The Deligne-Mumford and the Incidence Variety Compactifications of the Strata of $\Omega\mathcal{M}_g$.

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March 16, 2015

Abstract

The main goal of this work is to construct and study a reasonable compactification of the strata of the moduli space of Abelian differentials. This allows us to compute the Kodaira dimension of some strata of the moduli space of Abelian differentials. The main ingredients to study the compactifications of the strata are a version of the plumbing cylinder construction for differential forms and an extension of the parity of the connected components of the strata to the differentials on curves of compact type. We study in detail the compactifications of the hyperelliptic minimal strata and of the odd minimal stratum in genus three.

Contents

1 Introduction.

Let $\mathcal{M}_g$ be the moduli space of algebraic curves of genus g. In the early 1980s Harris and Mumford ([HM82]) proved that $\mathcal{M}_g$ is of general type for $g \geq 24$. They used in a crucial way the compactification of $\mathcal{M}_g$ proposed by Deligne and Mumford at the end of the 1960s ([DM69]). This compactification is the moduli space $\overline{\mathcal{M}}_g$ of stable algebraic curves of arithmetic genus g.

More recently, the moduli space of nonzero holomorphic differentials $\Omega\mathcal{M}_g$ and its projectivisation $\mathbb{P}\Omega\mathcal{M}_g$ have gained great interest, coming in particular from the theory of dynamical systems (see [Zor06]). The moduli space $\Omega\mathcal{M}_g$ has a natural stratification given by the orders of the zeros of the differentials. For a given tuple $(k_1, \cdots, k_n)$ of positive numbers such that $\sum k_i = 2g-2$, we define the stratum

$$\Omega\mathcal{M}_g(k_1, \cdots, k_n) := \left\{ (X, \omega) : X \in \mathcal{M}_g, \quad \operatorname{div}(\omega) = \sum_{i=1}^{n} k_i Z_i \right\},$$

and their images in $\mathbb{P}\Omega\mathcal{M}_g$ are denoted by $\mathbb{P}\Omega\mathcal{M}_g(k_1, \cdots, k_n)$. In analogy with $\mathcal{M}_g$, it is likely that a good compactification of $\mathbb{P}\Omega\mathcal{M}_g$ should help us to compute the Kodaira dimension of the strata of $\mathbb{P}\Omega\mathcal{M}_g$.

In this paper, we first introduce and study two compactifications of the strata of the moduli space of Abelian differentials. This allows us to compute the Kodaira dimension of some of these strata. The last sections are devoted to the study of the hyperelliptic minimal strata and the non hyperelliptic minimal stratum in genus three.

1.1 The incidence variety compactification.

The notion of Abelian differentials can be generalised to the case of stable curves by the notion of stable differentials. Therefore, we can prolong $\Omega\mathcal{M}_g$ above $\overline{\mathcal{M}}_g$ simply by looking at the moduli space of stable differentials $\Omega\overline{\mathcal{M}}_g$. The closure of the strata inside $\Omega\overline{\mathcal{M}}_g$ are called the *Deligne-Mumford compactifications* of these strata. The main drawback of this method is the loss of information. Indeed, a non vanishing stable differential may vanish on some irreducible components of the stable curve, losing completely the information on this component.

In order to keep track of more information, we introduce in Section 2 another compactification for the strata. Let us define the closure of the *ordered closed incidence variety* $\mathbb{P}\Omega\overline{\mathcal{M}}_{g,n}^{\mathrm{inc}}(k_1, \cdots, k_n)$ inside the moduli space of marked (semi) stable differentials by

$$\overline{\left\{ (X, \omega, Z_1, \cdots, Z_n) : (X, Z_1, \cdots, Z_n) \in \mathcal{M}_{g,n}, \quad \sum_{i=1}^{n} k_i Z_i = \operatorname{div}(\omega) \right\}}.$$

Now there is an action of a subgroup $\mathfrak{S}$ of $\mathfrak{S}_n$ permuting the zeros of same order. The *incidence variety compactification* of $\Omega\mathcal{M}_g(k_1, \cdots, k_n)$ is given by

$$\mathbb{P}\Omega\overline{\mathcal{M}}_{g,\{n\}}^{\mathrm{inc}}(k_1, \cdots, k_n) := \mathbb{P}\Omega\overline{\mathcal{M}}_{g,n}^{\mathrm{inc}}(k_1, \cdots, k_n)/S.$$

The interior of the incidence variety compactification is isomorphic to the strata $\mathbb{P}\Omega\mathcal{M}_g(k_1, \cdots, k_n)$. But we show that its closure contains in general much more information than $\mathbb{P}\Omega\overline{\mathcal{M}}_g(k_1,\cdots,k_n)$. The following theorem illustrates this point in the case of the principal stratum (see Theorem 2.6). Let us denote the projection from the incidence variety compactification to the Deligne-Mumford compactification of the principal stratum by

$$\pi : \mathbb{P}\Omega\overline{\mathcal{M}}^{\mathrm{inc}}_{g,\{2g-2\}}(1,\cdots,1) \to \mathbb{P}\Omega\overline{\mathcal{M}}_g(1,\cdots,1).$$

Theorem 1.1. *The fibre of π is positive dimensional above the locus of differentials (X,ω), where X is a reducible stable curve of genus $g \geq 2$ with two irreducible components connected by one node and ω vanishes on one component.*

In order to study the incidence variety compactification, we introduce some tools.

In Section 3, we develop the theory of *limit differentials*, which has a flavour of limit linear series. More precisely, we associate to a family of differentials a limiting object consisting of a collection of meromorphic differentials parametrised by the irreducible components of the special curve. For a given component X_i, the differential is obtained by rescaling the family in such a way that it converges on X_i (see Definition 3.2).

To construct examples of limit differentials, we extend the classical plumbing cylinder construction of curves to the case of differentials (see Lemma 3.13). In particular, this allows us to give necessary and sufficient conditions to be a limit differential for an important case (see Theorem 3.15). However, they are not sufficient in full generality and it remains unclear how to characterise general limit differentials (see nevertheless Lemma 3.20 and Lemma 3.18).

The second main ingredients are the notions of *spin structure* on (semi) stable curves and of *Arf invariant*. They allow us to generalise the notion of parity of smooth differential to some stable differentials in Section 4. In the case of curves of compact type, we associate a canonical spin structure to a stable pointed differential (see Definition 4.10). Using this notion, we show that the parity of the spin structure above the curves of compact type is invariant by deformations (see Theorem 4.12).

Theorem 1.2. *Let $n \geq 3$ and $(X,\omega,Z_1,\cdots,Z_n)$ be a differential in the closure of the stratum $\Omega\mathcal{M}^{\mathrm{inc}}_{g,\{n\}}(2l_1,\cdots,2l_n)$. Then the parity of the spin structure $\mathcal{L}_\omega$ associated to ω is ϵ if and only if $(X,\omega,Z_1,\cdots,Z_n)$ is in the closure of $\Omega\mathcal{M}^{\mathrm{inc}}_{g,\{n\}}(2l_1,\cdots,2l_n)^\epsilon$.*

The notion of spin structure does not seems to be the right one for the irreducible pointed differentials. However, in this case, we show that the Arf invariant can be generalised (see Definition 4.18) in such a way that it stays constant by deformation (see Theorem 4.19).

It would be very interesting to extend this invariant to the whole boundary of the incidence variety compactifications. But we show that, unfortunately, this invariant cannot be extend to the whole incidence variety compactification of the strata (see Corollary 1.8).

1.2 The Kodaira dimension of strata.

One of the main motivation for a good compactification of the strata of the moduli space of Abelian differentials is the computation of their Kodaira dimensions. In recent works Farkas and Verra computed the Kodaira dimension of the moduli space of spin structures and Bini, Fontonari and Viviani computed the Kodaira dimension of the universal Picard variety. They followed the path opened by Harris and Mumford for the moduli space of curves. In particular, they used in an essential way a nice compactification of these spaces constructed by Cornalba in the first case and Caporaso in the second.

A second way to compute the Kodaira dimension of algebraic spaces is to use the theory initiated by Iitaka. We can obtain information about the Kodaira dimension of the total space of an algebraic bundle using knowledge about the Kodaira dimension of the base and of a generic fibre.

Using these methods, we want to compute the Kodaira dimension of the strata S of the moduli space of Abelian differentials for which the forgetful map $\pi : S \to \mathcal{M}_g$ is generically surjective. We give a complete description of these strata and, more precisely, the dimension of the image of every connected component of each stratum.

Theorem 1.3. *Let $g \geq 2$ and S be a connected component of the stratum $\Omega\mathcal{M}_g(k_1, \cdots, k_n)$. The dimension of the projection of S by the forgetful map $\pi : \Omega\mathcal{M}_g \to \mathcal{M}_g$ is*

$$\dim(\pi(S)) = \begin{cases} 2g-1 & \text{if } S = \Omega\mathcal{M}_g(2d, 2d)^{\mathrm{hyp}} \\ 3g-4 & \text{if } S = \Omega\mathcal{M}_g(2, \cdots, 2)^{\mathrm{even}} \\ 2g-2+n & \text{if } n < g-1 \text{ and } S \neq \Omega\mathcal{M}_g(2d, 2d)^{\mathrm{hyp}} \\ 3g-3 & \text{if } n \geq g-1 \text{ and the parity of } S \text{ is not even} \end{cases}$$

Using this theorem and the fact that the Kodaira dimension of a finite cover is not smaller than the Kodaira dimension of the base, we deduce the Kodaira dimension of the strata of projective dimension $3g-3$, when $\mathcal{M}_g$ is of general type (see Corollary 5.9).

Theorem 1.4. *The connected strata $\mathbb{P}\Omega\mathcal{M}_g(k_1, \cdots, k_{g-1})$ are of general type for $g = 22$ and $g \geq 24$.*

Moreover, for a fibre space $f : X \to Y$ there is the well known inequality $\kappa(X) \leq \dim(Y) + \kappa(X_y)$ for a generic fibre X_y of f. This gives the Kodaira dimension of the strata which impose few conditions. Indeed, by showing that a generic fibre of the forgetful map has negative Kodaira dimension, we obtain the following result (see Theorem 5.10).

Theorem 1.5. *For any $g \geq 2$, let $(k_1, \cdots, k_n)$ be a tuple of positive numbers of the form $(k_1, \cdots, k_l, 1, \cdots, 1)$ with $k_i \geq 2$ for $i \leq l$ such that*

$$\sum_{i=1}^{n} k_i = 2g-2 \text{ and } \sum_{i=1}^{l} k_i \leq g-2.$$

Then the Kodaira dimension of the stratum $\mathbb{P}\Omega\mathcal{M}_g(k_1, \cdots, k_n)$ is $-\infty$.

The Iitaka conjecture has been proved by Eckart Viehweg for the fibre spaces $f : X \to Y$, where Y is of general type. So, a similar method could be used to determine the Kodaira dimension of the strata for which the forgetful map is generically surjective to $\mathcal{M}_g$, when $\mathcal{M}_g$ is of general type. However, this method is more subtle for the remaining strata and we can only prove that the strata $\mathbb{P}\Omega\mathcal{M}_g(g-1,1,\cdots,1)$ are of general type when $\mathcal{M}_g$ is of general type (see Proposition 5.13).

