} \zeta_{1,k}^{s'+(k+1)t'} q_{1,k}^{s'+kt'}. \tag{31}$$

The same estimate as for elliptic Jacobi forms yields that Hilbert Jacobi forms are indeed holomorphic sections of $\mathcal{J}^{\chi}_{\kappa,m}(A_{d^2})$.

6.4. **Theta functions.** We recall the definition of the classical (Siegel) theta-functions. We use the convention that $x = (x_1, x_2)$ and γ_i are row vectors while and $v = (v_1, v_2)^T$ is a column vector. Let

$$\theta\left[\begin{smallmatrix}\gamma_1\\ \gamma_2\end{smallmatrix}\right] : \begin{cases} \mathbb{H}_g \times \mathbb{C}^g & \to \quad \mathbb{C} \\ (Z,v) & \mapsto \quad \displaystyle\sum_{x \in \mathbb{Z}^g + \frac{1}{2}\gamma_1} \mathbf{e}\left(\tfrac{1}{2}xZx^T + x(v + \tfrac{1}{2}\gamma_2^T)\right). \end{cases} \tag{32}$$

be the *theta function with half-integral characteristic* $\gamma = (\gamma_1, \gamma_2) \in \mathbb{Z}^2$. The evaluation of a theta-function at $v = 0$ is called *theta constant*. The theta-function (and the characteristic (γ_1, γ_2)) is called *odd* if $\gamma_1 \gamma_2^T$ is odd and *even* otherwise. Odd theta constants vanish identically as functions in Z. The theta constants are modular forms of weight $1/2$ for the subgroup $\Gamma(4,8)$ of $\mathrm{Sp}(2g,\mathbb{Z})$, non-zero if and only if (γ_1, γ_2) is even.

For a matrix $M = \left(\begin{smallmatrix} A & B \\ C & E\end{smallmatrix}\right) \in \mathrm{Sp}(2g,\mathbb{Z})$ and a vector $\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^{2g}$ the theta function transforms (see [BL04]) as

$$\begin{aligned} &\theta\left[\begin{smallmatrix}(M\gamma)_1\\ (M\gamma)_2\end{smallmatrix}\right](M(Z), (CZ+E)^{-T}v) = \\ &\qquad \theta\left[\begin{smallmatrix}\gamma_1\\ \gamma_2\end{smallmatrix}\right](Z,v) \cdot \chi_\theta(M) \cdot \det(CZ+E)^{1/2} \mathbf{e}(\tfrac{1}{2}v^T(CZ+E)^{-1}Cv), \end{aligned} \tag{33}$$

$$\begin{aligned} &\theta\left[\begin{smallmatrix}\gamma_1\\ \gamma_2\end{smallmatrix}\right](Z, v + Z\lambda_1^T + \lambda_2^T) = \\ &\qquad \theta\left[\begin{smallmatrix}\gamma_1\\ \gamma_2\end{smallmatrix}\right](Z,v) \cdot \mathbf{e}(\tfrac{\gamma_1}{2}\lambda_2^T - \tfrac{\gamma_2}{2}\lambda_1^T - \tfrac{1}{2}\lambda_1 Z \lambda_1^T - v^T\lambda_1^T). \end{aligned} \tag{34}$$

Here, χ_θ is a multiplier, which takes values in the 8-th roots of unity, and M acts on the characteristic by

$$\begin{aligned} (M\gamma)_1 &= E\gamma_1^T - C\gamma_2^T + (CE^T)_0 \\ (M\gamma)_2 &= -B\gamma_1^T + A\gamma_2^T + (AB^T)_0, \end{aligned}$$

where $(S)_0 = (s_{11}, \ldots, s_{gg})$ denotes the diagonal vector of a matrix $S \in \mathbb{R}^{g \times g}$.

We are interested in *Hilbert theta functions* (with half-integral characteristics), the pullback of the Siegel theta-function for $g = 2$ to $\mathbb{H}^2 \times \mathbb{C}^2$ via the modular embedding $\widetilde{\psi}$ defined in Section 4. Concretely, these theta functions are given as the power series

$$\begin{aligned} \vartheta\left[\begin{smallmatrix}\widetilde{\gamma}_1\\ \widetilde{\gamma}_2\end{smallmatrix}\right](z,u) := \psi^*\theta\left[\begin{smallmatrix}\gamma_1\\ \gamma_2\end{smallmatrix}\right](z,u) &= \sum_{x \in \mathbb{Z}^2 + \frac{\gamma_1}{2}} \mathbf{e}(\tfrac{1}{2}xAz^*A^Tx^T + x(Au + \tfrac{1}{2}\gamma_2^T)) \\ &= \sum_{x \in (\mathbb{Z}^2 + \frac{\gamma_1}{2})A} \mathbf{e}(\tfrac{1}{2}xz^*x^T + x(u + \tfrac{1}{2}B\gamma_2^T)) \\ &= \sum_{x \in \mathfrak{o}^\vee_{d^2} + \frac{\widetilde{\gamma}_1}{2}} \mathbf{e}\big(\mathrm{tr}_{K/\mathbb{Q}}(\tfrac{1}{2}(x^2z + 2x(u + \tfrac{1}{2}\widetilde{\gamma}_2)))\big) \end{aligned} \tag{35}$$

where $\widetilde{\gamma}_1 = \gamma_1 A \in \mathfrak{o}_{d^2}^\vee$, and $\widetilde{\gamma}_2 = \gamma_2 B^T \in \mathfrak{o}_{d^2}$. We first analyze the action of Γ_{d^2} on characteristics.

Lemma 6.6. *The set of even theta characteristics decomposes under the action of Γ_{d^2} into two orbits*

$$\begin{aligned} E_0 &= \left\{ \left[\begin{smallmatrix}(0,0)\\(0,0)\end{smallmatrix}\right], \left[\begin{smallmatrix}(1,0)\\(0,1)\end{smallmatrix}\right], \left[\begin{smallmatrix}(1,0)\\(0,0)\end{smallmatrix}\right], \left[\begin{smallmatrix}(0,0)\\(0,1)\end{smallmatrix}\right] \right\} \\ E_2 &= \left\{ \left[\begin{smallmatrix}(1,1)\\(0,0)\end{smallmatrix}\right], \left[\begin{smallmatrix}(1,1)\\(1,1)\end{smallmatrix}\right], \left[\begin{smallmatrix}(0,0)\\(1,1)\end{smallmatrix}\right], \left[\begin{smallmatrix}(0,1)\\(1,0)\end{smallmatrix}\right], \left[\begin{smallmatrix}(0,1)\\(0,0)\end{smallmatrix}\right], \left[\begin{smallmatrix}(0,0)\\(1,0)\end{smallmatrix}\right] \right\} \end{aligned}$$

for d even and into $O_3 = \left\{ \left[\begin{smallmatrix}(0,1)\\(1,0)\end{smallmatrix}\right] \right\}$ and

$$O_1 = \left\{ \left[\begin{smallmatrix}0,0\\0,0\end{smallmatrix}\right], \left[\begin{smallmatrix}1,0\\0,0\end{smallmatrix}\right], \left[\begin{smallmatrix}0,0\\1,0\end{smallmatrix}\right], \left[\begin{smallmatrix}0,1\\0,0\end{smallmatrix}\right], \left[\begin{smallmatrix}0,0\\0,1\end{smallmatrix}\right], \left[\begin{smallmatrix}1,0\\0,1\end{smallmatrix}\right], \left[\begin{smallmatrix}1,1\\0,0\end{smallmatrix}\right], \left[\begin{smallmatrix}0,0\\1,1\end{smallmatrix}\right], \left[\begin{smallmatrix}1,1\\1,1\end{smallmatrix}\right] \right\}$$

for d odd.

The labeling of the orbits is consistent with the notation for spin structures for the reducible locus, as we will see in Section 7. The odd theta characteristics form two orbits for d odd and one orbit for d even, but we will not need this fact.

Proof. Recall that in $g = 2$ an even theta characteristic can be written as a sum of three (out of six) odd theta characteristics, and that this representation is unique up to passing to the complementary triple. Odd theta characteristics correspond to Weierstraß points and they have been normalized in Proposition 2.1 globally, i.e. in a way that is invariant under Γ_{d^2}. For d odd the alternating sum of the three Weierstrass points in one fiber is the distinguished even theta characteristic. For d even there are two kinds of triples: four triples (and their complements) can be formed by picking one Weierstraß point out of each pair from Proposition 2.1. Six triples (and their complements) can be formed by picking both Weierstraß point from such a pair and a third point. These correspond to the orbits E_0 and E_2 respectively.

It is easy to show that these orbits do not decompose further by exhibiting appropriate elements of Γ_{d^2} and the transformations

$$\begin{aligned} \widetilde{\gamma}_1 &\mapsto \widetilde{(M\widetilde{\gamma})}_1 = \widetilde{\gamma}_1 e^* - \widetilde{\gamma}_2 c^* + (B^T c^* e^* B)_0^T A \\ \widetilde{\gamma}_2 &\mapsto \widetilde{(M\widetilde{\gamma})}_2 = -\widetilde{\gamma}_1 b^* + \widetilde{\gamma}_2 a^* + (A a^* b^* A^T)_0^T B^T, \end{aligned}$$

where $M = \left(\begin{smallmatrix} a & b \\ c & e \end{smallmatrix}\right) \in \widetilde{\Gamma}_{d^2}$ that follow from (33) and the definition of the modular embedding. □

Proposition 6.7. *The Hilbert theta functions are Hilbert Jacobi forms of weight $(\frac{1}{2}, \frac{1}{2})$ and index $(\frac{1}{2}, \frac{1}{2})$ for some subgroup of finite index in $\widetilde{\Gamma}_{d^2}$.*

For d odd, one of the Hilbert theta functions is a Hilbert Jacobi form for the full group $\widetilde{\Gamma}_{d^2}$. With our choice of B and the modular embedding, this is $\vartheta\left[\begin{smallmatrix}(0,1)\\(1,0)\end{smallmatrix}\right]$.

Proof. The group $\widetilde{\Gamma}_{d^2}$ acts on ϑ by

$$\begin{aligned} \vartheta\left[\begin{smallmatrix}\widetilde{\gamma}_1\\ \widetilde{\gamma}_2\end{smallmatrix}\right](z,u) = \vartheta&\left[\begin{smallmatrix}(M\widetilde{\gamma})_1\\ (M\widetilde{\gamma})_2\end{smallmatrix}\right](M(z), (c^*z+e^*)^{-1}(u + z^* r_1^T + r_2^T)) \\ &\cdot \prod_{i=1}^{2} (c^{(i)} z_i + e^{(i)})^{-1/2} \cdot \mathbf{e}\Big(\tfrac{1}{2}\mathrm{tr}_{K/\mathbb{Q}}(r_1^2 z + 2 r_1 u)\Big) \\ &\cdot \mathbf{e}\Big(-\tfrac{1}{2}\mathrm{tr}_{K/\mathbb{Q}}((u + z^* r_1^T + r_2^T)^T (cz+e)^{-1} c (u + z^* r_1^T + r_2^T))\Big) \\ &\cdot \chi_\theta(\Psi(M))^{-1} \cdot \mathbf{e}\big(\mathrm{tr}_{K/\mathbb{Q}}(\tfrac{\widetilde{\gamma}_1}{2} r_2 - \tfrac{\widetilde{\gamma}_2}{2} r_1)\big) \end{aligned} \tag{36}$$

where $(M, r) = \left(\left(\begin{smallmatrix} a & b \\ c & e \end{smallmatrix}\right), (r_1, r_2)\right) \in \widetilde{\Gamma}_{d^2}$. This proves the claim on the weight and the index. The second statement follows from the previous lemma. □

Last, we list the theta characteristics and their images under the transformation $\widetilde{\gamma}_1 = \gamma_1 A$, respectively $\widetilde{\gamma}_2 = \gamma_2 B^T$. The first row is multiplied by d for convenience.