To conclude, we compute the Kodaira dimension of both odd (Corollary 5.17) and even (Proposition 5.15) components of the strata $\mathbb{P}\Omega\mathcal{M}_g(2,\cdots,2)$ and of the hyperelliptic component of $\mathbb{P}\Omega\mathcal{M}_g(g-1,g-1)$ (Proposition 5.14).

1.3 Examples.

We conclude this work by the explicit description of the incidence variety compactification of some strata. We focus on the minimal strata $\mathbb{P}\Omega\mathcal{M}_g(2g-2)$. In genus two, there is only one stratum $\mathbb{P}\Omega\mathcal{M}_2(2)$ and this stratum has many interpretations. For example, it can be seen as the Weierstrass divisor in $\mathcal{M}_{2,1}$ or the moduli space of even spin structures.

More generally, the hyperelliptic strata $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{hyp}}(2g-2)$ are very special and can be studied with specific tools. They are studied in Section 6 and the main result is Theorem 6.7 where we show that the fibres of the forgetful map from the incidence variety compactification of $\Omega\mathcal{M}_g^{\mathrm{hyp}}(2g-2)$ to the Weierstrass locus of hyperelliptic curves inside $\mathcal{M}_{g,1}$ are projective spaces.

To be more concrete, let us describe an important locus in the incidence variety compactification of the hyperelliptic minimal strata (see Theorem 6.9).

Theorem 1.6. *Let X be the union of a smooth curve $\widetilde{X}$ of genus $g-1$ and a projective line attached to $\widetilde{X}$ at the points N_1 and N_2.*

Then (X,ω,Z) is in the incidence variety compactification of the minimal hyperelliptic stratum $\Omega\mathcal{M}_{g,1}^{\mathrm{inc}}(2g-2)^{\mathrm{hyp}}$ if and only if the point Z is in the exceptional divisor coming from the blow-up, and the differential ω is the stable differential with a zero of order $g-2$ at both N_1 and N_2.

The first non hyperelliptic minimal stratum is $\mathbb{P}\Omega\mathcal{M}_3^{\mathrm{odd}}(4)$. The description of the boundary of this stratum gives us the opportunity to illustrate most of the tools developed in this paper.

Let us define a *generic curve in the divisor δ_i* to be a curve in the divisor δ_i with a single node. The description of the boundary of $\mathbb{P}\Omega\mathcal{M}_3^{\mathrm{odd}}(4)$ above the set of curves stably equivalent to generic curves in δ_0 and δ_1 is given in Corollary 7.9 and Corollary 7.5. For example, the pointed stable differentials in the boundary of $\mathbb{P}\Omega\mathcal{M}_{3,1}^{\mathrm{odd}}(4)$ such that the projection to $\overline{\mathcal{M}}_3$ is stably equivalent to a generic curve of the divisor δ_0 is given by the following theorem.

Theorem 1.7. *Let (X,ω,Z) be a stable pointed differential in $\mathbb{P}\Omega\overline{\mathcal{M}}_{3,1}^{\mathrm{inc}}(4)^{\mathrm{odd}}$ such that X is the union of a smooth curve $\widetilde{X}$ of genus two and a projective line which meet at two distinct points N_1 and N_2.*

Then (X,ω,Z) satisfies that $Z\in\mathbb{P}^1$, the restriction of ω to $\mathbb{P}^1$ vanishes and the restriction of ω to $\widetilde{X}$ is of one of the following two forms.

- *The restriction of ω to $\widetilde{X}$ is an holomorphic differential with a zero of order two at N_1.*
- *The restriction of ω to $\widetilde{X}$ is an holomorphic differential with two simple zeros at N_1 and N_2.*

This theorem together with Theorem 1.6 implies that the incidence variety compactifications of the hyperelliptic and odd connected components of $\mathbb{P}\Omega\mathcal{M}_{3,1}(4)$ intersect each other (see Corollary 7.10).

Corollary 1.8. *Let X be the union of a curve $\widetilde{X}$ of genus two and a projective line glue together at a pair of points of $\widetilde{X}$ conjugated by the hyperelliptic involution. Let $Z \in E$ and ω be a differential which vanishes on E and has two single zeros at the points which form the nodes on $\widetilde{X}$.*

Then the pointed differential (X, ω, Z) is in $\Omega\overline{\mathcal{M}}_{3,1}^{\mathrm{inc}}(4)^{\mathrm{hyp}}$ and $\Omega\overline{\mathcal{M}}_{3,1}^{\mathrm{inc}}(4)^{\mathrm{odd}}$.

2 The Incidence Variety Compactification of the Strata of the Moduli Space of Differentials.

The projectivisation of the Hodge bundle over the moduli space of curves $\mathbb{P}\Omega\mathcal{M}_g$ has a natural compactification given by the moduli space of stable differentials $\mathbb{P}\Omega\overline{\mathcal{M}}_g$. The first idea in order to compactify a stratum is to take its closure inside $\mathbb{P}\Omega\overline{\mathcal{M}}_g$. This is called the *Deligne-Mumford compactification* of the stratum. However, this compactification loses lots of information. To keep track of more information we introduce in Definition 2.2 another compactification $\mathbb{P}\Omega\overline{\mathcal{M}}_{g,\{n\}}^{\mathrm{inc}}(k_1, \cdots, k_n)$ via the closure of the strata inside the moduli space of marked differentials. This compactification of the strata will be called the *incidence variety compactification* of the stratum. The end of this section is devoted to the study of the spaces $\mathbb{P}\Omega\overline{\mathcal{M}}_{g,\{n\}}^{\mathrm{inc}}(k_1, \cdots, k_n)$. We show in Theorem 2.5 and Theorem 2.6 that this compactification contains much more information at the boundary than the one given by the closure inside $\mathbb{P}\Omega\overline{\mathcal{M}}_g$.

In this section, all spaces we consider will be complex orbifolds.

Background on moduli spaces. We begin this section by recalling some basic facts and notations about various moduli spaces. The *moduli space of curves of genus g*, denoted by $\mathcal{M}_g$, is the space of complex structures on a curve of genus g. The *moduli space of n-pointed curves* is denoted by $\mathcal{M}_{g,n}$. It is well known since Riemann (see for example [GH94]) that the dimension of $\mathcal{M}_{g,n}$ is $3g - 3 + n$.

A modular compactification of $\mathcal{M}_{g,n}$ is given by the *moduli space $\overline{\mathcal{M}}_{g,n}$ of n-marked stable curves.* This compactification is called the *Deligne-Mumford compactification* of the moduli space of n-marked curves. Recall that a *stable curve* is a connected nodal curve for which each irreducible component of the normalisation has not an Abelian fundamental group. The *dual graph* of a stable curve X of genus g, denoted by $\Gamma_{\mathrm{dual}}(X)$, is the weighted graph such that the vertices correspond to the irreducible components of X, the edges correspond to its nodes and the weight at a vertex is given by the geometric genus of the corresponding component.

The *moduli space of nonzero holomorphic 1-forms* $\Omega\mathcal{M}_g$ or *Hodge bundle of* $\mathcal{M}_g$ parameterises pairs (X, ω), where X is a smooth curve of genus g and ω is a nonzero holomorphic 1-form on X. Remark that the space $\Omega\mathcal{M}_g$ is sometimes denoted by $\mathcal{H}_g$ in the literature (for example [Zor06], [EMZ03],...). We will never use this notation due to the risk of confusion with the notation of the hyperelliptic locus inside $\mathcal{M}_g$ (see Section 6).

The space $\Omega\mathcal{M}_g$ has a natural stratification by the multiplicities of zeros of ω. Let $(k_1, \cdots, k_n)$ be a n-tuple of strictly positive numbers such that $\sum_{i=1}^{n} k_i = 2g - 2$. The stratum $\Omega\mathcal{M}_g(k_1, \cdots, k_n)$ is the subspace of $\Omega\mathcal{M}_g$ consisting of equivalence pairs (X, ω), where ω has n distinct zeros of respective orders $(k_1, \cdots, k_n)$. In particular, for $g \geq 2$ the following decomposition holds (see for example [Zor06]):

$$\Omega\mathcal{M}_g = \coprod_{\substack{n \in \{1, \cdots, 2g-2\} \\ 2g-2 \geq k_1 \geq \cdots \geq k_n \geq 1, \ \sum k_i = 2g-2}} \Omega\mathcal{M}_g(k_1, \cdots, k_n). \tag{1}$$

The notion of differentials extends to the case of (semi) stable curves in the following way. A *stable differential* on a stable curve X is a meromorphic 1-form ω on X which is holomorphic outside of the nodes of X and has at worst simple poles at the nodes and the two residues at a node are opposite. Alternatively, the stable differentials could be defined as the global sections of the *dualizing sheaf* ω_X of X (see [HM98]). We can now extend the Hodge bundle $\Omega\overline{\mathcal{M}}_g$ above $\overline{\mathcal{M}}_g$. The space $\Omega\overline{\mathcal{M}}_g$ is the *moduli space of stable differentials* of genus g.

Since the definition of stable differential extends readily to the case of semi stable curves, we can extend this notion to the case of stable marked curves.

Definition 2.1. A *marked stable differential* $(X, \omega, Q_1, \cdots, Q_n)$ *of genus* g is the datum of a stable n-marked curve $(X, Q_1, \cdots, Q_n)$ in $\overline{\mathcal{M}}_{g,n}$ and a stable differential ω on X.

The *moduli space of marked stable differentials* will be denoted by $\Omega\overline{\mathcal{M}}_{g,n}$. It is the Hodge bundle above the moduli space of marked curves $\overline{\mathcal{M}}_{g,n}$. Its restriction to the locus of smooth n-marked curves is the *moduli space of n-marked Abelian differentials* and is denoted by $\Omega\mathcal{M}_{g,n}$.

There is a natural $\mathbb{C}^*$-action on the moduli space of Abelian differentials given by

$$\mathbb{C}^* \times \Omega\mathcal{M}_g \to \Omega\mathcal{M}_g : (\alpha, (X, \omega)) \mapsto (X, \alpha\omega). \tag{2}$$

The quotient of $\Omega\mathcal{M}_g$ under this action is denoted by $\mathbb{P}\Omega\mathcal{M}_g$. Remark that this action preserves the stratification of $\Omega\mathcal{M}_g$ and the images of $\Omega\mathcal{M}_g(k_1, \cdots, k_n)$ inside $\mathbb{P}\Omega\mathcal{M}_g$ are well defined and are denoted by $\mathbb{P}\Omega\mathcal{M}_g(k_1, \cdots, k_n)$. Moreover, the group $\mathbb{C}^*$ acts in a similar way on $\Omega\mathcal{M}_{g,n}$ and we denote the quotient under this action by $\mathbb{P}\Omega\mathcal{M}_{g,n}$.

The Incidence variety compactification of the strata of $\Omega\mathcal{M}_g$. In order to compactify the strata of $\Omega\mathcal{M}_g$, we define the *ordered incidence variety* $\mathbb{P}\Omega\mathcal{M}^{\mathrm{inc}}_{g,n}(k_1, \cdots, k_n)$ to be the subspace of the moduli space of n-marked differentials given by

$$\left\{ (X, \omega, Z_1, \cdots, Z_n) : \ \mathrm{div}\,(\omega) = \sum_{i=1}^{n} k_i Z_i \right\} \subset \mathbb{P}\Omega\mathcal{M}_{g,n}. \tag{3}$$

Moreover, the *closed ordered incidence variety*, denoted by $\mathbb{P}\Omega\overline{\mathcal{M}}^{\mathrm{inc}}_{g,n}(k_1, \cdots, k_n)$, is defined as the closure of the ordered incidence variety inside $\mathbb{P}\Omega\overline{\mathcal{M}}_{g,n}$.

In general, there exists a subgroup of $\mathfrak{S}_n$ acting non-trivially on the closed ordered incidence variety $\mathbb{P}\Omega\overline{\mathcal{M}}^{\mathrm{inc}}_{g,n}(k_1, \cdots, k_n)$. Namely, if $k_i = k_j$ for $i \neq j$, then the transposition (i, j) acts on $\mathbb{P}\Omega\overline{\mathcal{M}}^{\mathrm{inc}}_{g,n}(k_1, \cdots, k_n)$ by permuting the points Z_i and Z_j. Let $\mathfrak{S}$ be the subgroup of $\mathfrak{S}_n$ generated by these transpositions. It is easy to see that $\mathfrak{S} \cong \prod \mathfrak{S}_{l_i}$, where $l_i := \#\{j | k_j = i\}$ is the number of indices j such that the order k_j is equal to i.

Definition 2.2. Let $\Omega\mathcal{M}_g(k_1,\cdots,k_n)$ be a stratum of $\Omega\mathcal{M}_g$ and let S be one of its connected components. The *incidence variety compactification of* S is

$$\mathbb{P}\Omega\overline{\mathcal{M}}^{\mathrm{inc}}_{g,\{n\}}(k_1,\cdots,k_n) := \mathbb{P}\Omega\overline{\mathcal{M}}^{\mathrm{inc}}_{g,n}(k_1,\cdots,k_n)/\mathfrak{S}. \tag{4}$$

A triple $(X,\omega,Z_1,\cdots,Z_n)\in\mathbb{P}\Omega\overline{\mathcal{M}}^{\mathrm{inc}}_{g,\{n\}}(k_1,\cdots,k_n)$ will be called a *pointed differential* or a *pointed flat surface.*

Remark that the notions of pointed differentials and marked differentials (see Definition 2.1) do not coincide.

Let us remark that the closed ordered incidence variety is a suborbifold of $\mathbb{P}\Omega\overline{\mathcal{M}}_{g,n}$. Therefore the incidence variety compactification of every stratum is an orbifold as the quotient of an orbifold by a finite group.

The forgetful map. There is a natural *forgetful map* between the incidence variety compactification and the corresponding stratum. Before defining this map on the whole compactification, we restrict ourself to its restriction above the smooth pointed differentials. This restriction is given by

$$\begin{aligned}\varphi : \mathbb{P}\Omega\mathcal{M}^{\mathrm{inc}}_{g,\{n\}}(k_1,\cdots,k_n) &\to \mathbb{P}\Omega\mathcal{M}_g(k_1,\cdots,k_n)\\ (X,\omega,Z_1,\cdots,Z_n) &\mapsto (X,\omega).\end{aligned}$$

This map turns out to be an isomorphism.

Lemma 2.3. *The forgetful map*

$$\varphi : \mathbb{P}\Omega\mathcal{M}^{\mathrm{inc}}_{g,\{n\}}(k_1,\cdots,k_n) \to \mathbb{P}\Omega\mathcal{M}_g(k_1,\cdots,k_n) \tag{5}$$

is an isomorphism of orbifolds.

In particular, this lemma clearly implies that the dimension of the incidence variety compactification $\mathbb{P}\Omega\overline{\mathcal{M}}^{\mathrm{inc}}_{g,\{n\}}(k_1,\cdots,k_n)$ is $2g-2+n$.

Proof. It suffices to show that there exists an inverse ψ to φ. Let (X,ω) be a smooth differential with zeros of order $(k_1,\cdots,k_n)$. We denote by $Z_1,\cdots,Z_n$ the corresponding zeros.

Let us define the map

$$\begin{aligned}\widetilde{\psi} : \mathbb{P}\Omega\mathcal{M}_g(k_1,\cdots,k_n) &\to \mathbb{P}\Omega\mathcal{M}^{\mathrm{inc}}_{g,n}(k_1,\cdots,k_n)\\ (X,\omega) &\mapsto (X,\omega,Z_1,\cdots,Z_n).\end{aligned}$$

We define ψ by the composition of $\widetilde{\psi}$ with the quotient by the action of $\mathfrak{S}$. It is a routine to prove that both maps are inverse to each other. $\square$

We extend the map $\varphi : \mathbb{P}\Omega\mathcal{M}^{\mathrm{inc}}_{g,\{n\}}(k_1,\cdots,k_n) \to \mathbb{P}\Omega\mathcal{M}_g(k_1,\cdots,k_n)$ at the boundary of the strata. Let $(X',\omega',Z_1,\cdots,Z_n)\in\mathbb{P}\Omega\overline{\mathcal{M}}^{\mathrm{inc}}_{g,\{n\}}(k_1,\cdots,k_n)$ be a pointed differential. We denote by X the image of X' by the forgetful map $\pi:\overline{\mathcal{M}}_{g,n}\to\overline{\mathcal{M}}_g$. Moreover, for every irreducible component X_i of X, the corresponding irreducible component of X' is denoted by X_i'. We obtain a differential ω on X in the following way. The restriction of ω on every irreducible component X_i of X is the differential $\omega'|_{X_i'}$.

This is clearly an extension of the forgetful map φ, and it remains to show that the image of this extension lies in $\mathbb{P}\Omega\overline{\mathcal{M}}_g$.

Lemma 2.4. *The forgetful map*

$$\varphi : \mathbb{P}\Omega\overline{\mathcal{M}}^{\mathrm{inc}}_{g,\{n\}}(k_1, \cdots, k_n) \to \mathbb{P}\Omega\overline{\mathcal{M}}_g(k_1, \cdots, k_n)$$
$$(X', \omega', Z_1, \cdots, Z_n) \mapsto (X, \omega)$$

described in the preceding paragraph is well defined. More precisely, the pair (X, ω) is a stable differential.

Proof. The forgetful map $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_g$ is well defined. Hence it is enough to show that the differential ω is stable.

Let E be an exceptional component of X' and X'_E be the curve obtained from X' by blowing down E. We denote by ω'_E the restriction of the form ω' on X'_E. We can suppose that the nodal points of E are 0 and ∞.

Then the restriction of the form ω' on E is either zero or of the differential $\frac{a\mathrm{d}z}{z}$ for some $a \in \mathbb{C}^*$.

If the restriction is zero, then it is clear that ω'_E is stable on X'_E.

If the differential on E is given $\frac{a\mathrm{d}z}{z}$. Since the residues of the differential at the nodes have to sum up to zero, the residues at the points of the node are $-a$ and a. This implies that ω'_E is still a stable differential on X'_E.

If we take E_1 and E_2 two exceptional components, we can easily verify that $X'_{E_1E_2} = X'_{E_2E_1}$ and $\omega_{E_1E_2} = \omega_{E_2E_1}$. So by induction on the set of exceptional components, the limit ω is a well defined stable differential on X. And it is easy to check that this limit coincides with the form given by the map φ. $\square$

Closure of the principal stratum. In this paragraph, we show that the incidence variety compactification contains much more information than the Deligne-Mumford compactification at the boundary of the principal stratum. This part uses the results of Section 3 and in particular Theorem 3.15.

Theorem 2.5. *Let $(X, \omega) \in \mathbb{P}\Omega\mathcal{M}_g(2g-2)$ be a differential in the minimal stratum. This differential is in the boundary of principal stratum $\mathbb{P}\Omega\mathcal{M}_g(1, \cdots, 1)$ and the dimension of the fibre of the forgetful map*

$$\pi : \mathbb{P}\Omega\overline{\mathcal{M}}^{\mathrm{inc}}_{g,\{2g-2\}}(1, \cdots 1) \to \mathbb{P}\Omega\overline{\mathcal{M}}_g(1, \cdots, 1)$$

above (X, ω) is $\max(0, 2g-4)$.

Proof. Let $(X, \omega, Z) \in \mathbb{P}\Omega\mathcal{M}^{\mathrm{inc}}_{g,1}(2g-2)$ be a pointed differential of genus g and $(\mathbb{P}^1, Q_1, \cdots, Q_{2g-2}, P)$ be a marked rational curve. There exists a meromorphic differential with a single zero at all the Q_i and a pole of order $2g$ at P. Indeed, this differential is given up to scalar multiplication by

$$\eta := \frac{\prod_i (z - Q_i)}{(z-P)^{2g}} \mathrm{d}z.$$

Let us glue the curve X with this rational curve via the identification of Z with P. It is easy to verify that we can apply Theorem 3.15 in order smooth this differential. The differential that we obtain has $2g-2$ simple zeros. This shows that the pointed differential

$$\left(X \cup \mathbb{P}^1/Z \sim P, (\omega, 0), Q_1, \cdots, Q_{2g-2}\right)$$

is an element of $\mathbb{P}\Omega\overline{\mathcal{M}}^{\mathrm{inc}}_{g,\{2g-2\}}(1, \cdots, 1)$ for any tuple $(Q_1, \cdots, Q_{2g-2}, P)$. A simple dimension count concludes the proof. $\square$

We now prove an analogous result for the absolute boundary of the stratum $\mathbb{P}\Omega\overline{\mathcal{M}}^{\mathrm{inc}}_{g,\{2g-2\}}(1,\cdots 1)$ for curves in the divisor δ_i, for $i \geq 2$.

Theorem 2.6. *The fibre of the forgetful map*

$$\pi : \mathbb{P}\Omega\overline{\mathcal{M}}^{\mathrm{inc}}_{g,\{2g-2\}}(1,\cdots 1) \to \mathbb{P}\Omega\overline{\mathcal{M}}_g(1,\cdots,1)$$

is positive dimensional over a differential (X,ω), where X is a generic curve in δ_i for $i \geq 1$ and ω vanishes on one component of X.

Proof. Let $(X := X_1 \cup X_2/N_1 \sim N_2, \omega)$ be a differential of genus $g \geq 2$ in $\mathbb{P}\Omega\overline{\mathcal{M}}_g(1,\cdots,1)$ and suppose that $\omega|_{X_1} = 0$. Then, the component X_1 contains more than $2g_1 - 2$ marked points. The map $h : X_1^{(k)} \to \mathcal{J}(X_1)$ from the symmetric product of X_1 to the Jacobian of X_1 given by

$$(Q_1,\cdots,Q_k) \mapsto \mathcal{O}_{X_1}\left(\sum_i Q_i - (k - 2g_1 + 2)N_1\right)$$

is surjective. Hence the dimension of the fibre of π at (X,ω) is at least $k - g_1$. Such divisors are canonical and since there is no residue at N_1, we apply Theorem 3.15 to conclude that every such differential can be smoothed in $\mathbb{P}\Omega\mathcal{M}_g(1,\cdots,1)$. $\square$

We are going to present some other results about the closure of the minimal hyperelliptic strata in Section 6 and of the closure of $\mathbb{P}\Omega\mathcal{M}^{\mathrm{odd}}_{3,1}(4)$ in Section 7.