$\left[\begin{smallmatrix}\gamma_1\\ \gamma_2\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0,0\\0,0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1,0\\0,0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0,0\\1,0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0,1\\0,0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0,0\\0,1\end{smallmatrix}\right]$
$\left[\begin{smallmatrix}d\widetilde{\gamma}_1\\ \widetilde{\gamma}_2\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0,0\\0,0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}d,0\\0,0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0,0\\1,1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}-1,1\\0,0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0,0\\0,d\end{smallmatrix}\right]$
$\left[\begin{smallmatrix}\gamma_1\\ \gamma_2\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1,0\\0,1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0,1\\1,0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1,1\\0,0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0,0\\1,1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1,1\\1,1\end{smallmatrix}\right]$
$\left[\begin{smallmatrix}d\widetilde{\gamma}_1\\ \widetilde{\gamma}_2\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}d,0\\0,d\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}-1,1\\1,1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}d-1,1\\0,0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0,0\\1,d+1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}d-1,1\\1,d+1\end{smallmatrix}\right]$

TABLE 1. Even theta characteristics under base change

$\left[\begin{smallmatrix}\gamma_1\\ \gamma_2\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1,0\\1,0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1,1\\1,0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1,0\\1,1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0,1\\0,1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1,1\\0,1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0,1\\1,1\end{smallmatrix}\right]$
$\left[\begin{smallmatrix}d\widetilde{\gamma}_1\\ \widetilde{\gamma}_2\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}d,0\\1,1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}d-1,1\\1,1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}d,0\\1,d+1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}-1,1\\0,d\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}d-1,1\\0,d\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}-1,1\\1,d+1\end{smallmatrix}\right]$

TABLE 2. Odd theta characteristics under base change

6.5. **The divisor of a Hilbert Jacobi form.** In this section, we determine the class of the bundle of Hilbert Jacobi forms in terms of the pullbacks of the Hodge bundles $\pi^*\lambda_i$ and the zero sections $N^{(i)}$, that is we complete the proof of Theorem 6.1.

The plan is to reduce the weight and index of any Hilbert Jacobi form to zero with the help of the following two functions, whose divisor class we can compute.

Lemma 6.8. *The function* $\vartheta_d^{(i)}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] : \mathbb{H}^2 \times \mathbb{C}^2 \to \mathbb{C}$, *given by*

$$\vartheta_d^{(i)}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](z,u) = \sum_{x\in\mathbb{Z}} \mathbf{e}\big(\tfrac{1}{2}(x+\tfrac{1}{2})^2 \tfrac{z_i}{d} + (x+\tfrac{1}{2})(u_i+\tfrac{1}{2})\big)$$

as a pullback of a one-variable theta function, is a Hilbert Jacobi form for $\widetilde{\Gamma}_{d^2}$ *of weight* κ *with* $\kappa_j = \frac{1}{2}\delta_{ij}$ *and index* $(m^{(1)}, m^{(2)})$ *where* $m^{(j)} = \frac{d}{2}\delta_{ij}$. *Its divisor is*

$$\mathrm{div}\vartheta_d^{(i)}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] = N^{(i)} + \frac{d}{8}D^{(i)}.$$

Proof. One immediately deduces from the theta transformation formula that

$$\vartheta_d^{(i)}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]((M,r)(z,u)) \cdot j^{(i)}_{\frac{1}{2},\frac{d}{2}}((M,r),z,u) = \mathbf{e}(\tfrac{1}{2}r_2^{(i)} - \tfrac{1}{2}dr_1^{(i)}) \cdot \chi_\theta^{(i)}(M) \cdot \vartheta_d^{(i)}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](z,u),$$

for $(M,r) \in \widetilde{\Gamma}_{d^2}$, where $\chi_\theta^{(i)}(M) := \chi_\theta(\mathrm{diag}(d^{-1},1)M^{(i)}\,\mathrm{diag}(d,1))$, and where χ_θ denotes the multiplier introduced in the 1-dimensional theta transformation formula (34).

For the divisor calculation we may focus on the case $i = 1$. At the boundary divisor $D^{(1)}$, which in the local coordinates $(\zeta_{1,k}, q_{1,k}, z_2, u_2)$ of Lemma 5.6 is given by $q_{1,k} = 0$, we have the Fourier development

$$\begin{aligned}\vartheta_d^{(1)}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] &= \sum_{x\in\mathbb{Z}} q_1^{d/2(x+1/2)^2} \zeta_1^{d(x+1/2)} \cdot \mathbf{e}(\tfrac{1}{2}(x+\tfrac{1}{2}))\\ &= \sum_{x\in\mathbb{Z}} q_{1,k}^{d/2(x+1/2)^2 + kd(x+1/2)} \zeta_{1,k}^{d/2(x+1/2)^2+(k+1)d(x+1/2)} \cdot \mathbf{e}(\tfrac{1}{2}(x+\tfrac{1}{2}))\end{aligned}$$

Thus, the vanishing order of $\vartheta_d^{(1)}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ at $q_{1,k}=0$ as *a function* is given by

$$\begin{aligned}\min_{x\in\mathbb{Z}} \tfrac{d}{2}(x+\tfrac{1}{2})^2 + kd(x+\tfrac{1}{2}) &= \tfrac{d}{2}\big(\min_{x\in\mathbb{Z}} x^2 + (1+2k)x + \tfrac{1}{4} + k\big)\\ &= \tfrac{d}{2}\big(\min_{x\in\mathbb{Z}}(x+\tfrac{1}{2}+k)^2 - (\tfrac{1}{2}+k)^2 + \tfrac{1}{4} + k\big)\\ &= \tfrac{d}{2}\big(\min_{x\in\mathbb{Z}}(x+\tfrac{1}{2}+k)^2 - k^2\big)\\ &= \tfrac{d}{8} - k^2\tfrac{d}{2}\end{aligned}$$

Using (30), we see that the vanishing order as *a section* of the bundle of Hilbert Jacobi forms is $\frac{d}{8}$. Thus,

$$C = \mathrm{div}\vartheta_d^{(1)}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] - \tfrac{d}{8}D^{(1)}$$

is a divisor on A_{d^2}, whose support is disjoint from the boundary.

The divisor of the classical theta function $\theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ on $E[d]^\circ = \mathbb{H}\times\mathbb{C}/(\Gamma(d)\ltimes(d\mathbb{Z})^2)$ is equal to d^2-times the zero-section. This relation persists under passing to the quotient by the conjugate group $\widetilde{\Gamma}(d)_d$ via the equivariant isomorphism $(z,u)\mapsto(\frac{z}{d},u)$. Thus

$$\mathcal{O}_{(E(d)_d^\circ)^2}(\mathrm{div}\vartheta_d^{(1)}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]) \cong \mathcal{O}_{(E(d)_d^\circ)^2}(d^2N^{(1)}_{X(d)^2}).$$

Therefore,

$$\begin{aligned}\deg(\widetilde{\tau})C &= \widetilde{\tau}_*\widetilde{\tau}^*C\\ &= \widetilde{\tau}_*\mathcal{O}_{(E(d)_d^\circ)^2}(\mathrm{div}\vartheta_d^{(1)}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]) - \deg(\widetilde{\tau})\tfrac{d}{8}D^{(1)}\\ &= d^2\widetilde{\tau}_*N^{(1)}_{X(d)^2} - \deg(\widetilde{\tau})\tfrac{d}{8}D^{(1)}\\ &= d^2\Delta_d N^{(1)} - \deg(\widetilde{\tau})\tfrac{d}{8}D^{(1)}\,,\end{aligned}$$

which together with $\deg(\widetilde{\tau}) = d^2\Delta_d$ implies the claim. □

Lemma 6.9. *The pullback of the one-variable η-function $\eta^{(i)}:\mathbb{H}^2\times\mathbb{C}^2\to\mathbb{C}$, given by*

$$\eta^{(i)}(z,u) = \mathbf{e}(\tfrac{z_i}{24d})\prod_{n=1}^{\infty}\big(1-\mathbf{e}(n\tfrac{z_i}{d})\big),$$

is a Hilbert Jacobi form for $\widetilde{\Gamma}_{d^2}$ of weight (κ_1,κ_2), where $\kappa_j=\frac{1}{2}\delta_{ij}$, and index $(0,0)$ with divisor

$$\mathrm{div}\eta^{(i)} = \frac{d}{24}D^{(i)}.$$

Proof. From the well-known one-dimensional transformation formula one deduces

$$\eta^{(i)}\big|_{\kappa,0}[M,r] = \chi_\eta(M^{(i)})\cdot\eta^{(i)}$$

where the multiplier χ_η takes values in the 24-th roots of unity. At $D^{(i)}_{\infty,k}$, the function $\eta^{(i)}$ can be written as

$$\eta^{(i)} = q_i^{d/24}\prod_{n=1}^{\infty}(1-q_i^{dn}) = q_{i,k}^{d/24}\zeta_{i,k}^{d/24}\prod_{n=1}^{\infty}(1-q_{i,k}^{dn}\zeta_{i,k}^{dn}).$$

and the rightmost term does not vanish at $q_{i,k}=0$. □

Proof of Theorem 6.1. Let f be a Hilbert-Jacobi form of weight $\kappa=(\kappa_1,\kappa_2)$ and index $m=(m',m'')$. Let $g^{(i)}$, $i=1,2$ be the pullback via pr_i of a modular form form of weight $24d\ell\kappa_i$ for $\Gamma(1)_d$. The function

$$\Big(\big(\vartheta_d^{(1)}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\big)^{2m'}\big(\vartheta_d^{(2)}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\big)^{2m''}(\eta^{(1)})^{-2m'}(\eta^{(2)})^{-2m''}\Big)^{24\ell}g^{(1)}g^{(2)}\cdot f^{-24d\ell}$$

has trivial automorphic factor. Hence, it descends to a meromorphic function on $A^\circ_{d^2}$, and one checks that its extension to A_{d^2} is also meromorphic. Therefore, we can obtain an explicit divisor linear equivalent to f by computing the divisors of the different factors of the product.

Using the above lemmas, we have

$$\begin{aligned}\operatorname{div} f \sim \frac{1}{d}\Big(&2m'(N^{(1)} + \tfrac{d}{8}D^{(1)}) + 2m''(N^{(2)} + \tfrac{d}{8}D^{(2)}) - 2m'\tfrac{d}{24}D^{(1)} - 2m''\tfrac{d}{24}D^{(2)}\\ &+ d\kappa_1\pi^*\lambda_1 + d\kappa_2\pi^*\lambda_2\Big)\\ = {}& \kappa_1\pi^*\lambda_1 + \kappa_2\pi^*\lambda_2 + \tfrac{2m'}{d}N^{(1)} + \tfrac{2m''}{d}N^{(2)} + \tfrac{m'}{6}D^{(1)} + \tfrac{m''}{6}D^{(2)}\,.\end{aligned}$$

Applying $D^{(i)} = \frac{12}{d}\pi^*\lambda_i$ yields the claim. □

7. The reducible locus

Let $P^\circ_{d^2} \subset X^\circ_{d^2}$ be the *reducible locus*, i.e. the locus of points corresponding to abelian surfaces that are isomorphic to a product of elliptic curves.

Proposition 7.1. *The closure P_{d^2} of the reducible locus has the divisor class*

$$[P_{d^2}] \;=\; (5 - \tfrac{6}{d})(\lambda_1 + \lambda_2)\,.$$

in $\mathrm{CH}^1(X_{d^2})$*. If* $d \equiv 1 \bmod 2$*, its spin components have the divisor classes*

$$\begin{aligned}[P_{d^2,\varepsilon=3}] &= (\tfrac{1}{2} - \tfrac{3}{2d})(\lambda_1 + \lambda_2)\\ [P_{d^2,\varepsilon=1}] &= (\tfrac{9}{2} - \tfrac{9}{2d})(\lambda_1 + \lambda_2)\,.\end{aligned}$$

If $d \equiv 0 \bmod 2$*, its spin components have the divisor classes*

$$\begin{aligned}[P_{d^2,\varepsilon=0}] &= (2 - \tfrac{6}{d})(\lambda_1 + \lambda_2)\\ [P_{d^2,\varepsilon=2}] &= 3(\lambda_1 + \lambda_2)\,.\end{aligned}$$

Corollary 7.2. *The spin components of the reducible locus have Euler characteristic*

$$\begin{aligned}\chi(P^\circ_{d^2,\varepsilon=3}) &= -\tfrac{1}{288}(d-3)\tfrac{\Delta_d}{d}\\ \chi(P^\circ_{d^2,\varepsilon=1}) &= -\tfrac{1}{32}(d-1)\tfrac{\Delta_d}{d}\end{aligned}$$

if d is odd, and

$$\begin{aligned}\chi(P^\circ_{d^2,\varepsilon=0}) &= -\tfrac{1}{72}(d-3)\tfrac{\Delta_d}{d}\\ \chi(P^\circ_{d^2,\varepsilon=2}) &= -\tfrac{1}{48}\Delta_d\end{aligned}$$

if $d > 2$ is even.

This fits with the total count $\chi(P^\circ_{d^2}) = -\frac{1}{144}(5d-6)\frac{\Delta_d}{d}$ obtained by several authors, see e.g. [Bai07, Formula (2.23)].

Given a theta function with characteristic, we write

$$\vartheta_0\left[\begin{smallmatrix}\tilde\gamma_1\\ \tilde\gamma_2\end{smallmatrix}\right](z) = \vartheta\left[\begin{smallmatrix}\tilde\gamma_1\\ \tilde\gamma_2\end{smallmatrix}\right](z,0)$$

for the corresponding *theta constant.* Mumford shows ([Mum83, § 8]) that the reducible locus is cut out by the product of all even theta constants and this product vanishes to order one there.

If $d \equiv 1 \bmod 2$, we define

$$\vartheta_{0,\varepsilon=3} = \vartheta_0\left[\begin{smallmatrix}0,1\\ 1,0\end{smallmatrix}\right] \qquad \text{and} \qquad \vartheta_{0,\varepsilon=1} = \prod_{\left[\begin{smallmatrix}\gamma_1\\ \gamma_2\end{smallmatrix}\right]\in O_1} \vartheta_0\left[\begin{smallmatrix}\gamma_1\\ \gamma_2\end{smallmatrix}\right].$$

These functions are, by the description of the action of Γ_{d^2} on characteristics in Section 6.4, modular forms for the full group Γ_{d^2} of weight $(\frac{1}{2},\frac{1}{2})$, respectively of weight $(\frac{9}{2},\frac{9}{2})$. If $d\equiv 0 \bmod 2$, define

$$\vartheta_{0,\varepsilon=0} = \prod_{\left[\begin{smallmatrix}\gamma_1\\ \gamma_2\end{smallmatrix}\right]\in E_0} \vartheta_0\left[\begin{smallmatrix}\gamma_1\\ \gamma_2\end{smallmatrix}\right] \qquad \text{and} \qquad \vartheta_{0,\varepsilon=2} = \prod_{\left[\begin{smallmatrix}\gamma_1\\ \gamma_2\end{smallmatrix}\right]\in E_2} \vartheta_0\left[\begin{smallmatrix}\gamma_1\\ \gamma_2\end{smallmatrix}\right]$$

Again by the calculations in Section 6.4 these four functions are Hilbert modular forms of weight $(2,2)$ in the first case and $(3,3)$ in the second. The zero loci of these modular forms correspond to the spin components of the reducible locus.

Lemma 7.3. *In the open part $X^\circ_{d^2}$ the components of the reducible locus are vanishing loci of the modular forms*

$$P^\circ_{d^2,\varepsilon=3} \;=\; \{\vartheta_{0,\varepsilon=3}=0\}, \qquad \textit{respectively} \qquad P^\circ_{d^2,\varepsilon=1} \;=\; \{\vartheta_{0,\varepsilon=1}=0\}$$

for d odd and

$$P^\circ_{d^2,\varepsilon=0} \;=\; \{\vartheta_{0,\varepsilon=0}=0\}, \qquad \textit{respectively} \qquad P^0_{d^2,\varepsilon=2} \;=\; \{\vartheta_{0,\varepsilon=2}=0\}$$

for d even.

Proof. In the case of a smooth genus two curve, the function $\vartheta=\vartheta\left[\begin{smallmatrix}0,0\\ 0,0\end{smallmatrix}\right]$ vanishes at all odd 2-torsion points, since translating ϑ by such a point gives a theta function with odd characteristic. Consequently, the odd 2-torsion points are the Weierstrass points. This identification extends to reducible curves.

A 2-torsion point $\left[\begin{smallmatrix}\gamma_1\\ \gamma_2\end{smallmatrix}\right]$ is integral, i.e. has the same image under the origami map as the node, if and only if its base change $\left[\begin{smallmatrix}d\widetilde{\gamma}_1\\ \widetilde{\gamma}_2\end{smallmatrix}\right]$ has $\left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right]$ as first column. So the number of integral Weierstrass points in the vanishing locus of $\vartheta_0\left[\begin{smallmatrix}\mu_1\\ \mu_2\end{smallmatrix}\right]$ is the number of odd theta characteristics that have $\left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right]$ as first column after adding $\left[\begin{smallmatrix}d\widetilde{\mu}_1\\ \widetilde{\mu}_2\end{smallmatrix}\right]$.

The claim now follows from inspecting Table 2. □

Lemma 7.4. *For $d\equiv 1 \bmod 2$, we have*

$$\begin{aligned} P_{d^2,\varepsilon=3} &= \tfrac{1}{2}(\lambda_1+\lambda_2) - \tfrac{1}{8}(R^{(1)}+R^{(2)}) \\ P_{d^2,\varepsilon=1} &= \tfrac{9}{2}(\lambda_2+\lambda_2) - \tfrac{3}{8}(R^{(1)}+R^{(2)})\,. \end{aligned}$$

For $d\equiv 0 \bmod 2$, we have:

$$\begin{aligned} P_{d^2,\varepsilon=0} &= 2(\lambda_1+\lambda_2) - \tfrac{1}{2}(R^{(1)}+R^{(2)}) \\ P_{d^2,\varepsilon=2} &= 3(\lambda_2+\lambda_2)\,. \end{aligned}$$

Proof. Let $\vartheta_0\left[\begin{smallmatrix}\gamma_1\\ \gamma_2\end{smallmatrix}\right]$ be an even theta constant. Using the Fourier development, we have

$$\vartheta_0\left[\begin{smallmatrix}\widetilde{\gamma}_1\\ \widetilde{\gamma}_2\end{smallmatrix}\right] = \mathbf{e}(\mathrm{tr}_{K/\mathbb{Q}}(\widetilde{\gamma}_1\widetilde{\gamma}_2)) \sum_{s'\equiv -s''} q_1^{1/2(s'+d\widetilde{\gamma}_1')^2} q_2^{1/2(s''+d\widetilde{\gamma}_1'')^2} \mathbf{e}(\mathrm{tr}(s\tfrac{\widetilde{\gamma}_2}{d}))\,.$$

By symmetry, we may concentrate on the first boundary, which is locally given by $q_1=0$. The minimal q_1-exponent appearing is

$$\min_{s'\in\mathbb{Z}} \tfrac{1}{2}(s'+d\widetilde{\gamma}_1')^2 = \begin{cases} \frac{1}{8}, & \text{if } d\widetilde{\gamma}_1'\equiv 1 \bmod 2 \\ 0, & \text{if } d\widetilde{\gamma}_1'\equiv 0 \bmod 2 \end{cases}.$$

Thus, $\vartheta_0\left[\begin{smallmatrix}\gamma_1\\ \gamma_2\end{smallmatrix}\right]$ vanishes at $R^{(i)}$ to the order $\frac{1}{8}\varepsilon(\gamma)$, where for $\gamma\in\frac{1}{d}\mathbb{Z}$, we set $\varepsilon(\gamma)=1$, if $d\gamma\equiv 1 \bmod 2$ and $\varepsilon(\gamma)=0$ else. The claim now follows using Table 1. □

Proof of Proposition 7.1 and Corollary 7.2. Proposition 7.1 follows from the preceding lemmas and formula (7). Since the components of the reducible locus are all Kobayashi geodesics, the Euler characteristic can be computed by integration against ω_1. Consequently,

$$\begin{aligned} \chi(P^\circ_{d^2,\varepsilon=1}) &= \int_{P_{d^2,\varepsilon=1}} -\omega_1 & &= -\tfrac{1}{2}\big(\tfrac{9}{2}-\tfrac{9}{2d}\big)\int_{X_{d^2}} \omega_1\wedge\omega_2 \\ &= -\tfrac{1}{2}\big(\tfrac{9}{2}-\tfrac{9}{2d}\big)\,\chi(X_{d^2}) & &= -\tfrac{1}{32}(d-1)\tfrac{\Delta_d}{d} \end{aligned}$$

since $\chi(X_{d^2}) = \frac{1}{72}\Delta_d$. The calculation for the other spin components and for d even is the same. □

8. Arithmetic Teichmüller curves in $\Omega\mathcal{M}_2$

In this section we describe loci in the universal covering of $A^\circ_{d^2}$ in terms of theta functions, their derivatives and the torsion sections with the following properties. First, they are invariant under the covering group and hence they descend to loci in $A^\circ_{d^2}$. Second, their images in the pseudo-Hilbert modular surfaces are the Teichmüller curves we are interested in, or rather a union of these.

For this purpose we take for d odd the unique even Hilbert theta function $\vartheta = \vartheta\left[\begin{smallmatrix}0,1\\1,0\end{smallmatrix}\right]$ whose characteristic is invariant under Γ_{d^2} (see Section 6.4), and for d even we take one of the Hilbert theta function with even characteristic in the orbit E_0, say $\vartheta = \vartheta\left[\begin{smallmatrix}0,0\\0,0\end{smallmatrix}\right]$. We let

$$U : \mathbb{H}^2 \times \mathbb{C}^2 \to A^\circ_{d^2} \tag{37}$$

be the universal covering map.