3 Limit Differentials and Plumbing Cylinders.

In order to remedy the disadvantage of stable differentials that may vanish on some components, we introduce the notion of *limit differential.* It is, in a sense, similar to the notion of limit linear series, but for families of pointed differentials in a stratum (see Definition 3.2). In particular, a limit differential is a collection of differentials parametrised by the set of irreducible components of a marked curve (such collection will be called *candidate differential*). None differential of this collection identically vanish, but the price to pay is to allow some poles of order greater than one at the nodes.

This notion is interesting only if the following conditions are satisfied. First, this notion should be manageable, at least for important cases. In particular, we should be able to exhibit limit differentials. To produce examples, we extend the classical plumbing cylinder construction from the case of curves to the case of differentials (see Lemma 3.13). This allows us to give necessary and sufficient conditions for being a limit differential in an important case (see Theorem 3.15). Two of these conditions are easily stated: a limit differential (X,ω) must satisfy the *compatibility condition* at every node N_i of X

$$\mathrm{ord}_{N_{i,1}}(\omega) + \mathrm{ord}_{N_{i,2}}(\omega) = -2,$$

and the *residue condition* at every node where ω has simple poles

$$\mathrm{Res}_{N_{i,1}}(\omega) + \mathrm{Res}_{N_{i,2}}(\omega) = 0.$$

We refer to Theorem 3.15 for the other conditions. The general case is much more complicated, and we show that the above conditions are not sufficient. However, see Lemma 3.18 for a necessary condition and Lemma 3.20 for a sufficient one (both of them being non-optimal).

Second, it should be possible to deduce information on the incidence variety compactification of the strata of $\Omega\mathcal{M}_g$ from the limit differentials. We connect the two notions for important cases in Proposition 3.23. We will use this relationship intensively in Section 6 and Section 7.

Limit Differentials. Before defining the notion of limit differential, we prove a preliminary result about families of pointed differentials. This allows us to introduce the notion of *scaling*.

Lemma 3.1. *Let*

$$\left(f:\mathscr{X}\to\Delta^*,\mathscr{W}:\Delta^*\to\omega_{\mathscr{X}/\Delta^*},\mathscr{Z}_1,\cdots,\mathscr{Z}_n:\Delta^*\to\mathscr{X}\right)$$

be a family of pointed differentials inside the stratum $\Omega\overline{\mathcal{M}}_{g,n}^{\mathrm{inc}}(k_1,\cdots,k_n)$ *and let* $(X,\omega,Z_1,\cdots,Z_n)$ *be its stable limit. Then, for every irreducible component* X_i *of* X *there exists a unique* $r_i\in\mathbb{Z}$ *such that for a generic section* $s:\Delta^*\to\mathscr{X}$ *with* $\bar{s}(0)\in X_i$ *we have*

$$\lim_{t\to 0} t^{r_i}\mathscr{W}(t,s(t))\neq 0. \tag{6}$$

Moreover, every map $\alpha_i:\Delta\to\mathbb{C}$ *satisfying this property is given by*

$$t^{r_i}(1+t\mathbb{C}[t]).$$

The map t^{r_i} *is called the* scaling *of the component* X_i *for this family.*

The stable limit of the family of differentials is given by

$$\lim_{t\to 0}\left(\alpha(t)\mathscr{W}(t)\right),$$

where α *is a scaling such that for every scaling* α_i *the quotient* α/α_i *is bounded at the origin.*

Let X be a (semi) stable curve, we denote by $\mathfrak{Irr}(X)$ the set of irreducible components of X.

Proof. Let us define the meromorphic map

$$h:\mathscr{X}\to\mathbb{C},(t,x)\mapsto\mathscr{W}(t,x),$$

where $\mathscr{W}$ is seen as a section of $\mathcal{O}_{\mathscr{X}}\left(\sum k_i\mathscr{Z}_i\right)$. In particular, the map h is of the form $h(x,t)=h(t)$, where h is never vanishing. Let us denote its meromorphic continuation on $\bar{\mathscr{X}}$ by $\bar{h}$. The divisor of $\bar{h}$ is of the form

$$\operatorname{div}\left(\bar{h}\right)=\sum_{X_i\in\mathfrak{Irr}(X)} l_iX_i,$$

where $l_i\in\mathbb{Z}$. This implies that $\alpha_i:=t^{-l_i}$ is a scaling for X_i. The uniqueness and the general description of the map α_i having this property clearly follows from this description.

Now, let α be a map such that $\lim_{t\to 0}(\alpha(t)\mathscr{W}(t))$ is a non vanishing stable differential and α_i be the scaling of any component of X. By definition α is the scaling of some component of X. Let us show that the quotient $\frac{\alpha}{\alpha_i}$ is bounded in a neighbourhood of 0. For any section $s : \Delta^* \to \mathscr{X}$ we have the equality

$$\alpha(x)\mathscr{W}(t, s(t)) = \frac{\alpha(t)}{\alpha_i(t)}\alpha_i(t)\mathscr{W}(t, s(t)).$$

Hence, if $\frac{\alpha}{\alpha_i}$ is not bounded at the origin, then the limit is not bounded on the smooth locus of X_i. In particular, the limit is not a stable differential. □

Now we introduce the notion of limit differential of a family of pointed differentials.

Definition 3.2. A *limit differential* $(X, \omega, Z_1, \cdots, Z_n)$ *of type* $(k_1, \cdots, k_n)$ is a tuple such that there exists a family of pointed differentials

$$\left(f : \mathscr{X} \to \Delta^*, \mathscr{W} : \Delta^* \to \omega_{\mathscr{X}/\Delta^*}, \mathscr{Z}_1, \cdots, \mathscr{Z}_n : \Delta^* \to \mathscr{X}\right)$$

inside $\Omega\mathcal{M}^{\mathrm{inc}}_{g,\{n\}}(k_1, \cdots, k_n)$ which satisfies the two following properties.

First, the marked curve $(X, Z_1, \cdots, Z_n)$ is the stable limit of the family $(\mathscr{X}, \mathscr{Z}_1, \cdots, \mathscr{Z}_n)$.

Second, for every irreducible component X_i of the curve X and for every section $s : \Delta^* \to \mathscr{X}$, we have

$$\lim_{t\to 0}\alpha_i(t)\mathscr{W}(t, s(t)) = \omega(\bar{s}(0)),$$

where α_i is the scaling of X_i.

The set of limit differentials of type $(k_1, \cdots, k_n)$ modulo the usual action of the group $\mathfrak{S} \subset \mathfrak{S}_n$ (see Section 2) is denoted by $K\overline{\mathcal{M}}^{\mathrm{lim}}_{g,\{n\}}(k_1, \cdots, k_n)$.

A limit differential is a collection of never identically zero meromorphic differentials parametrised by the set of irreducible components of a stable marked curve. Moreover, the sum of the smooth parts of the divisors of these differentials is given by $\sum k_i Z_i$. In order to avoid confusion with the stable pointed differentials, we call such objects *candidate differentials* of type $(k_1, \cdots, k_n)$.

We now give necessary conditions for a candidate differential to be a limit differential.

Lemma 3.3. *Let* $(X, \omega, Z_1, \cdots, Z_n)$ *be a limit differential and* $N_1 \sim N_2$ *be a node of the curve* X*. Then the differential* ω *satisfies the* Compatibility Condition

$$\mathrm{ord}_{N_1}(\omega) + \mathrm{ord}_{N_2}(\omega) = -2. \tag{7}$$

Moreover, if the orders of ω *at* N_1 *and* N_2 *are* -1*, then the differential* ω *satisfies the* Residue Condition

$$\mathrm{Res}_{N_1}(\omega) + \mathrm{Res}_{N_2}(\omega) = 0. \tag{8}$$

Proof. Let $(f : \mathscr{X} \to \Delta^*, \mathscr{W}, \mathscr{Z}_1, \cdots, \mathscr{Z}_n)$ be a family of pointed differentials which converges to the limit differential $(X, \omega, Z_1, \cdots, Z_n)$. Let U be a neighbourhood of the node $N_1 \sim N_2$ in $\bar{\mathscr{X}}$. Without loss of generality, we can assume that U satisfies the following properties. First, the intersections $\mathscr{Z}_i \cap U$

are empty for every $i \in \{1, \cdots, n\}$. In particular, the only possible zeros and poles of $\mathscr{W}|_U$ are contained in $X|_U$. Second, there exists a coordinate system (x, y, t) of an open subset of Δ^3 containing the origin such that

$$U := \{xy = t^a\}, \tag{9}$$

where $a \geq 1$. Moreover, we can suppose that $X|_U$ is given by the equation $\{xy = 0\}$. In the rest of the proof, we denote by X_x, X_y and X_U the subset of U of respective equations $\{y = 0\}$, $\{x = 0\}$ and $\{xy = 0\}$.

We pick a differential η that generates $\Omega^1_U / f^*(\Omega^1_\Delta)$ and that vanishes nowhere on U, for example

$$\eta := \frac{x\mathrm{d}x - y\mathrm{d}y}{x^2 + y^2}.$$

For $t \neq 0$, its restriction to the curve $\mathscr{X}_t$ is a differential without zeros or poles. For $t = 0$, its restriction to the component X_x (resp. X_y) has a unique simple pole at N_1 (resp. N_2) with residue 1 (resp. -1).

Since η generates $\Omega^1_U / f^*(\Omega^1_\Delta)$, the family of differentials $\mathscr{W}|_{U \setminus X_U}$ is given by

$$\mathscr{W} = h \cdot \eta,$$

where h is a meromorphic function with neither poles nor zeros in $U \setminus X_U$. By multiplying the function h by a power of t^a, we obtain a new family of differentials proportional to $\mathscr{W}$ on $U \setminus X_U$. In particular, we can suppose that h is holomorphic on U and vanishes on at most one component of X_U. This new family will still be denoted by $\mathscr{W}$ and the holomorphic function by h.

We have two cases to consider. The first one is the case where h is invertible on U. In this case the limit differential of $\mathscr{W}$ on X_U is simply a scaling of the restriction of η on X_U. Hence the residues of ω at N_1 and N_2 are respectively $h(0)$ and $-h(0)$. In particular, in this case, both the compatibility and the residue conditions are satisfied.

The second case is where h vanishes on one component. Without loss of generality, we can suppose that $h|_{X_y} \equiv 0$ and $h|_{X_x} \not\equiv 0$. By the Weierstrass preparation theorem, the function h can be written as

$$h(x, y) = \left(x^d + h_1(y)x^{d-1} + \cdots + h_d(y)\right) \widetilde{h}(x, y), \tag{10}$$

where $\widetilde{h}$ is invertible and the h_i are holomorphic maps vanishing at the origin. Moreover, since by hypothesis the divisor of h is a multiple of X_y, we deduce that the functions h_i are identically zero. Hence the function h is of the form

$$h(x, y) = x^d \cdot \widetilde{h}(x, y). \tag{11}$$

This implies that restriction ω_x of ω to the component X_x is given by

$$\left. \left(x^d \cdot \widetilde{h}(x, y) \cdot \frac{x\mathrm{d}x - y\mathrm{d}y}{x^2 + y^2} \right) \right|_{X_x} = x^d \cdot \widetilde{h}(x, 0) \frac{\mathrm{d}x}{x}.$$

By rescaling the family of differentials $\mathscr{W}$ by the function $(t^a)^{-d}$, we find that the restriction ω_y of ω to the component X_y is given by

$$y^{-d} \cdot \widetilde{h}(0, y) \frac{-\mathrm{d}y}{y}.$$

In particular, since $\widetilde{h}(0, 0) \in \mathbb{C}^*$, the sum of the orders of ω_x and ω_y at the origin is -2. $\square$

It is convenient to formulate a byproduct of our proof as a separate lemma.