8.1. **The stratum $\Omega\mathcal{M}_2(1,1)$.** We fix a torsion order $m \in \mathbb{N}$ and define $\widetilde{O}_m(1,1)$, the *lifted origami locus* for the stratum $\Omega\mathcal{M}_2(1,1)$. These are points on the theta divisor, where the derivative of theta in the u_2-direction vanishes and whose first coordinate projects to an m-torsion point. Formally,

$$\widetilde{O}_m(1,1) = \Big\{(z,u)\in\mathbb{H}^2\times\mathbb{C}^2 \;:\; \vartheta(z,u)=0,\; \frac{\partial\vartheta}{\partial u_2}(z,u)=0,\; (z,u)\in U^{-1}(N^{(1)}_{m-\mathrm{tor}})\Big\}. \tag{38}$$

The transformation properties of theta functions imply that the images of the lifted origami loci are closed (in fact algebraic) subsets of the (open) universal families.

Lemma 8.1. *The images $O^\circ_m(1,1) = U(\widetilde{O}_m(1,1))$ for any $m\in\mathbb{N}$ are closed subsets of $A^\circ_{d^2}$.*

We are ultimately interested in their closures in the compactified universal family.

Definition 8.2. *The* origami locus $O_m(1,1)$ *is the closure in A_{d^2} of $O^\circ_m(1,1)$).*

In this section, we show that the π-push forward of $O_m(1,1)$ is a union of arithmetic Teichmüller curves in $\Omega\mathcal{M}_2(1,1)$ plus possibly some spurious parts of the reducible locus and of arithmetic Teichmüller curves in $\Omega\mathcal{M}_2(2)$ if $m=1,2$.

Theorem 8.3. *Let $m\in\mathbb{N}$, $m>1$. If $m\equiv 0 \bmod 2$, then*

$$\pi_* O_{2m}(1,1) = 2T_{d,M=m}.$$

If $m\equiv 1 \bmod 2$, then

$$\begin{aligned} &\pi_*O_{2m}(1,1) = 2T_{d,M=m,\varepsilon=1}, && \pi_*O_m(1,1) = 2T_{d,M=m,\varepsilon=3}, && \textit{for } d \textit{ odd} \\ &\pi_*O_{2m}(1,1) = 2T_{d,M=m,\varepsilon=2}, && \pi_*O_m(1,1) = 2T_{d,M=m,\varepsilon=0}, && \textit{for } d \textit{ even.} \end{aligned}$$

The case $m=1$ is special in that we also hit Teichmüller curves in $\Omega\mathcal{M}_2(2)$ and parts of the reducible locus by $\pi_*O_1(1,1)$.

Theorem 8.4. *The push-forward of the origami locus decomposes as*

$$\begin{array}{llr}
\pi_* O_1(1,1) = 2[T_{d,M=1,\varepsilon=3}] + 3[W_{d^2,\varepsilon=3}] + & [P_{d^2,\varepsilon=3}] & d \text{ odd} \\
\pi_* O_2(1,1) = 2[T_{d,M=1,\varepsilon=1}] + 3[W_{d^2,\varepsilon=1}] + & [P_{d^2,\varepsilon=1}] & d \text{ odd} \\
\pi_* O_1(1,1) = 2[T_{d,M=1,\varepsilon=0}] \phantom{+ 3[W_{d^2,\varepsilon=0}]} + & [P_{d^2,\varepsilon=0}] & d \text{ even} \\
\pi_* O_2(1,1) = 2[T_{d,M=1,\varepsilon=2}] + 3[W_{d^2,\varepsilon=2}] + & [P_{d^2,\varepsilon=2}] & d \text{ even}
\end{array}$$

We start the proofs with the closedness lemma.

Proof of Lemma 8.1. The vanishing locus of a Hilbert Jacobi form is closed, since it is a closed subset of $\mathbb{H}^2 \times \mathbb{C}^2$ and since the automorphy factor is a product of non-zero terms. This applies for the full group $\widetilde{\Gamma}_{d^2}$ for d odd, and for a subgroup of finite index in $\widetilde{\Gamma}_{d^2}$ that stabilizes the characteristic (see Section 9.4) for d even. Arguing for this subgroup is sufficient since the image of a closed set under a finite map is again closed.

The torsion condition is also closed. It remains to treat the derivative of the theta function. We define $\chi(M,r) = \chi_\theta(\Psi(M,r))\mathbf{e}(\mathrm{tr}_{K/\mathbb{Q}}(\frac{\widetilde{\gamma}_1}{2}r_2 - \frac{\widetilde{\gamma}_2}{2}r_1)$. Restricted to points (z,u) where $\vartheta(z,u) = 0$ (and hence also $\vartheta((M,r)(z,u))$ we obtain for all $(M,r) \in \widetilde{\Gamma}_{d^2}$ by differentiating the equation defining modularity (see Proposition 6.7) and using the definition of the action in (12) that

$$\begin{aligned}
\frac{\partial \vartheta}{\partial u_2}((z,u)) &= \frac{\partial}{\partial u_2}\Big(\vartheta((M,r)(z,u))\, \widetilde{\jmath}_{(\frac12,\frac12),(\frac12,\frac12)}((M,r),z,u)\, \chi(M,r)\Big) \\
&= \frac{\partial \vartheta}{\partial u_2}((M,r)(z,u))\big(c^{(2)}z_2 + e^{(2)}\big)^{-1}\, \widetilde{\jmath}_{(\frac12,\frac12),(\frac12,\frac12)}((M,r),z,u)\, \chi(M,r) \\
&= \frac{\partial \vartheta}{\partial u_2}((M,r)(z,u))\, \widetilde{\jmath}_{(\frac12,\frac32),(\frac12,\frac12)}((M,r),z,u)\, \chi(M,r)\,. \qquad (39)
\end{aligned}$$

Consequently, the automorphy factor is here again a product of non-zero terms and the vanishing locus is well-defined and closed as a subset of $A^\circ_{d^2}$ for both parities of d. $\square$

As first step towards the theorems of this section, we show that the origami maps are normalized in the sense of Proposition 2.1 for the two theta functions we need.

Lemma 8.5. *For fixed $z \in \mathbb{H}^2$, let $\Theta_z\left[\begin{smallmatrix}\gamma_1\\ \gamma_2\end{smallmatrix}\right]$ denote the curve in $A_{d^2,z}$ given by $\vartheta\left[\begin{smallmatrix}\gamma_1\\ \gamma_2\end{smallmatrix}\right] = 0$. The covering*

$$pr_i : \Theta_z\left[\begin{smallmatrix}\gamma_1\\ \gamma_2\end{smallmatrix}\right] \to E_{(z_i/d,1)}$$

is normalized, if and only if

$$\left[\begin{smallmatrix}\gamma_1\\ \gamma_2\end{smallmatrix}\right] \in E_0, \ \text{if } d \equiv 0 \bmod 2 \qquad \text{resp.} \qquad \left[\begin{smallmatrix}\gamma_1\\ \gamma_2\end{smallmatrix}\right] = \left[\begin{smallmatrix}0,1\\ 1,0\end{smallmatrix}\right], \ \text{if } d \equiv 1 \bmod 2.$$

Proof. The function $\vartheta = \vartheta\left[\begin{smallmatrix}0,0\\ 0,0\end{smallmatrix}\right]$ vanishes at all odd 2-torsion points, since translating ϑ by such a point gives a theta function with odd characteristic. Since $\Theta_z = \Theta_z\left[\begin{smallmatrix}\widetilde{\gamma}_1\\ \widetilde{\gamma}_2\end{smallmatrix}\right]$ is a symmetric divisor with respect to $[-1]$, the translates by $z^*\widetilde{\gamma}_1^T + \widetilde{\gamma}_2^T$ of the odd 2-torsion points are precisely the 6 Weierstraß points on Θ_z. The claim now follows by inspecting Table 2. $\square$

Proof of Theorems 8.3 and 8.4. A point $z \in X^\circ_{d^2}$ lies in the support of $\pi_* O_m(1,1)$ if and only if it has a preimage $y \in A_{d^2,z}$ such that $y \in \Theta_z$, such that y is a ramification point of $p_1 : \Theta_z \to E_{z_1/d,1}$, or alternatively a zero of the first eigendifferential $\omega_1 = \pi_1^*\omega_E$, and such that y is mapped to a m-torsion point in $E_{z_1/d,1}$.

If y is a ramification point of order 2, then it is a fixed point of the hyperelliptic involution, so it is a Weierstraß point. Consequently $z \in W_{d^2}$ and such a point has a unique preimage in $O_m(1,1)$.

Suppose that y is a ramification point of order 1 and that Θ_z is a smooth curve. Then two zeros of ω_1 are exchanged by the hyperelliptic involution σ, and σ descends to the elliptic involution (see Proposition 2.1). Hence the images of the ramification points differ by a torsion point on $E_{z_1/d,1}$ and z lies on some $T_{d,M,\varepsilon}$. The torsion order of the corresponding minimal covering is m or $m/2$, depending on $m \mod 4$, on d and ε, as explained in Section 2.1. This implies the set-theoretic assignment of the various $T_{d,M,\varepsilon}$ to the push-forwards of the $O_m(1,1)$. In each of the cases there are two possible points y for the same z.

If Θ_z is a singular curve, then it is reducible, and its components are two elliptic curves E_1, E_2 joined at a node, since $\pi_* O_m(1,1)$ is the closure of a subvariety in $X^\circ_{d^2}$ for any m, and hence the Jacobian of a generic point of its support is compact. On each E_i ($i = 1,2$), the projection p_1 is still non-constant (since p_1 and the projection to the kernel of p_1 deform over all of $X^\circ_{d^2}$, otherwise the splitting as product of elliptic curves would deform to all of $X^\circ_{d^2}$), and thus an unramified covering. Consequently, $\frac{\partial \vartheta}{\partial u_2}$ never vanishes at a smooth point of Θ_z, while it does vanish at the singular point of Θ_z (even both partial derivatives of ϑ vanish).

The node y is a 2-torsion point different from the six odd Weierstraß points, i.e. it is an even 2-torsion point. Consequently, its p_1-image is a 2-torsion point and there is no contribution from the reducible locus, except for $m = 1$ and $m = 2$.

Suppose first that d is odd, hence $\vartheta = \vartheta\left[\begin{smallmatrix}0,1\\1,0\end{smallmatrix}\right](z)$. If the node is mapped to zero, then it is an even two-torsion point with the property that after translating by $\left[\begin{smallmatrix}0,1\\1,0\end{smallmatrix}\right]$ its p_1-image is zero, i.e. in the eigenform coordinates of the second row of the Table 1 the first column of the point is zero. By inspecting the table we see that there is only one possibility, $\left[\begin{smallmatrix}0,1\\1,0\end{smallmatrix}\right]$ itself. This implies that $y = 0$ and that z is in the vanishing locus of the corresponding theta constant, i.e. $\vartheta_0\left[\begin{smallmatrix}0,1\\1,0\end{smallmatrix}\right](z) = 0$. By Lemma 7.3 this is the defining equation of $P_{d^2,\varepsilon=3}$.

Similarly, precisely the odd theta characteristics in O_9 are mapped after translation by $\left[\begin{smallmatrix}0,1\\1,0\end{smallmatrix}\right]$ to a primitive 2-torsion point. By Lemma 7.3 this implies that $P_{d^2,\varepsilon=3}$ is contained in $\pi(O_2(1,1))$.

Suppose next that d is even and $\vartheta = \vartheta\left[\begin{smallmatrix}0,0\\0,0\end{smallmatrix}\right](z)$. Precisely the odd theta characteristics in E_0 are mapped (after translation by zero and base change) to a first column equal to zero, while those in E_2 are mapped to a primitive 2-torsion point. Together with Lemma 7.3 this explains the setwise distribution of the reducible locus among $\pi(O_1(1,1))$ and $\pi(O_2(1,1))$.