Lemma 3.4. *Let $(\mathscr{X}, \mathscr{W}, \mathscr{Z}_1, \cdots, \mathscr{Z}_n)$ be a family of pointed differentials which converges to the limit differential $(X, \omega, Z_1, \cdots, Z_n)$. Let N be a node between the irreducible components X_i and X_j (which may coincide), and suppose that the equation of $\mathscr{X}$ around N is $xy = t^a$ for some $a \geq 1$.*

If $\mathrm{ord}_N(\omega|_{X_i}) = k \geq -1$, then the scaling α_i and α_j of X_i and X_j satisfy the equality

$$\frac{\alpha_i}{\alpha_j} = (t^a)^{k+1}. \tag{12}$$

As an application we can prove that a limit differential $(X, \omega, Z_1, \cdots, Z_n)$ is uniquely determined up to multiplicative constants by $(X, Z_1, \cdots, Z_n)$.

Corollary 3.5. *Let $(X, Z_1, \cdots, Z_n)$ be a marked curve in the image of the incidence variety compactification $\Omega\overline{\mathcal{M}}^{\mathrm{inc}}_{g,\{n\}}(k_1, \cdots, k_n)$ by the forgetful map. Then there exists a limit differential on $(X, Z_1, \cdots, Z_n)$ of type $(k_1, \cdots, k_n)$. Moreover for any two of such limit differentials ω and ω' there exist constants $c_i \in \mathbb{C}^*$ such that*

$$\left.\frac{\omega}{\omega'}\right|_{X_i} = c_i,$$

for every irreducible component X_i of X.

Proof. Let X_i be an irreducible component of X which corresponds to a leaf of the dual graph of X. Let $Z_{i,1}, \cdots, Z_{i,n_i}$ be the marked points in X_i. Then the restriction of ω to X_i has zeros of order $k_{i,j}$ at $Z_{i,j}$ and at most one other zero or a unique pole which has to be located at the node of X_1. Moreover, the order at the node is imposed by the fact that the degree of $\omega|_{X_i}$ is $2g_i - 2$. Hence $\omega|_{X_i}$ is uniquely determined up to a multiplicative constant.

Now we continue this process on the irreducible components adjacent to the preceding components. The order at the nodes with the previous components are determined by the compatibility condition (7) and the order at the marked points Z_l are k_l. Hence it follows that the order at the last node is imposed by the condition on the degree of ω.

Iterating this process we show that there is at most one limit differential (up to multiplication) on $(X, Z_1, \cdots, Z_n)$. And since $(X, Z_1, \cdots, Z_n)$ lies in the projection of $\Omega\overline{\mathcal{M}}^{\mathrm{inc}}_{g,\{n\}}(k_1, \cdots, k_n)$, there exists at least one limit differential on this curve. □

There is a global obstruction to smooth a candidate differential which satisfies the compatibility condition and the residue condition.Let us look first at a very simple example.

Example 3.6. Let X be irreducible with one node and the differential ω has a zero of order k and a pole of order $k+2$ at the node. It follows from Lemma 3.4 that the differential cannot be smoothed. Indeed, the scaling of an irreducible component is unique for a given family of differentials. But in this case, by Lemma 3.4, the scaling α of X satisfies $\frac{\alpha}{\alpha} = (t^a)^{k+1}$ for an $a \geq 1$, which is absurd.

Let us now introduce some definitions.

Definition 3.7. Let (X, ω) be a candidate differential and N a node of X. The *order* of N relatively to ω is

$$\mathrm{ord}(N) := \max_{i=1,2}(\mathrm{ord}_{N_i}(\omega)).$$

Definition 3.8. Let $(X, \omega, Z_1, \cdots, Z_n)$ be a candidate differential. The *dual graph* Γ_ω of (X, ω) is the partially directed weighted graph given by the following data.

- The graph coincides with the dual graph of X.
- An edge is directed from the component with the zero to the component with the pole of ω and no orientation in the case of simple poles.
- The weight $w(e)$ of an edge e is one greater than the order of the corresponding node (see Definition 3.7).

Example 3.9. The graphs of the curves of Example 3.6 and Example 3.17 are drawn in Figure 1.

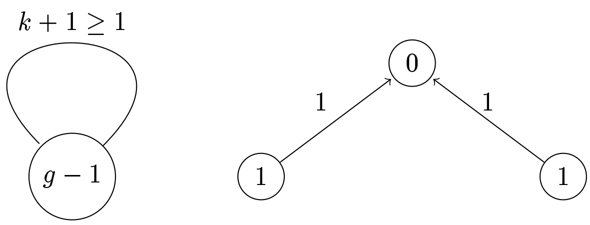


Figure 1

Definition 3.10. Let Γ be an partially oriented graph. A *path* γ is a finite continuous sequence of pairs $\{(e_i, \alpha_i)\}_{i\in\{1,\cdots,l\}}$, where e_i is an edge of Γ and $\alpha_i \in \{0, \pm 1\}$ is 0 if the edge has no orientation, 1 if the direction coincide with the orientation of e_i and -1 otherwise. Such path γ will be denoted by

$$\gamma := \sum_{i=1}^{l} \alpha_i e_i.$$

We now give another property which is satisfied by the limit differentials. Let us recall that $\mathcal{N}_X$ denotes the set of nodes of a curve X.

Lemma 3.11. *Let $(X, \omega, Z_1, \cdots, Z_n)$ be a limit differential. There exists a tuple $(\epsilon_1, \cdots, \epsilon_r) \in (\Delta^*)^{\mathcal{N}_X}$ such that for every closed path $\gamma = \sum_{i=1}^{l} \alpha_i e_i$ in the dual graph of (X, ω) the equation*

$$\prod_{i=1}^{l} \epsilon_{j_i}^{\alpha_i w(e_i)} = 1 \tag{13}$$

is satisfied, where the node corresponding to e_i is N_{j_i}.

Proof. Let $(\mathscr{X},\mathscr{W},\mathscr{Z}_1,\cdots,\mathscr{Z}_n)$ be a family converging to the limit differential $(X,\omega,Z_1,\cdots,Z_n)$. Let $\gamma=\sum_{i=1}^{l}\alpha_i e_i$ be a closed path in the dual graph Γ_ω of the limit differential (X,ω), starting at the vertex v_1 and ending at the vertex $v_{l+1}=v_1$. We denote the node corresponding to e_i by N_i. We suppose that the local equation of $\bar{\mathscr{X}}$ around N_i is given by $xy=t^{a_{N_i}}$. We denote by ω_{V_j} the restriction of ω to the irreducible component X_{V_j} of X corresponding to V_j. We can suppose (maybe after rescaling) that the family of differentials $\mathscr{W}$ converges to ω_{V_1} on X_{V_1}. It follows from Lemma 3.4 that the scaling of X_{V_2} for $\mathscr{W}$ is $(t^{a_{N_1}})^{\alpha_1 w(e_1)}$. Therefore the family of differentials

$$(t^{a_{N_1}})^{\alpha_1 w(e_1)}\,\mathscr{W}$$

converges to ω_{V_2}. Looking at the node N_2, the family

$$(t^{a_{N_2}})^{\alpha_2 w(e_2)}\,(t^{a_{N_1}})^{\alpha_1 w(e_1)}\,\mathscr{W}$$

converges to ω_{V_3}. We iterate this process until $i=l$ and we obtain that the family of differentials

$$\prod_{i=1}^{l}(t^{a_{N_i}})^{\alpha_i w(e_i)}\mathscr{W} \tag{14}$$

converges to ω_{V_1}. By uniqueness of the scaling for a given irreducible component, the following equation is satisfied

$$\prod_{i=1}^{l}(t^{a_{N_i}})^{\alpha_i w(e_i)}=1. \tag{15}$$

In particular, the tuple $(t^{a_{N_1}},\cdots,t^{a_{N_r}})\in(\Delta^*)^{\mathcal{N}_X}$ satisfies Equation (13) for every closed path γ in the dual graph of (X,ω). $\square$

Plumbing Cylinder Construction. We develop the theory of plumbing cylinders in two steps. First, we introduce the *plumbing cylinder construction* at a single node. Second, we use it to smooth some limit differentials which will be called *plumbable differentials.*

Before extending the plumbing cylinder construction to the case of differentials, let us recall this classical result known since (at least) Klein. For a simple proof of the polar case, which extends to the holomorphic case, see [dSG10, Encadré III.2].

Lemma 3.12. *Let ω be a differential on a Riemann surface X and $Q\in X$. Let k be the order and a_{-1} be the residue of ω at Q.*

There exists an open neighbourhood U of Q and a coordinate z on U such that $z(Q)=0$ and:

If $k\leq -2$, the differential $\omega|_U$ is given by the equation $\left(z^k+\frac{a_{-1}}{z}\right)\mathrm{d}z$.

If $k=-1$, the differential $\omega|_U$ is given by the equation $\frac{a_{-1}}{z}\mathrm{d}z$.

If $k\geq 0$, the differential $\omega|_U$ is given by the equation $z^k\mathrm{d}z$.

These equations are called the local normal form *of ω at Q.*

We can now describe the Plumbing cylinder construction.

Lemma 3.13 (Plumbing cylinder construction.). *Let $V := \{z \in \mathbb{C} : |z| < 1\}$ and $W := \{w \in \mathbb{C} : |w| < 1\}$ be two discs in $\mathbb{C}$ and $U = V \cup W$ identified at their origins.*

Let $(a, b, k) \in \mathbb{C}^2 \times \mathbb{Z}$ be a triple of the form $(0, 0, -1)$ or $(1, -1, k)$ for $k \neq -1$ and let a_{-1} be a complex number. We define the differential ω on $U \setminus 0$ by

$$\omega|_V = az^k \mathrm{d}z + \frac{a_{-1}}{z}\mathrm{d}z, \text{ and } \omega|_W = \frac{b}{w^{(k+2)}}\mathrm{d}w - \frac{a_{-1}}{w}\mathrm{d}w.$$

Then there exists a differential form η on the cylinder of parameter ϵ

$$A_\epsilon := \left\{(x, y) \in \mathbb{C}^2 : xy = \epsilon, |x| < 1, |y| < 1\right\}, \tag{16}$$

and a biholomorphism

$$\varphi : U \setminus B(0, \sqrt{|\epsilon|}) \to A_\epsilon \setminus \{(x, y) \in A_\epsilon : |x| = |y|\}, \tag{17}$$

satisfying the following two properties.

i) The pair (A_ϵ, η) is a flat cylinder (i.e., η has no zeros or poles in A_ϵ).

ii) The restrictions of the pull back of η by φ are

$$\varphi^*(\eta)|_{V\setminus B(0,\sqrt{|\epsilon|})} = az^k\mathrm{d}z + \epsilon^{k+1}\frac{a_{-1}}{z}\mathrm{d}z \tag{18}$$

and

$$\varphi^*(\eta)|_{W\setminus B(0,\sqrt{|\epsilon|})} = \epsilon^{k+1}\omega|_{W\setminus B(0,\sqrt{|\epsilon|})}. \tag{19}$$

Proof. First, we prove the result in the cases $k = -1$ and $a_{-1} = 0$.