It remains to determine the multiplicities of $O_m(1,1)$ at the components lying over the curves $T_{d,M,\varepsilon}$, $W_{d^2,\varepsilon}$ and $P_{d^2,\varepsilon}$. We start with $W_{d^2,\varepsilon}$. Fix q_1, an M-torsion point u_1 and shift the remaining coordinates, so that in the new coordinates the point will be at $\widetilde{z_2} = 0$ and $\widetilde{u_2} = 0$. The fiber of the origami locus is cut out by $\widetilde{\vartheta}(\widetilde{z_2},\widetilde{u_2}) = 0$ and $\partial_{u_2}\widetilde{\vartheta}(\widetilde{z_2},\widetilde{u_2}) = 0$ for some function $\widetilde{\vartheta}$, which is odd as a function of $\widetilde{u_2}$. This implies that the multiplicity of the fiber is two, hence the multiplicity of the component is a multiple of two. Now we consider the fiber with (z_1, u_1) varying, choosing locally (z_2, u_2) so the the first two conditions of the origami locus are satisfied. Since locally near the critical point three branches of the map p_1 come together, the multiplicity of the component is divisible by three. Taking the factor $1/2$ from the quotient stacks into account, this implies that the multiplicity of $W_{d^2,\varepsilon}$ is three.[3]

[3]A priori,this argument shows that the multiplicity is at least three. Similarly, the arguments in the subsequent paragraphs show that the coefficients on the right hand sides are at least what is written in Theorem 8.4 resp. Theorem 8.3. Since we know the total count by an independent argument, see Proposition 9.9, the multiplicities cannot be larger.

Near $T_{d,M,\varepsilon}$ the branching argument for p_1 gives multiplicity two. Two preimages and the stacky factor $1/2$ give in total the coefficient two in Theorem 8.4.

Near $P_{d^2,\varepsilon}$ the fiber is singular near the preimage point z, hence besides $\partial_{u_2}\vartheta$ also $\partial_{u_1}\vartheta$ vanishes there. This implies multiplicity at least two, hence at least one, with stacky factor $1/2$ taken into account. $\square$

8.2. **The stratum $\Omega\mathcal{M}_2(2)$.** We need the following theorem from [Bai07] to subtract the contribution of the curves W_d^ε that appear in Theorem 8.4.

Theorem 8.6. *The classes in* $\mathrm{CH}^1(X_{d^2})$ *of the Teichmüller curves generated by reduced square-tiled surfaces in* $\Omega\mathcal{M}_2(2)$ *are*

$$[W_{d^2}^{\varepsilon=3}] = \tfrac{3}{2}(1-\tfrac{3}{d})\lambda_1 + \tfrac{9}{2}(1-\tfrac{3}{d})\lambda_2 \quad \textit{and} \quad [W_{d^2}^{\varepsilon=1}] = \tfrac{3}{2}(1-\tfrac{1}{d})\lambda_1 + \tfrac{9}{2}(1-\tfrac{1}{d})\lambda_2$$

for d odd and

$$[W_{d^2}] = [W_{d^2}^{\varepsilon=2}] = 3(1-\tfrac{2}{d})\lambda_1 + 9(1-\tfrac{2}{d})\lambda_2$$

for $d > 2$ even. Consequently, the number w_d^ε of reduced square-tiled surfaces in $\Omega\mathcal{M}_2(2)$ with spin ε is

$$w_d^{\varepsilon=3} = \frac{3}{16}(d-3)\tfrac{\Delta_d}{d} \qquad w_d^{\varepsilon=1} = \frac{3}{16}(d-1)\tfrac{\Delta_d}{d}$$
$$w_d^{\varepsilon=2} = \frac{3}{8}(d-2)\tfrac{\Delta_d}{d}$$

where the first line corresponds to d odd and the second to $d > 2$ even.

The counting part of this theorem was proven in [LR06], the class in $\mathrm{CH}^1(X_{d^2})$ was first determined in [Bai07].

We sketch how one could prove this theorem, at least without distinguishing the components, with a similar setup as for the stratum $\Omega\mathcal{M}_2(1,1)$. We define $\widetilde{O}(2)$, the lifted origami locus for the stratum $\Omega\mathcal{M}_2(2)$ to be

$$\widetilde{O}(2) = \Big\{(z,u) \in \mathbb{H}^2 \times \mathbb{C}^2 \,:\, \vartheta(z,u) = 0, \quad \frac{\partial\vartheta}{\partial u_2}(z,u) = 0, \quad \frac{\partial^2\vartheta}{\partial u_2^2} = 0\Big\}.$$

The transformation properties of theta functions imply again that $O^\circ(2) = U(\widetilde{O}(2))$ is closed in $A_{d^2}^\circ$. The origami locus $O(2)$ is defined as the closure in A_{d^2} of $O^\circ(2))$. With similar arguments as above one can show that the push-forward of the origami locus O_2 is supported on $W_{d^2}^\varepsilon$. To prove Theorem 8.6 from here it remains to determine the multiplicity of this push-forward and compute the class of $\pi_* O(2)$ as a triple intersection, following the proof for $\pi_*(O_m(1,1))$ given in the next section.

9. Intersection products

We now can complete the proof of Theorem 1.3. For this purpose we prove in Theorem 9.2 how to subtract from a triple intersection of divisors on A_{d^2} suitable boundary components in order to compute the class of the pushforward of the origami locus $O_m(1,1)$. As technical steps it remains to actually perform triple intersection of the geometric divisors appearing on the right hand side of the class computation in Theorem 6.1 (see Proposition 9.3) and to compute these boundary contribution.

In this section, we restrict to the case d odd. The additional computations that have to be performed for even d are briefly discussed in Section 9.4. We continue to denote by ϑ the unique Hilbert theta function with even characteristic fixed by Γ_{d^2}. It gives rise to a section of the Hilbert-Jacobi bundle $\mathcal{J}_\vartheta = \mathcal{J}_{\kappa,m}(\widetilde{\Gamma}_{d^2})$ with

$\kappa = m = (1/2, 1/2)$, and therefore to a Cartier divisor divϑ on A_{d^2}. The associated Weil divisor [divϑ] can be written as

$$[\mathrm{div}\vartheta] = \Theta + B(\vartheta)$$

where $B(\vartheta)$ is a linear combination of boundary components and Θ has no support at the boundary. We view Θ as element of $\mathrm{CH}^1(A_{d^2})$.

Let $|\Theta|$ denote the support of Θ and let $i : |\Theta| \to A_{d^2}$ be the inclusion. We can compare the intersection numbers on Θ and A_{d^2} since Θ is a reduced (and in fact irreducible) subvariety.

The next condition in the definition of the origami locus is the vanishing of the theta derivative. On Θ this function is a section of the restriction of a bundle on A_{d^2}, whose class we already computed. Recall the definition of U from (37)

Proposition 9.1. *The function $\frac{\partial\vartheta}{\partial u_2}$ restricted to $U^{-1}(|\Theta|)$ descends to a well-defined global meromorphic section $\partial\vartheta$ of $i^*\mathcal{J}_{\partial\vartheta}$, where $\mathcal{J}_{\partial\vartheta}$ is the bundle of Hilbert Jacobi forms $\mathcal{J}_{\kappa,m}(\widetilde{\Gamma}_{d^2})$ with $\kappa = (\frac{1}{2}, \frac{3}{2}), m = (\frac{1}{2}, \frac{1}{2})$.*

Proof. This follows immediately from the calculation in (39). □

To the Cartier divisor div$\partial\vartheta$ we associate the Weil divisor [div$\partial\vartheta$]. It is a sum

$$[\mathrm{div}\partial\vartheta] = D\Theta + B(\partial\vartheta)$$

where $B(\partial\vartheta) \in \mathrm{CH}^1(|\Theta|)$ is a linear combination of boundary components of Θ, and $D\Theta$ has no support on the boundary.

Finally, in the definition of the origami locus, we have to intersect with the torsion condition. This may also result in components, that lie entirely in the boundary. We have to subtract this contribution, that is, in $\mathrm{CH}^*(|\Theta|)$, we can write

$$i^*[O_m(1,1)] = D\Theta.i^*(N^{(1)}_{m-\mathrm{tor}}) - B_m(N)$$

where $B_m(N)$ is supported in the boundary of $|\Theta|$ since by definition $O_m(1,1)$ has no support on the boundary.

Theorem 9.2. *For d odd, the class of the origami locus in $\mathrm{CH}^*(A_{d^2})$ can be computed as*

$$\begin{aligned}[O_m(1,1)] \;=\; & \mathrm{c}_1(\mathcal{J}_\vartheta).\,\mathrm{c}_1(\mathcal{J}_{\partial\vartheta}).N^{(1)}_{m-\mathrm{tor}} - B(\vartheta).\,\mathrm{c}_1(\mathcal{J}_{\partial\vartheta}).N^{(1)}_{m-\mathrm{tor}} \\ & - N^{(1)}_{m-\mathrm{tor}}.i_*B(\partial\vartheta) - i_*B_m(N)\,. \end{aligned} \tag{40}$$

Proof. Since Θ is reduced, the pushforward of $D\Theta$ by i is

$$i_*D\Theta = \mathrm{c}_1(\mathcal{J}_{\partial\vartheta}).\Theta - i_*B(\partial\vartheta).$$

by the projection formula. Thus,

$$\begin{aligned}[O_m(1,1)] + i_*B_m(N) &= i_*(i^*N^{(1)}_{m-\mathrm{tor}}.D\Theta) \\ &= N^{(1)}_{m-\mathrm{tor}}.\,\mathrm{c}_1(\mathcal{J}_{\partial\vartheta}).\Theta \;-\; N^{(1)}_{m-\mathrm{tor}}.i_*B(\partial\vartheta)\,.\end{aligned}$$

Now plug in $\Theta = \mathrm{c}_1(\mathcal{J}(\vartheta)) - B(\vartheta)$ to obtain the claim. □

9.1. **Triple intersections.** The divisors of Jacobi forms have been expressed in term of the zero section divisors $N^{(i)}$, the pullbacks of Hodge bundles $\pi^*\lambda_i$. The evaluation of intersection products of those divisors and with the boundary divisors $D^{(i)}$ is manageable since many triple intersections have π-pushforward equal to zero.

Proposition 9.3. *The π-pushforward of a triple intersection between any of $N^{(i)}$, $\pi^*\lambda_i$, and $D^{(i)}$ is given by*

$$\pi_*(N^{(1)}.N^{(2)}.\pi^*\lambda_i) = d^2\lambda_i \qquad \pi_*(N^{(1)}.N^{(2)}.D^{(i)}) = d^2R^{(i)}$$

$$\pi_*((N^{(1)})^2.N^{(2)}) = -d^2\lambda_1 \qquad \pi_*(N^{(1)}.(N^{(2)})^2) = -d^2\lambda_2$$

and is zero for all triples that do not agree with any of the above up to permutation.

Proof. The divisors $\pi^*\lambda_i$ and $D^{(i)}$ are vertical, i.e. their π-images are divisors, while the $N^{(i)}$ are horizontal, i.e. $\pi|_{N_i}$ is surjective. Consequently, any intersection of three divisors meeting properly, among which two are vertical, consists of 1-cycles along which π is of relative dimension ≥ 1, hence their π-pushforward is zero. We may use linear equivalence in the base to ensure that the proper intersection hypothesis holds for any of the intersections $N^{(i)}.\pi^*\lambda_j.\pi^*\lambda_k$ for $i,j,k \in \{1,2\}$.

The intersection $N^{(1)}.N^{(2)}$ is the closure of the projection of

$$\left\{(z,u) \in \mathbb{H}^2 \times \mathbb{C}^2 \mid u \in \operatorname{diag}(\tfrac{z_1}{d}, \tfrac{z_2}{d})\mathbb{Z}^2 + \mathbb{Z}^2\right\}$$

to A_{d^2}. In each fiber, this is a group of order d^2, the kernel of the projection to $\mathbb{C}/(\frac{1}{d}z_1, 1)\mathbb{Z}^2 \times \mathbb{C}/(\frac{1}{d}z_2, 1)\mathbb{Z}^2$. Thus, $\pi_*(N^{(1)}.N^{(2)}) = d^2\pi_*N = d^2[X_{d^2}]$, where N is the zero section of $\pi : A_{d^2} \to X_{d^2}$. This gives all the intersection products with π-pullbacks as stated.

It remains to treat intersections of π-pullbacks with $(N^{(i)})^2$. Since $(N^{(i)})^2$ is represented by the pullback via pr_i of a zero-cycle $E(d)_d$, its intersection with any of the vertical divisors is a cycle on which π is of relative dimension one, hence again its π-pushforward is zero.

For the remaining two cases stated in the last line of the lemma we start with $\varpi_*(N^2_{X(d)}) = -\lambda_{X(d)}$, as in the proof of Proposition 6.2. This directly implies that

$$(\varpi \times \varpi)_*\big((N^{(1)}_\square)^2.N^{(2)}_\square\big) = -\lambda^{(1)}_\square,$$

using the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{CH}^*_{\mathbb{Q}}(E(d)_d) \otimes \mathrm{CH}^*_{\mathbb{Q}}(E(d)_d) \arrow[r] \arrow[d, "\varpi_* \otimes \varpi_*"'] & \mathrm{CH}^*_{\mathbb{Q}}(E(d)^2_d) \arrow[d, "(\varpi\times\varpi)_*"] \\
\mathrm{CH}^*_{\mathbb{Q}}(X(d)_d) \otimes \mathrm{CH}^*_{\mathbb{Q}}(X(d)_d) \arrow[r] & \mathrm{CH}^*_{\mathbb{Q}}(X(d)^2_d)
\end{tikzcd}$$

and the fact that $N^{(2)}_\square$ is the pullback of a section the second elliptic fibration. The same argument gives

$$(\varpi \times \varpi)_*((\mu_* N^{(1)}_\square)^2.\nu_* N^{(2)}_\square) = -\lambda^{(1)}_\square$$

for any translates by torsion sections μ and ν. Now

$$\begin{aligned}
\pi_*((N^{(1)})^2.N^{(2)}) &= \frac{1}{d^2\Delta_d}\pi_*\widetilde{\tau}_*((\widetilde{\tau}^*N^{(1)})^2\,.\,\widetilde{\tau}^*N^{(2)}) \\
&= \frac{1}{d^2\Delta_d}\tau_*(\varpi\times\varpi)_*\left(\sum_{\mu\in T}\sum_{\nu\in T}(\mu_*N^{(1)}_{X(d)})^2\,.\,\nu_*N^{(2)}_{X(d)}\right) \\
&= \frac{d^2}{\Delta_d}\tau_*(-\lambda^{(1)}_{X(d)}) \;=\; -d^2\lambda^{(1)}
\end{aligned}$$

where $T = \mathfrak{o}^\vee_{d^2} \oplus \mathfrak{o}_{d^2}/(\mathbb{Z}^2 \oplus d\mathbb{Z}^2)$ is a torsion subgroup of order d^2 and where we used that for $\mu,\mu' \in T$ we have $(\mu_*N^{(1)}_{X(d)}\cdot\mu'_*N^{(1)}_{X(d)}) = 0$ unless $\mu = \mu'$. □

9.2. **Boundary contributions.** In this section we collect all the boundary contributions that appear in Theorem 9.2. Together with the results from Section 8 this allows us to conclude the proof of the main Theorem 1.3 for d odd. The proofs of the boundary statements appear in the next section.

Proposition 9.4. *For d odd the boundary contribution of* $\operatorname{div}\vartheta$ *in* $\mathrm{CH}^1(A_{d^2})$ *is*

$$B(\vartheta) = \tfrac{1}{8}\left(D^{(1)} + D^{(2)}\right).$$

Proposition 9.5. *For d odd the boundary contribution of* $\operatorname{div}\partial\vartheta$ *in* $\mathrm{CH}^2(A_{d^2})$ *is equal to*

$$B(\partial\vartheta) = \tfrac{1}{8}\big(D^{(1)} + D^{(2)}\big).\,\mathrm{c}_1(\mathcal{J}_\vartheta)\,. \tag{41}$$

Proposition 9.6. *For d odd the push-forward of the boundary contribution $B_m(N)$ is equal to*

$$\pi_* B_m(N) = \begin{cases} R^{(2)} & \text{if} m = 1 \\ 0 & \text{else} \end{cases}\,. \tag{42}$$

Proof of Theorem 1.3. There are several cases to be discussed.
Case $M > 1$ odd, spin $\varepsilon = 3$. In this case $2[T_{d,M,\varepsilon=3}] = [\pi_* O_M(1,1)]$ by Theorem 8.3. The first contribution to this is, according to Proposition 6.2 and Theorem 9.2, equal to

$$\begin{aligned}\pi_*(\mathrm{c}_1(\mathcal{J}_\vartheta).\,\mathrm{c}_1(\mathcal{J}_{\partial\vartheta}).N^{(1)}_{M-\mathrm{tor}}) \;&=\; \pi_*\Big(\big((\tfrac{1}{2}+\tfrac{1}{d})\pi^*\lambda_1 + (\tfrac{1}{2}+\tfrac{1}{d})\pi^*\lambda_2 + \tfrac{1}{d}N^{(1)} + \tfrac{1}{d}N^{(2)}\big).\\ &\quad.\,\big((\tfrac{1}{2}+\tfrac{1}{d})\pi^*\lambda_1 + (\tfrac{3}{2}+\tfrac{1}{d})\pi^*\lambda_2 + \tfrac{1}{d}N^{(1)} + \tfrac{1}{d}N^{(2)}\big)\;.\;\tfrac{\Delta_M}{M}\big(N^{(1)}+\lambda_1\big)\Big)\\ &=\; d\,\tfrac{\Delta_M}{M}\big((1+\tfrac{2}{d})\lambda_1 + (2+\tfrac{1}{d})\lambda_2\big)\,.\end{aligned} \tag{43}$$

Next,

$$\begin{aligned}\pi_*(B(\vartheta).\,\mathrm{c}_1(\mathcal{J}_{\partial\vartheta}).N^{(1)}_{M-\mathrm{tor}}) \;&=\; \pi_*\Big(\big((\tfrac{1}{2}+\tfrac{1}{d})\pi^*\lambda_1 + (\tfrac{3}{2}+\tfrac{1}{d})\pi^*\lambda_2 + \tfrac{1}{d}N^{(1)} + \tfrac{1}{d}N^{(2)}\big).\\ &\quad.\,\tfrac{\Delta_M}{M}\big(N^{(1)}+\lambda_1\big)\,.\,\tfrac{1}{8}(D^{(1)}+D^{(2)})\Big)\\ &=\; \tfrac{1}{8}\,d\,\tfrac{\Delta_M}{M}\,\big(R^{(1)}+R^{(2)}\big)\\ &=\; d\,\tfrac{\Delta_M}{M}\,\Big(\tfrac{3}{2d}\lambda_1 + \tfrac{3}{2d}\lambda_2\Big)\,.\end{aligned} \tag{44}$$

By Proposition 9.5 we get

$$\begin{aligned}\pi_*(N^{(1)}_{M-\mathrm{tor}}.B(\partial\vartheta)) \;&=\; \pi_*(N^{(1)}_{m-\mathrm{tor}}.\,\mathrm{c}_1(\mathcal{J}_\vartheta).\tfrac{1}{8}(D^{(1)}+D^{(2)}))\\ &=\; d\,\tfrac{\Delta_M}{M}\,\Big(\tfrac{3}{2d}\lambda_1 + \tfrac{3}{2d}\lambda_2\Big)\,.\end{aligned} \tag{45}$$

Since $\pi_*(B_M(N)) = 0$ for $M > 1$ we find altogether

$$[\pi_* O_M(1,1)] \;=\; d\,\tfrac{\Delta_M}{M}\,\Big((1-\tfrac{1}{d})\lambda_1 + (2-\tfrac{2}{d})\lambda_2\Big),$$

and this completes the first case.

Case $M > 1$ odd, spin $\varepsilon = 1$. Since in this case $2\,[T_{d,M,\varepsilon=1}] = [\pi_* O_{2M}(1,1)]$ and since $N^{(1)}_{2M-\mathrm{tor}} = \frac{3\Delta_M}{M}N^{(1)}$ all the contributions are multiplied by three compared to the previous calculation, and this proves the second case.

Case M even. Recall that there is no spin distinction in this case. Now $2[T_{d,M,\varepsilon=0}] = [\pi_* O_{2M}(1,1)]$ and for M even the number of primitive $2M$-torsion points is $4\frac{\Delta_M}{M}$. Hence all the contributions are 4 times larger than in the corresponding cases for M odd and spin $\varepsilon = 3$, completing the discussion of this case.

It remains to discuss the subcases for $M = 1$.

Case $M=1$, spin $\varepsilon=3$. We compute as in (43), (44) and (45), taking into account that $N^{(1)}_{1-\mathrm{tor}}$ has *no* λ_1-contribution (as $N^{(1)}_{m-\mathrm{tor}}$ had it according to Proposition 6.2),

$$\begin{aligned}
\pi_*(\mathrm{c}_1(\mathcal{J}_\vartheta).\,\mathrm{c}_1(\mathcal{J}_{\partial\vartheta}).N^{(1)}_{1-\mathrm{tor}}) &= d\left(\lambda_1 + (2+\tfrac{1}{d})\lambda_2\right), \\
\pi_*(\mathrm{c}_1(\mathcal{J}_{\partial\vartheta}).N^{(1)}_{1-\mathrm{tor}}.B(\vartheta)) &= d\left(\tfrac{3}{2d}\lambda_1 + \tfrac{3}{2d}\lambda_2\right) \\
\pi_*(N^{(1)}_{1-\mathrm{tor}}.B(\partial\vartheta)) &= d\left(\tfrac{3}{2d}\lambda_1 + \tfrac{3}{2d}\lambda_2\right).
\end{aligned}$$

Since $\pi_*(i_*B_1(N)) = R^{(2)} = \frac{12}{d}\lambda_2$ we find

$$\pi_* O_1(1,1) = (d-3)\lambda_1 + \tfrac{2}{d}(d^2-d-6)\lambda_2.$$

Subtracting the contributions from the reducible locus (see Proposition 7.1) and from $W_{d,\varepsilon=3}$ (see Theorem 8.6) according to Theorem 8.4 gives the claim.