Let $\epsilon \in \Delta^*$ and define the following spaces:

$$A_\epsilon = \left\{(x, y) \in \mathbb{C}^2; |x| < 1, |y| < 1, xy = \epsilon\right\},$$
$$A'_\epsilon = A_\epsilon \setminus \{(x, y) \in A_\epsilon; |x| = |y|\},$$

and

$$\begin{aligned} B'_\epsilon &= B'_{V,\epsilon} \cup B'_{W,\epsilon} \\ &= \left\{z \in V; |z| > \sqrt{|\epsilon|}\right\} \cup \left\{w \in W; |w| > \sqrt{|\epsilon|}\right\}. \end{aligned}$$

The biholomorphism φ is given by the two following restrictions (see Figure 2):

$$\begin{aligned} \varphi_{V,\epsilon} &: B'_{V,\epsilon} \to A'_\epsilon, \quad z \mapsto \left(z, \frac{\epsilon}{z}\right), \\ \varphi_{W,\epsilon} &: B'_{W,\epsilon} \to A'_\epsilon, \quad w \mapsto \left(\frac{\epsilon}{w}, w\right). \end{aligned}$$

Let us now define the differential form η on A_ϵ to be the restriction of the differential of $\mathbb{C}^2$ of equation

$$\frac{x^{k+1}}{x^2 + y^2}\left(x\mathrm{d}x - y\mathrm{d}y\right). \tag{20}$$

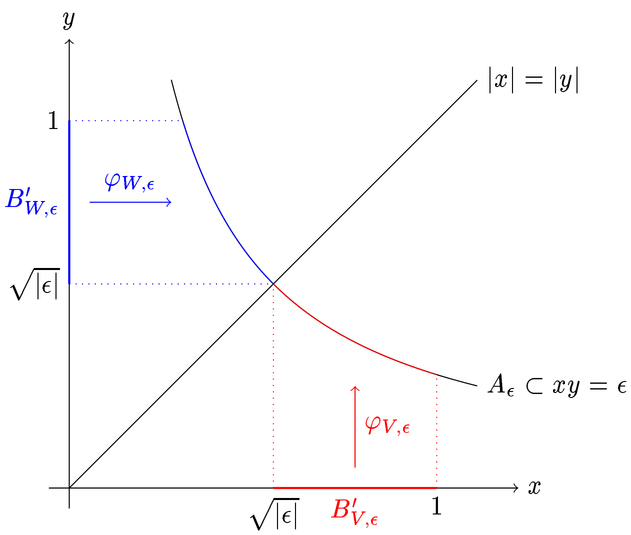


Figure 2: The maps $\varphi_{V,\epsilon}$ and $\varphi_{W,\epsilon}$.

It is clear that η does not vanishes on A_ϵ. Therefore (A_ϵ, η) is a flat cylinder.

It remains to compute the pull backs of η by $\varphi_{V,\epsilon}$ and $\varphi_{W,\epsilon}$. It is easily verified that the push forward of ∂_z via $\varphi_{V,\epsilon}$ and ∂_w via $\varphi_{W,\epsilon}$ are respectively

$$\partial_x - \frac{y}{x}\partial_y, \text{ and } -\frac{x}{y}\partial_x + \partial_y.$$

Hence the pull backs by φ of η on $B'_{V,\epsilon}$ and $B'_{W,\epsilon}$ are:

$$\varphi^*_{V,\epsilon}(\eta) = z^k \mathrm{d}z, \tag{21}$$

$$\varphi^*_{W,\epsilon}(\eta) = -\frac{\epsilon^{k+1}}{w^{k+2}}\mathrm{d}w. \tag{22}$$

In the case $k = -1$, it suffices to multiply this η by a_{-1} to obtain all the residues.

Now we prove the general result: let us suppose that $k \neq -1$ and $a_{-1} \neq 0$. The biholomorphism φ is of course given by the same formula. One can easily verify that the differential η is the restriction to A_ϵ of the differential

$$\frac{x^{k+1} - \epsilon^{k+1}a_{-1}}{x^2 + y^2}\left(x\mathrm{d}x - y\mathrm{d}y\right). \tag{23}$$

□

Let us now define the subset of the set of limit differentials which can be obtained by plumbing the nodes.

Definition 3.14. We say that a limit differential $(X, \omega, Z_1, \cdots, Z_n)$ is *plumbable* if there exists a family of pointed limit differentials

$$\left(f :: \mathscr{X} \to \Delta^*, \mathscr{W} : \Delta^* \to \omega_{\mathscr{X}/\Delta^*}, \mathscr{Z}_1, \cdots, \mathscr{Z}_n : \Delta^* \to \mathscr{X}\right),$$

satisfying the following conditions.

- The tuple $(X, \omega, Z_1, \cdots, Z_n)$ is the limit differential of this family.
- For every node N_i, there exists a neighbourhood $\mathcal{U}_i$ of N_i not containing any other node or marked point Z_i satisfying the following properties:

  the complement of the union of the $\mathcal{U}_i$ is

  $$\mathscr{X} \setminus \bigcup_i \mathcal{U}_i = \left(X \setminus \bigcup_i U_i \right) \times \Delta,$$

  where U_i denotes the restriction of $\mathcal{U}_i$ on X;

  the sections $\mathscr{Z}_i$ are given by $Z_i \times \Delta$; and

  the differentials $(\mathcal{U}_{i,t}, \mathscr{W}(t)|_{\mathcal{U}_i})$ are given by the plumbing cylinder construction at N_i with a parameter $\epsilon_i(t)$.

The set of pointed plumbable differentials of type $(k_1, \cdots, k_n)$ modulo the action of $\mathfrak{S} \subset \mathfrak{S}_n$ (see Section 2) is denoted by $K\overline{\mathcal{M}}^{\mathrm{plb}}_{g,\{n\}}(k_1, \cdots, k_n)$.

We now prove that the conditions given in Lemma 3.3 and Lemma 3.11 characterise limit differentials without poles of order ≥ 2 with a nonzero residue. Let us recall that for a curve X, we denote by $\mathcal{N}_X$ the set of nodes of X. Moreover, let e_i be an edge in the dual graph of (X, ω) (see Definition 3.8), we denote by $w(e_i)$ the weight of e_i (which is one greater than the order of the corresponding node).

Theorem 3.15. *Let $(X, \omega, Z_1, \cdots, Z_n)$ be a candidate differential which has no residue at the poles of order $k \geq 2$.*

If $(X, \omega, Z_1, \cdots, Z_n)$ satisfies the three conditions,

i) The Compatibility Condition (Equation (7))

$$\mathrm{ord}_{N_1}(\omega) + \mathrm{ord}_{N_2}(\omega) = -2,$$

at every node $N_1 \sim N_2$ of X.

ii) The Residue Condition (Equation (8))

$$\mathrm{Res}_{N_1}(\omega) + \mathrm{Res}_{N_2}(\omega) = 0,$$

at every node $N_1 \sim N_2$ of X.

iii) There exists a tuple $(\epsilon_1, \cdots, \epsilon_r) \in (\Delta^*)^{\mathcal{N}_X}$ satisfying Equation (13), i.e.

$$\prod_{i=1}^{l} \epsilon_{j_i}^{\alpha_i w(e_i)} = 1$$

for every closed path $\gamma := \sum_{i=1}^{l} \alpha_i e_i$ in the dual graph of (X, ω).

then $(X, \omega, Z_1, \cdots, Z_n)$ is a plumbable differential.

Proof. Let $(X, \omega, Z_1, \cdots, Z_n)$ be a candidate differential which has no residue at the poles of order $k \geq 0$ and let $N_1, \cdots, N_m \in \mathcal{N}_X$ be the nodes of X.

It is easily verified that if the parameters $(\epsilon_1, \cdots, \epsilon_r)$ satisfy Equation (13) for any closed path, then the same holds for $(\epsilon_1^{1/t}, \cdots, \epsilon_r^{1/t})$ for any $t \in \Delta^*$. Hence it suffices to show that $(X, \omega, Z_1, \cdots, Z_n)$ can be plumbed using the parameters $(\epsilon_1, \cdots, \epsilon_r)$ of Theorem 3.15.

According to [ACG11, page 184], there exist neighbourhoods U_i of N_i which contains neither any other node nor any point Z_i. They may be chosen as the unions of the discs $V_i = \{z_i \in \mathbb{C}; |z_i| < 1\}$ and $W_i = \{w_i \in \mathbb{C}; |w_i| < 1\}$, identified at their origins. Moreover, since the compatibility condition and the residue condition are satisfied, we can suppose that the respective restrictions of ω to V_i and W_i are of the form $a_i z_i^{k_i} \mathrm{d}z_i$ and $b_i w_i^{-(k_i+2)} \mathrm{d}w_i$, where a_i and b_i are not zero and $a_i = -b_i$ if $k_i = -1$.

We define for each node N_i the following spaces:

$$A_i = \left\{(x_i, y_i) \in \mathbb{C}^2; |x_i| < 1, |y_i| < 1, x_i y_i = \epsilon_i\right\},$$

$$A_i' = A_i \setminus \{(x_i, y_i) \in A_i; |x_i| = |y_i|\},$$

and

$$B = \left(X \setminus \bigcup_i U_i\right) \bigcup \left(\bigcup_i B_i'\right),$$

where

$$\begin{aligned} B_i' &= B_{V_i}' \cup B_{W_i}' \\ &= \left\{z_i \in V_i; |z_i| > \sqrt{|\epsilon_i|}\right\} \cup \left\{w_i \in W_i; |w_i| > \sqrt{|\epsilon_i|}\right\}. \end{aligned}$$

Now the curve X' is the union of B and $\bigcup_i A_i$ with the space $\bigcup_i B_i'$ and $\bigcup_i A_i'$ identified via the embeddings:

$$\varphi_{V_i} : B_{V_i}' \to A_i, \quad z_i \mapsto \left(z_i, \tfrac{\epsilon_i}{z_i}\right)$$

$$\varphi_{W_i} : B_{V_i}' \to A_i, \quad w_i \mapsto \left(\tfrac{\epsilon_i}{w_i}, w_i\right).$$

We denote the image of φ_{V_i} by A_i^V and the image of φ_{W_i} by A_i^W.

Let us remark that the connected components of $X \setminus \bigcup_i U_i$ and $X' \setminus \bigcup_i A_i$ are canonically biholomorphic. We denote these connected components by $\widetilde{X}_j$ and the corresponding irreducible components of X by X_j. Moreover, the set of cylinders A_i^V and A_i^W which are at the boundary of $\widetilde{X}_j$ is denoted by $\mathcal{C}(\widetilde{X}_j)$.

According to Lemma 3.13, there exist differentials $\omega_1', \cdots, \omega_r'$ on the cylinders $A_1, \cdots, A_r$ which are proportional to ω on A_i^V and A_i^W. More precisely, if $\omega_i' = \alpha_i \omega$ for an $\alpha_i \in \mathbb{C}^*$ on A_i^V, then $\omega_i' = \alpha_i \epsilon_i^{\pm(k_i+1)} \omega$ on A_i^W. The fact that the constants of proportionality are distinct is the key point in the rest of the proof.