Case $M=1$, spin $\varepsilon=1$. Since $N^{(1)}_{2-\mathrm{tor}} = 3(N^{(1)}+\lambda_1)$ and since in this case $\pi_*(B_2)=0$ we get as in (43), (44) and (45), that

$$\pi_*(O_2(1,1)) = (3d-3)\lambda_1 + 6(d-1)\lambda_2.$$

Again, subtracting the contributions from the reducible locus (see Proposition 7.1) and from $W_{d,\varepsilon=1}$ (see Theorem 8.6) according to Theorem 8.4 gives the claim. □

9.3. **Intersection with the boundary: proofs.** We will deduce Proposition 9.4 from the following result. We compute the vanishing order of the theta function for general k and general characteristics, and later specialize to the unique theta function invariant under the whole group $\widetilde{\Gamma}_{d^2}$.

Proposition 9.7. *The vanishing order at the boundary divisor $D^{(i)}_{\infty,k}$ of the theta function $\vartheta\left[\begin{smallmatrix}\widetilde{\gamma}_1\\ \widetilde{\gamma}_2\end{smallmatrix}\right]$ considered as a* function *on the infinite chain of rational lines*

$$\begin{cases} \frac{1}{8} - \frac{1}{2}k^2, & \text{if } \; d\widetilde{\gamma}_1^{(i)} \equiv 1 \bmod 2 \\ -\frac{1}{2}k^2, & \text{if } \; d\widetilde{\gamma}_1^{(i)} \equiv 0 \bmod 2 \end{cases}$$

Proof. By symmetry, we may focus on the case $i=1$ and compute the vanishing order of $\vartheta\left[\begin{smallmatrix}\widetilde{\gamma}_1\\ \widetilde{\gamma}_2\end{smallmatrix}\right]$ as *a function* at $q_{1,k}=0$. In the second line, we use the substitution $s=dx$, so that the summation is over all $s\in\mathbb{Z}^2$ with $s'\equiv -s'' \bmod d$. We let $\eta_i = \frac{1}{2}\widetilde{\gamma}_i$.

$$\begin{aligned}
\vartheta\left[\begin{smallmatrix}\widetilde{\gamma}_1\\ \widetilde{\gamma}_2\end{smallmatrix}\right](z,u) &= \sum_{x\in\mathfrak{o}^\vee_{d^2}} \mathbf{e}\left(\tfrac{1}{2}\mathrm{tr}_{K/\mathbb{Q}}((x+\eta_1)^2 z + 2(x+\eta_1)(u+\eta_2))\right) \\
&= \sum_{s'\in\mathbb{Z}} \mathbf{e}\left(\tfrac{1}{2}(s'+d\eta_1')^2\tfrac{z_1}{d^2} + (s'+d\eta_1')(\tfrac{u_1}{d}+\tfrac{\eta_2'}{d})\right) \\
&\quad\cdot \sum_{s''\equiv -s'(d)} \mathbf{e}\left((\tfrac{1}{2}(s''+d\eta_1'')^2\tfrac{z_2}{d^2} + (s''+d\eta_1'')(\tfrac{u_2}{d}+\tfrac{\eta_2''}{d})\right) \\
&= \mathbf{e}(\mathrm{tr}_{K/\mathbb{Q}}(\eta_1\eta_2)) \sum_{s'\in\mathbb{Z}} q_1^{1/2\cdot(s'+d\eta_1')^2} \zeta_1^{s'+d\eta_1'} \mathbf{e}(s'\tfrac{\eta_2'}{d}) \\
&\quad\cdot \sum_{s''\equiv -s'(d)} q_2^{1/2(s''+d\eta_1'')^2} \zeta_2^{s''+d\eta_1''} \mathbf{e}(s''\tfrac{\eta_2''}{d}) \\
&= \mathbf{e}(\mathrm{tr}_{K/\mathbb{Q}}(\eta_1\eta_2)) \sum_{s'\in\mathbb{Z}} q_{1,k}^{1/2\cdot(s'+d\eta_1')^2 + k(s'+d\eta_1')} \zeta_{1,k}^{1/2\cdot(s'+d\eta_1')^2+(k+1)(s'+d\eta_1')} \\
&\quad\cdot \mathbf{e}(s'\tfrac{\eta_2'}{d}) \sum_{s''\equiv -s'(d)} q_2^{1/2(s''+d\eta_1'')^2} \zeta_2^{s''+d\eta_1''} \mathbf{e}(s''\tfrac{\eta_2''}{d})
\end{aligned}$$

Note that $d\eta_1' \in \frac{1}{2}\mathbb{Z}$. We let $\varepsilon(d\eta_1') = 1$, if $d\eta_1'$ is half-integral and 0 if it is integral. In this notation, the smallest $q_{1,k}$-exponent appearing in the development of $\vartheta\left[\begin{smallmatrix}\widetilde{\gamma}_1\\ \widetilde{\gamma}_2\end{smallmatrix}\right]$ is given by

$$\begin{aligned}
&\min\left\{\tfrac{1}{2}(s+d\eta_1')^2 + k(s+d\eta_1') \mid s\in\mathbb{Z}\right\}\\
&= \tfrac{1}{2}\min_{s\in\mathbb{Z}}\left[s^2 + 2s(d\eta_1'+k) + d\eta_1'(2k+d\eta_1')\right]\\
&= \tfrac{1}{2}\min_{s\in\mathbb{Z}}\left[\left(s+d\eta_1'+k\right)^2\right] - \tfrac{1}{2}(d\eta_1'+k)^2 + \tfrac{1}{2}d\eta_1'(2k+d\eta_1')\\
&= \tfrac{1}{8}\varepsilon(d\eta_1') - \tfrac{1}{2}k^2\,.
\end{aligned}$$

This implies the claim, once we have checked that the corresponding coefficient is indeed non-zero. We may restrict to the chart $k=0$. If $\varepsilon(d\eta_1')=0$, then the minimum is attained only once for $s=-d\eta_1'$ and the coefficient is a power of $\zeta_{1,0}$-power times a non-zero power series in q_2 and ζ_2. This coefficient does not vanish for generic $(\zeta_{1,0},q_2,\zeta_2)$. If $\varepsilon(d\eta_1')=1$, then the minimum is attained twice, for $s'+d\eta_1' = \pm\frac{1}{2}$. The coefficient is of the form

$$\zeta_{1,k}^{1/8-k^2/2-k+1/2}A_1(q_2,\zeta_2) + \zeta_{1,k}^{1/8-k^2/2-k-1/2}A_1(q_2,\zeta_2)$$

for non-zero power series A_1 and A_2. This coefficient does not vanish for generic $(\zeta_{1,0},q_2,\zeta_2)$ either. □

Proof of Proposition 9.4. By Lemma 5.6 and (30), we can determine the vanishing order of a Hilbert Jacobi form near $D^{(i)}$ by its Fourier development in the coordinates $(\zeta_{1,k},q_{1,k},z_2,u_2)$, resp. $(z_1,u_1,\zeta_{2,k},q_{2,k})$, and then compare to the definition of local sections of Hilbert Jacobi forms in (30). Using this and plugging in the characteristic $\left[\begin{smallmatrix}\widetilde{\gamma}_1\\ \widetilde{\gamma}_2\end{smallmatrix}\right]$ invariant under Γ_{d^2} in the previous proposition yields the claim. □

For the proof of Proposition 9.5, let again ϑ denote the unique theta function invariant under $\widetilde{\Gamma}_{d^2}$. We develop ϑ and $\partial_2\vartheta$ with respect to the boundaries. To this end, we introduce for $i\in\mathbb{Z}$ the functions

$$\theta_{1,[i]} = \sum_{s'\,\equiv\,-i(d)} q_1^{\frac{1}{2}(s'-\frac{1}{2})^2}\zeta_1^{s'-\frac{1}{2}}\;\mathbf{e}\left(\tfrac{1}{2d}(s'+i)\right) \tag{46}$$

$$\theta_{2,[i]} = \sum_{s''\,\equiv\,-i(d)} q_2^{\frac{1}{2}(s''+\frac{1}{2})^2}\zeta_2^{s''+\frac{1}{2}}\mathbf{e}\left(\tfrac{1}{2d}(s''+i)\right)\cdot\mathbf{e}\left(\tfrac{2i-1}{4d}\right) \tag{47}$$

With the above notation, we expand ϑ and its derivative near a divisor $D^{(1)}_{\infty,k}$ lying over the first boundary $D^{(1)}$ as

$$\begin{aligned}
\vartheta &= q_{1,k}^{\frac{1}{8}-\frac{k^2}{2}}\zeta_{1,k}^{\frac{1}{8}-\frac{(k+1)^2}{2}}\cdot\Big(\quad\theta_{2,[-k]} + \quad\theta_{2,[-k+1]}\zeta_{1,k} + O(q_{1,k})\Big)\\
\partial_2\vartheta &= q_{1,k}^{\frac{1}{8}-\frac{k^2}{2}}\zeta_{1,k}^{\frac{1}{8}-\frac{(k+1)^2}{2}}\cdot\Big(\partial_{u_2}\theta_{2,[-k]} + \partial_{u_2}\theta_{2,[-k+1]}\zeta_{1,k} + O(q_{1,k})\Big)\,.
\end{aligned}\tag{48}$$

and near a divisor $D^{(2)}_{\infty,k}$ lying over the second boundary $D^{(2)}$

$$\begin{aligned}
\vartheta &= q_{2,k}^{\frac{1}{8}-\frac{k^2}{2}}\zeta_{2,k}^{\frac{1}{8}-\frac{(k+1)^2}{2}}\cdot\Big(\theta_{1,[-k-1]} + \zeta_{2,k}\theta_{1,[-k]} + O(q_{2,k})\Big)\\
\partial_2\vartheta &= q_{2,k}^{\frac{1}{8}-\frac{k^2}{2}}\zeta_{2,k}^{\frac{1}{8}-\frac{(k+1)^2}{2}}\cdot\Big(-\tfrac{1+2k}{2}\theta_{1,[-k-1]} + \tfrac{1-2k}{2}\zeta_{2,k}\theta_{1,[-k]} + O(q_{2,k})\Big)\,.
\end{aligned}\tag{49}$$

Proof of Proposition 9.5. We have to determine the boundary contribution of $\partial_2\vartheta$ on Θ, which is locally (using the chart $k=0$ and Proposition 9.7) given as the vanishing locus of $\vartheta\,/\,q_{j,0}^{1/8}\zeta_{j,0}^{1/8}$ for $j=1,2$. The factor $q_{j,0}^{1/8}\,\zeta_{j,0}^{1/8}$ gives, for both boundaries, the contribution claimed in (41). So we have to argue that the constant terms (in $q_{j,0}$) of the remaining factors of ϑ and $\partial\vartheta$ have no common factors. Since these terms are linear in $\zeta_{j,0}$, this holds if and only if

$$\det{}_1 \;=\; \begin{vmatrix} \theta_{2,[0]} & \theta_{2,[1]} \\ \partial_{u_2}\theta_{2,[0]} & \partial_{u_2}\theta_{2,[1]} \end{vmatrix} \neq 0 \quad\text{and}\quad \det{}_2 \;=\; \begin{vmatrix} \theta_{1,[0]} & \theta_{1,[1]} \\ -\frac{1}{2}\theta_{1,[0]} & \frac{1}{2}\theta_{1,[1]} \end{vmatrix} \neq 0.$$

Since

$$\begin{aligned}\theta_{2,[0]} &= \mathbf{e}\big(\tfrac{-1}{4d}\big)\; q_2^{\frac{1}{8}}\zeta_2^{\frac{1}{2}} + O\big(q_2^{\frac{1}{8}(2d-1)^2}\big) \qquad\qquad \text{and}\\ \theta_{2,[1]} &= \mathbf{e}\big(\tfrac{2d+1}{4d}\big)\, q_2^{\frac{1}{8}(2d-1)^2}\,\zeta_2^{\frac{1}{2}(-2d-1)} + O\big(q_2^{\frac{1}{8}(2d+3)^2}\big)\end{aligned}$$

the claim for $\det_1$ is easily checked using the beginning of the q_2-expansion and for $\det_2$ the claim follows similarly. □

Proof of Proposition 9.6. Suppose that $(z,u)\in\mathbb{H}^2\times\mathbb{C}^2$ projects to $N^{(1)}_{M-\mathrm{tor}}$ under the universal covering map U. This is the case iff $u_1=\frac{t_1}{d}z_1+t_2$ for some $t_1,t_2\in\frac{1}{M}\mathbb{Z}$ but there is no way to represent the point with $t_1,t_2\in\frac{1}{k}\mathbb{Z}$ for any k strictly dividing M. Such a point is mapped to $(\zeta_1,q_1)=(q_1^{t_1}\mathbf{e}(\frac{t_2}{d}),q_1)$.