To complete the proof, it suffices to show that we can extend the differentials ω_i' by a differential ω' on X' which is proportional to ω on every component $\widetilde{X}_i$. Observe that such a differential exists if and only if for every component $\widetilde{X}_j$ there exists a common constant of proportionality between ω_i' and ω for every cylinder A_i in $\mathcal{C}(\widetilde{X}_j)$.

Let us construct the constants of proportionality in the following way. Let X_1 be an irreducible component of X and $a_1 \in \mathbb{C}^*$. We impose that on every cylinder of $\mathcal{C}(\widetilde{X}_1)$ the relation between ω and ω'_i is given by $\omega'_i = a_1\omega$.

Let X_k be another irreducible component of X. For every path

$$\gamma_{1,k} = \sum_{i=1}^{l_k} \alpha_i^k e_i^k$$

from X_1 to X_k in the dual graph of (X,ω) we assign the following number

$$a_k^\gamma := a_1 \prod_{i=1}^{l_k} \epsilon_{j_i}^{\alpha_i^k w(e_i^k)}, \tag{24}$$

where $w(e_i^k)$ one greater than the order of the node corresponding to e_i^k.

It suffices to prove that under the third condition of Theorem 3.15 the a_k^γ do not depend on the choice of the path γ. Indeed, if this is the case there exists a differential ω' on X' which coincides with $a_k\omega$ on $\widetilde{X}_k$.

Let γ_1 and γ_2 be two paths from X_1 to X_2 in the dual graph of (X,ω). Then the number associated by Equation (24) to the concatenation $\gamma_1 \circ \gamma_2^{-1}$ is $a_k^{\gamma_1}(a_k^{\gamma_2})^{-1}$. Hence it suffices to show that $a_k^{\gamma_1}(a_k^{\gamma_2})^{-1} = a_1$ to conclude the proof. Let us denote the path $\gamma_1 \circ \gamma_2^{-1}$ by $\sum_{i=1}^{l} \alpha_i e_i$. Then by definition

$$a_1^\gamma = a_1 \prod_{i=1}^{l} \epsilon_{j_i}^{\alpha_i w(e_i)}.$$

Since the parameters ϵ_i satisfy Equation (13), this quantity is precisely a_1. $\square$

As an easy application of this theorem, we have the following remark.

Remark 3.16. Let (X,ω) be a holomorphic differential with at least one zero Z of order $k \geq 2$. Moreover, let $(\mathbb{P}^1, 0, 1, \infty)$ be a rational curve with three marked points and define the differential $\eta_i := z^i(z-1)^{k-i}\mathrm{d}z$ on $\mathbb{P}^1$. Attaching X to $\mathbb{P}^1$ via the identification of Z with ∞ and using the plumbing cylinder construction of Lemma 3.13, we obtain the construction of [EMZ03] for breaking up a zero of a differential into a pair of zeros.

An advantage of this construction is that it can be easily generalised to the case of breaking up a zero into more zeros. We use such a generalisation in the proof of Theorem 2.5.

As shown in Lemma 3.12, there exist differentials which have a pole of order $k \geq 2$ and a nonzero residue. If our candidate differential has such local behaviour at a node, then the conditions of Theorem 3.15 are not sufficient to be smoothable.

Example 3.17. Let (X,ω,Z) be a candidate differential of genus two such that $X := X_1 \cup \mathbb{P}^1 \cup X_2$, where $(X_1, \omega|_{X_1})$ and $(X_2, \omega|_{X_2})$ are two flat tori and the projective line has coordinate z such that it is attached to X_1 at 0 and to X_2 at ∞. Finally, the restriction of ω to $\mathbb{P}^1$ is $\omega_0 := \frac{(z-1)^2}{z^2}\mathrm{d}z$.

The differential (X,ω,Z) is not a limit differential. Otherwise, the differential $\mathscr{W}(t)$ of the family $\left(f : \mathscr{X} \to \Delta^*, \mathscr{W} : \Delta^* \to \omega_{\mathscr{X}/\Delta^*}, \mathscr{Z} : \Delta^* \to \mathscr{X}\right)$ would have a zero of order two at $\mathscr{Z}(t)$. Therefore, the point $\mathscr{Z}(t)$ would be a Weierstrass point of $\mathscr{X}(t)$. Since the limiting position of the Weierstrass points are the 2-torsion points of both elliptic curves, the curve $\mathscr{X}(t)$ would have seven Weierstrass points (see Theorem 6.5), a contradiction.

We do not give a complet characterisation of the limit differentials having poles of order greater or equal to 2 with a nonzero residue. However, the following lemma gives a necessary condition.

Lemma 3.18. *Let $(X, \omega, Z_1, \cdots, Z_n)$ be a limit differential. Let N be a node between the irreducible components X_1 and X_2 such that the local normal form at N of $\omega|_{X_1}$ is $z^k \mathrm{d}z$ and the one of $\omega|_{X_2}$ is $z^{-k-2} + \alpha z^{-1}\mathrm{d}z$. Then, there exists a differential η on X_1 with a pole of order -1 and residue $-\alpha$ at N which has no poles on the smooth locus of X_1.*

Proof. Let N be a node where the restriction of ω to one branch of the node has a pole of order $d+1 \geq 2$ with a nonzero residue. Let $(\mathscr{X}, \mathscr{W}, \mathscr{Z}_1, \cdots, \mathscr{Z}_n)$ be a family of pointed differentials which converges to $(X, \omega, Z_1, \cdots, Z_n)$. Let U be a neighbourhood of N in $\bar{\mathscr{X}}$ given inside Δ^3 by the equation

$$xy = t^a,$$

with $a \geq 1$. Let us suppose that the equation of the branch of N with the pole is $\{x = 0\}$ (denoted by X_y) and the restriction on it is

$$\omega_y = \frac{1}{y^d}(1 + \alpha y^d)\frac{-\mathrm{d}y}{y}$$

and that the family $\mathscr{W}$ converges to ω_y on X_y. We denote by η the differential $\frac{x\mathrm{d}x - y\mathrm{d}y}{x^2+y^2}$ on U. Observe that the family of differentials $\mathscr{W}$ is given by

$$\mathscr{W} = g \cdot \eta,$$

where g is a meromorphic function with no poles or zeros in $U \setminus X|_U$. The limit of $(t^a)^d \mathscr{W}$ to $\{y = 0\}$ (denoted by X_y) is

$$\omega_x = x^d \frac{\mathrm{d}x}{x}.$$

The limit of the family

$$\left(f - \frac{x^d}{(t^a)^d}\right) \cdot \eta$$

on X_x is $\alpha\frac{-\mathrm{d}x}{x}$. This form can be prolonged on the whole component X_1 containing X_x to a meromorphic form ω_1.

It remains to show that the poles of ω_1 cannot be located in the smooth locus of X_1. Let us denote the family which prolongs respectively $\left(g - \frac{x^d}{(t^a)^d}\right)\eta$ and $\frac{x^d}{(t^a)^d}\eta$ on $\mathscr{X}$ by $\mathscr{W}_1$ and $\mathscr{W}_2$. It follows from the fact that $\mathscr{W}_1$ converges to a meromorphic differential on X_1 and the equality

$$t^{ad}\mathscr{W} = t^{ad}\left(\mathscr{W}_1 + \mathscr{W}_2\right),$$

that the limit of $t^{ad}\mathscr{W}_2$ on X_1 is $\omega|_{X_1}$. And it follows by addition that the differential ω_1 has no poles on the smooth locus of X_1. $\square$

An interesting application is the fact that the zero of a differential in the strata $\Omega\mathcal{M}_g(2g-2)$ cannot converge to the node of a compact curve with two components.

Corollary 3.19. *Let $(X,\omega,Z) \in K\overline{\mathcal{M}}_g^{\mathrm{lim}}(2g-2)$ be a limit pointed differential with a single zero. Then (X,Z) is not the union of two components attached by a pointed projective line.*

Proof. If it was the case, then the restriction of the form ω to the projective line would be

$$\frac{(z-1)^{2g-2}}{z^{2g_1}}\mathrm{d}z$$

in a coordinate z, where the nodes are $z=0$ and $z=\infty$. This form has always a nonzero residue at the nodes. Let X_1 be another irreducible component. This would implies that X_1 has a differential with a single pole, which is of order one. $\square$

We now give conditions which are sufficient to smooth a candidate differential having poles of order ≥ 2 with nonzero residue. They are too strong to be necessary. Recall that for an irreducible component X_α of X we denote by $\mathcal{N}_{X_\alpha}$ the set of nodes of X meeting X_α. And if N is a node between X_α and X_β, we denote by N_α the point of N belonging to X_α.

Lemma 3.20. *If $(X,\omega,Z_1,\cdots,Z_n)$ is a candidate differential which satisfies the following properties, then it is plumbable.*

The Compatibility Condition (7)

$$\mathrm{ord}_{N_1}(\omega)+\mathrm{ord}_{N_2}(\omega) = -2$$

holds at every node of X. The Residue Condition (8)

$$\mathrm{Res}_{N_1}(\omega)+\mathrm{Res}_{N_2}(\omega) = 0,$$

holds at every node of X with a simple pole of ω.

There exists $(\epsilon_1,\cdots,\epsilon_n) \in (\Delta^*)^{\mathcal{N}_X}$ satisfying Equation (13) *for every closed path γ in the dual graph of (X,ω).*

There exists a differential form η_α and a constant $c_\alpha \in \mathbb{C}^*$ on every irreducible component X_α of X satisfying the following properties. The differential η_α has simple poles with residue $-a_i$ at every node $N_i \in \mathcal{N}_{X_\alpha}$ where $\omega|_{X_\beta}$ has a pole of order $k \geq 0$ with residue $a_i \neq 0$. At every other point Q in X_α, the residue of η_α is zero and

$$\mathrm{ord}_Q(\eta_\alpha) \geq \mathrm{ord}_Q(\omega).$$

At every simple pole N of η_α, the parameter ϵ_N satisfies

$$\epsilon_N^{k_N+1} = c_\alpha.$$

We will prove that on the open set corresponding to X_α, the smoothed differential is proportional to $\omega + c_\alpha\eta_\alpha$.

Proof. Let us first remark that if the parameters ϵ_i are small enough, then the orders of ω_α and $\omega_\alpha + c_\alpha\eta_\alpha$ coincide at every node of X. By replacing the ϵ_i by ϵ_i^r, for some $r > 1$, we can suppose that this is the case. Let N be a node of X between the components X_α and X_β. First suppose that the pole of ω at N has no residue. Then by plumbing the node N, we can find a differential which coincide with

$$\omega_\alpha + c_\alpha\eta_\alpha$$

on the part of the cylinder meting X_α and with

$$\epsilon_N^{\pm(k_N+1)}\left(\omega_\beta + c_\beta\eta_\beta\right)$$

on the other part of the cylinder. Now suppose that ω_β has a pole of order $k \geq 2$ with a nonzero residue. Then the plumbing cylinder construction gives a differential which coincide with

$$\omega_\alpha + \epsilon_N^{k_N+1}\eta_\alpha$$

on the part of the cylinder meting X_α and with

$$\epsilon_N^{k_N+1}\left(\omega_\beta + c_\beta\eta_\beta\right)$$

on the other part. Since by hypothesis $\epsilon_N^{k_N+1} = c_\alpha$, we can prolong this differential on the component X_α.