Near the boundary $D^{(1)}$ we inspect the expansion (48) with this specialization. Bearing in mind that $\zeta_{2,0}\neq 0$, already to first order in $q_{1,0}$ the only solution is $q_{2,0}=0$. Such a component vanishes under π_*, as claimed.

Near the boundary $D^{(2)}$ we inspect the expansion (49). With the substitution $r'=s'-1$ we find

$$\begin{aligned}\theta_{1,[-1]}(q_1^{t_1}\mathbf{e}(\tfrac{t_2}{d}),q_1) &= \sum_{r'\equiv 0(d)} q_1^{1/2(r'-1/2)^2+t_1(-r'+1/2)}\mathbf{e}\big(\tfrac{t_2}{d}(-r'+\tfrac{1}{2})\big)(-1)^{r'/d}\\ &= \sum_{r'\equiv 0(d)} q_1^{1/2(r'-1/2)^2+t_1(-r'+1/2)}\mathbf{e}\big(\tfrac{t_2}{2d}\big)(-1)^{r'/d}\end{aligned}$$

For $t_1=t_2=0$ this expression is equal to $\theta_{1,[0]}(q_1^{t_1}\mathbf{e}(\frac{t_2}{d}),q_1)$, hence $\det_2$ vanishes at $(q_1^{t_1}\mathbf{e}(\frac{t_2}{d}),q_1)$. One checks that the next term in the expansion (corresponding to $q_{1,0}^{1}$, since $q_{1,0}^{1/8}$ has been taken out) is non-zero, so that the multiplicity of this contribution is one, as claimed. Hence this point $t_1=t_2=0$ contributes a divisor to $B_1(N)$, whose π-pushforward equals $R^{(2)}$.

The substitution works for no other pair (t_1,t_2). In fact, one checks that $\det_1(q_1^{t_1}\mathbf{e}(\frac{t_2}{d}),q_1)$ has non-trivial q_1-expansion for any non-zero (t_1,t_2). This proves the claim. □

9.4. **Modifications for d even.** Let $d>2$ be even. In this case none of the even theta characteristics in E_0 is fixed by Γ_{d^2}. The vanishing locus of the product is a well-defined subvariety of $A^\circ_{d^2}$, but using this product in the definition of the origami locus in (38) does not quite work since when taking partial derivatives, the product rule introduces a lot of spurious components.

Consequently, one has to work here with the subgroup Γ'_{d^2} of Γ_{d^2} fixing the characteristic $\left[\begin{smallmatrix}0,0\\0,0\end{smallmatrix}\right]$. In fact, the subgroup $\Gamma'_{d^2}=\mathrm{diag}((d,-d),1)\cdot\Gamma'\cdot\mathrm{diag}((\frac{1}{d},-\frac{1}{d}),1)$ where

$$\Gamma'=\big\{A\in\mathrm{SL}_2(\mathbb{Z})^2 \mid A'\equiv A''\equiv I \bmod 2,\ A'\equiv A'' \bmod 2d\big\}\subset\mathrm{SL}_2(\mathfrak{o})\,.$$

of index 48 has this property. Again one can compactify the open family $\mathbb{H}^2\times\mathbb{C}^2/\widetilde{\Gamma}'_{d^2}$, where $\widetilde{\Gamma}'_{d^2}=\Gamma'_{d^2}\ltimes(\mathfrak{o}^\vee_{d^2}\oplus\mathfrak{o}_{d^2})$ by employing a toroidal compactification for a normal subgroup – in this case $\widetilde{\Gamma}'_d(2d) = \big(\Gamma(2d)_d \ltimes (\mathbb{Z} \oplus d\mathbb{Z})\big)^2$ will do the job. Unfortunately, the resulting morphism $A'_{d^2} \to A_{d^2}$ from this new compactification A'_{d^2} is not flat at the boundary – it maps a folded 2-gon to a folded 1-gon by contracting one of the curves. One thus cannot simply pull back the relations obtained in $\mathrm{Pic}_{\mathbb{Q}}(A_{d^2})$. Instead one has to rederive the formula for the class of a Hilbert-Jacobi form (Theorem 6.1), of the section of primitive ℓ-torsion points (Proposition 6.2), and compute the vanishing orders of the theta-function and its derivative (Section 9.2).

9.5. **Intersection products and Euler characteristics.** We first convert Theorem 1.3 into a statement about Euler characteristics.

Corollary 9.8. *The Euler characteristics of the arithmetic Teichmüller curve $T_{d,M,\varepsilon}$ are as follows. If $M > 1$ is odd, then*

$$\begin{aligned} \chi(T_{d,M,\varepsilon=3}) &= -\frac{1}{144}(d-1)\Delta_d\frac{\Delta_M}{M}, \\ \chi(T_{d,M,\varepsilon=1}) &= -\frac{1}{48}(d-1)\Delta_d\frac{\Delta_M}{M} \end{aligned} \tag{50}$$

If M is even, then $\chi(T_{d,M}) = -\frac{1}{6}(d-1)\Delta_d\frac{\Delta_M}{M}$.
If $M = 1$, then

$$\begin{aligned} \chi(T_{d,M,\varepsilon=3}) &= -\frac{1}{144}(d-3)(d-5)\frac{\Delta_d}{d}, \\ \chi(T_{d,M,\varepsilon=1}) &= -\frac{1}{48}(d-1)(d-3)\frac{\Delta_d}{d}. \end{aligned} \tag{51}$$

Proof. Pairing with ω_1 and integration, as in Corollary 7.2. □

Now we complete easily the proof of the counting theorem.

Proof of Theorem 1.1. Since $\chi(\mathbb{H}/\Gamma(1)) = -1/6$, the number of squares is minus six times the Euler characteristic. (This also holds if the curve is reducible.) □

For comparison we include the proof how to deduce the total count (i.e. without separating the spin components) from two results in the literature.

Proposition 9.9 ([Kan06, Theorem 3], [EMS03])**.** *The number of minimal degree d covers of an elliptic curve E' branched over the divisor $P + Q$ is*

$$\begin{aligned} \frac{1}{3}(d-1)\Delta_d, &\qquad \text{if } P \neq Q \\ \left(\frac{1}{6}(d-1) - \frac{1}{24}\frac{1}{d}(7d-6)\right)\Delta_d, &\qquad \text{if } P = Q \end{aligned} \tag{52}$$

Corollary 9.10. *The number of square-tiled surfaces in $\Omega\mathcal{M}_2(1,1)$ of degree d and torsion order $M \geq 2$ is given by*

$$\frac{1}{3}(d-1)\Delta_d\frac{1}{2M}\Delta_M.$$

Proof. Each such surface arises as a composition of an isogeny of degree M with a minimal cover with reduced branching divisor $P + Q$. There are four choices to normalize it in such a way that $P + Q$ becomes symmetric; they correspond to the choice of a square-root of $P - Q$. After normalization, $[2]P$ is of order M. Choose a basis of $H_1(E', \mathbb{Z})$ in order to make an identification with $\mathbb{Z}^2$. Thus the M-torsion points of E' are identified with $(\mathbb{Z}/M\mathbb{Z})^2$. Since $\mathrm{SL}_2(\mathbb{Z}/M\mathbb{Z})$ acts transitively on points of order M in $(\mathbb{Z}/M\mathbb{Z})^2$, and the stabilizer of one of these is of order M, there are $\Delta_M \frac{1}{M}$ points of order M on E'. There are 4 choices of a square-root of $[2]P$, but since P is determined by the covering only up to sign, this gives in total

$$\frac{1}{4} \cdot \frac{1}{2} \cdot 4 \cdot \frac{1}{M} \Delta_M \cdot \frac{1}{3}(d-1)\Delta_d$$

square-tiled surfaces of degree d and torsion order M. □

10. Notations

We summarize the notation used for pseudo-Hilbert modular surfaces, the universal families over these surfaces and their coverings.

$$\begin{aligned} K &= \mathbb{Q} \oplus \mathbb{Q} \\ \mathfrak{o}_{d^2} &= \{x = (x', x'') \in \mathbb{Z} \oplus \mathbb{Z} : x' \equiv x'' \bmod d\} \subset K \end{aligned}$$

Modular groups and pseudo-Hilbert modular groups

$$\begin{aligned} \Gamma(\ell) &= \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/(\ell))) \quad \text{with } \ell \in \mathbb{N} \\ \Gamma^1(d) &= \left\{ A \in \mathrm{SL}_2(\mathbb{Z}) \mid A \equiv \left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right) \bmod d \right\} \\ \Gamma^1(d)^{\pm} &= \Gamma^1(d) \cup \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \Gamma^1(d) \\ \Gamma(\ell)_d &= \mathrm{diag}(d,1) \cdot \Gamma(\ell) \cdot \mathrm{diag}(d^{-1},1) \end{aligned}$$

Semidirect products

$$\begin{aligned} \widetilde{\Gamma}(\ell)_d &= \mathrm{diag}(d,1) \cdot (\Gamma(\ell) \ltimes \ell\mathbb{Z}^2) \cdot \mathrm{diag}(d^{-1},1) \\ \widetilde{\Gamma}_{d^2} &= \mathrm{SL}(\mathfrak{o}_{d^2} \oplus \mathfrak{o}_{d^2}^{\vee}) \ltimes (\mathfrak{o}_{d^2}^{\vee} \oplus \mathfrak{o}_{d^2}) \end{aligned}$$

Open modular varieties

$X(d)^\circ = \mathbb{H}/\Gamma(d)$	the open modular curve with level-d-structure
$X(d)_d^\circ = \mathbb{H}/\Gamma(d)_d$	isomorphic to $X(d)^\circ$, uniformizing group conjugated
$X_{d^2}^\circ = \mathbb{H}^2/\Gamma_{d^2}$	the open pseudo-Hilbert modular surface

Open universal families

$E(d)_d^\circ = \mathbb{H} \times \mathbb{C}/\widetilde{\Gamma}(d)_d$	universal family of elliptic curves over $X(d)_d^\circ$
$A_{d^2}^\circ = \mathbb{H}^2 \times \mathbb{C}^2/\widetilde{\Gamma}_{d^2}$	universal family of abelian surfaces over $X_{d^2}^\circ$

Their compactifications are denoted by the same letter without $^\circ$

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