Finally, it follows from the fact that the parameters satisfy Equation (13) that the constants of proportionality are globally well defined (see the proof of Lemma 3.11 for details). □

Remark 3.21. As said before, the hypotheses of this lemma are to strong. A way to generalise it is to allow many differentials $(\eta_{\alpha,1}, \cdots, \eta_{\alpha,r})$ on a component X_α with distinct constants $c_{\alpha,i}$. We do not write the precise condition, because it becomes very technical and messy.

Let us remark that a necessary condition for the existence of the differentials η_α is that the following equation is satisfied

$$\sum_{N_i \in \mathcal{N}_{X_\alpha}} \mathrm{Res}_{N_{i,\beta_i}}(\omega) = 0, \tag{25}$$

where N_i is a node between X_α and a distinct component X_{β_i}. This condition should be necessary for every limit differentials, but we have no proof of it.

Relationship with the incidence variety compactification. To conclude this section, we show how to obtain a stable pointed differential from a limit differential without any pole of order $k \geq 2$ with a nonzero residue. We even prove that this map extends to the case of plumbable differentials satisfying the hypotheses of Lemma 3.20.

Definition 3.22. Let (X, ω) be a limit or stable differential. Let X_1 and X_2 be two irreducible components of X. We say that X_1 and X_2 are *polarly related by* ω if $X_1 = X_2$ or the differential ω has simple poles at the nodes between X_1 and X_2.

The equivalence classes of this relation are the *polarly related components* of (X, ω).

There is a map from the set of limit differentials to the space of marked stable differentials

$$\varphi : K\overline{\mathcal{M}}^{\lim}_{g,n}(k_1,\cdots,k_n) \to \Omega\overline{\mathcal{M}}_{g,n},$$

which is given by forgetting the differentials on the polarly related components which contain a pole of order ≥ 2.

Proposition 3.23. *Let $(X,\omega,Z_1,\cdots,Z_n)$ be a plumbable differential satisfying the hypotheses of Lemma 3.20. The marked differential $\varphi(X,\omega,Z_1,\cdots,Z_n)$ is contained in $\Omega\overline{\mathcal{M}}^{\mathrm{inc}}_{g,n}(k_1,\cdots,k_n)$.*

The point is to show that we do not have to forget the differentials more irreducible components of X.

Proof. Let $(X,\omega,Z_1,\cdots,Z_n)$ be a plumbable differential in the closure of the stratum $\Omega\mathcal{M}^{\mathrm{inc}}_{g,\{n\}}(k_1,\cdots,k_n)$. Let us first remark that we can suppose that the polarly related components are the irreducible components of X. Otherwise, we use the plumbing cylinder construction at the nodes with poles of order one. The resulting differential is still a plumbable differential satisfying the hypotheses of Lemma 3.20 and which lies in the closure of the stratum $\Omega\mathcal{M}^{\mathrm{inc}}_{g,\{n\}}(k_1,\cdots,k_n)$ if and only if the previous differential lay in its closure.

Let $(\epsilon_1 : \Delta^* \to \Delta^*,\cdots,\epsilon_n : \Delta^* \to \Delta^*)$ be parameters at the nodes of X which satisfy Equation (13). Let $c_i(t)$ and $\eta_i(t)$ be the constants and differentials satisfying the hypotheses of Lemma 3.20. The family of differentials which is obtained by plumbing the nodes with these parameters and such that the limit is a nonzero stable differential $\widetilde{\omega}$ is denoted by $(\mathscr{X},\mathscr{W},\mathscr{Z}_1,\cdots,\mathscr{Z}_n)$.

Let X_i be an irreducible component of X such that $\omega|_{X_i}$ is holomorphic, but the differential $\widetilde{\omega}|_{X_i}$ is identically zero. Let V_i be the subset of X_i which is the complement of $X_i \setminus \bigcup_j (U_j \cap X_i)$, where the U_j are the neighbourhoods of the nodes in which the plumbing take place. In $V_i \times \Delta$, the differential satisfies

$$\mathscr{W}|_{V_i\times\Delta} = h(t)\left(\omega|_{V_i} + c_i(t)\eta_i(t)\right),$$

where h is a function vanishing at the origin.

For every node $N_{i,j}$ of X_i, we replace the parameter $\epsilon_{N_{i,j}}$ by

$$h(t)^{1/(k_{N_{i,j}}+1)} \cdot \epsilon_{N_{i,j}}(t),$$

where $k_{N_{i,j}}$ is the order of the zero of ω at $N_{i,j}$. The parameters remain unchanged at the other nodes of X.

Let us show that these new parameters satisfy the conditions given by Equation (13). Let γ be a closed path in the dual graph of (X,ω). Let us denote the vertex corresponding to X_i in the dual graph of (X,ω) by V_i. Since γ is closed, it has the same number of edges which point to V_i as edges which come from V_i. Using the fact that the component X_i has only holomorphic nodes for ω, we deduce that an incoming edge and an outgoing edge of γ contribute together to Equation (13) by

$$\left(h(t)^{1/(k_{N_{i,j}}+1)}\epsilon_{N_{i,j}}\right)^{(k_{N_{i,j}}+1)} \cdot \left(h(t)^{1/(k_{N_{i,k}}+1)}\epsilon_{N_{i,k}}\right)^{-(k_{N_{i,k}}+1)},$$

which is clearly equal to

$$(\epsilon_{N_{i,j}})^{(k_{N_{i,j}}+1)} \cdot (\epsilon_{N_{i,k}})^{-(k_{N_{i,k}}+1)}.$$

So the contribution to Equation (13) of the new parameters at the nodes of X_i is the same as the contribution with the old ones. This implies that this equation remains satisfied by the new parameters. It is direct to check that the constants c_j coincide with the previous ones when $j \neq i$ and c_i is replaced by new constants c'_i. It is easily verified that we may keep the same differentials η_i.

According to Lemma 3.20, we can smooth the limit differential ω using these new parameters. We scale the family of differentials in such a way that the new one coincides with the old one on $V_j \times \Delta$, for every irreducible component X_j different from X_i. On the other hand, we claim that in a neighbourhood of V_i, we have

$$\mathscr{W}_{\mathrm{new}}|_{V_i \times \Delta} = \omega|_{V_i} + c'_i \eta_i,$$

for the family with the new parameters. Indeed, let X_k be an irreducible component which meets X_i at N_l. Let $h_j : \Delta \to \Delta$ denotes the function such that

$$\mathscr{W}_{\mathrm{new}}|_{V_j \times \Delta} = h_j(t)(\omega|_{V_j} + c'_j \eta_j).$$

Since for every irreducible component X_j distinct from X_i, this equation holds for the family $\mathscr{W}$, it follows from Lemma 3.13 that

$$\frac{h_j}{h} = (\epsilon_{N_l})^{(k_{N_l}+1)} \text{ and } \frac{h_j}{h_i} = h \cdot (\epsilon_{N_l})^{(k_{N_l}+1)}.$$

It follows from these two equation that $c_i = 1$ as claimed. This implies that the stable limit of this family restricts to ω on X_i and to $\widetilde{\omega}$ on the other irreducible components of X.

The Proposition follows by doing this procedure at every component of X where ω' vanishes but ω restricts to a holomorphic differential. □

4 Parity at the Boundary of the Strata.

The notion of theta characteristic is an essential tool for the description of the connected components of the strata of $\Omega\mathcal{M}_g$. Indeed, every stratum of the form $\Omega\mathcal{M}_g(2l_1, \cdots, 2l_n)$ has at least two connected components distinguished by the parity of the theta characteristic associated to the differential. It would be nice to show that this invariant can be extended for all limit differentials in the closure of such strata. However, we will show (see Corollary 7.10) that such extension is not possible in general. Indeed, the incidence variety compactifications of the even and the odd components of $\mathbb{P}\Omega\mathcal{M}_{3,1}(4)$ meet each other.

In this section, we will nevertheless extend this invariant to two important cases. In the first part, we treat the case of limit differentials above curves of compact type (see Theorem 4.12). This uses the theory of spin structures introduced by Cornalba, which we will recall at the beginning of this section. In the second part, we extend this invariant to the case of irreducible stable pointed differentials (see Theorem 4.19). For this purpose, we generalise the Arf invariant to such differentials (see Definition 4.17).

4.1 Differentials of Compact Type.

Let us begin this section by some preliminary paragraphs about line bundles on (semi) stable curves and Cornalba theory of spin structures.

Some basic facts about line bundles on stable curves. The material of this paragraph comes mostly from [ACG11] and [HM98]. Let us recall that the normalisation of a (semi) stable curve X is denoted by $\nu : \widetilde{X} \to X$. We denote by $\mathfrak{Irr}(X) := \{X_i\}$ the set of irreducible components of X and by $\nu_i : \widetilde{X_i} \to X_i$ the restriction of the normalisation to X_i. The set of nodes $\mathcal{N}_X$ of X is of cardinality $\mathfrak{n}$ and for each node N_i of X, its preimage by ν is $\{N_{i,1}, N_{i,2}\}$.

The key to describe the Picard group of X is the exact sequence

$$1 \to \mathcal{O}_X^* \to \nu_*\mathcal{O}_{\widetilde{X}}^* \xrightarrow{e} \prod_{N \in \mathcal{N}_X} \mathbb{C}_N^* \to 1, \tag{26}$$

where the map e is defined in the following way. For every $h \in \nu_*\mathcal{O}_{\widetilde{X}}^*$, the $\mathbb{C}_{N_i}^*$-component of $e(h)$ is $\frac{h(N_{i,1})}{h(N_{i,2})}$. The long exact sequence associated to the short exact sequence (26) is

$$1 \to \mathbb{C}^* \to (\mathbb{C}^*)^{|\mathfrak{Irr}(X)|} \to (\mathbb{C}^*)^{\mathfrak{n}} \to \mathrm{Pic(X)} \xrightarrow{\alpha^*} \mathrm{Pic}(\widetilde{\mathrm{X}}) \to 1. \tag{27}$$

The interpretation, from the right to the left, of this sequence is the following.

i) To describe a line bundle L on X it suffices to give a line bundle $\widetilde{L}$ on $\widetilde{X}$ and an identification $\varphi_{N_i} : \widetilde{L}_{N_{i,1}} \to \widetilde{L}_{N_{i,2}}$ of the fibres above the preimages of each node $N_i \in \mathcal{N}_X$. The second part of the data are usually called the *descent data* of L. Let us remark that the descent data can be interpreted as a condition for a section of L to be a lift of a section of $\widetilde{L}$.

ii) If $\widetilde{L}$ is trivial, a choice of trivialisation identifies each φ_N with a well defined non-zero complex number. So, two line bundles L_1 and L_2 such that $\widetilde{L_1} = \widetilde{L_2}$ differ only by a tuple of $\mathfrak{n}$ nonzero complex numbers.

iii) Let $\widetilde{L}$ be a line bundle on $\widetilde{X}$. If two $\mathfrak{n}$-tuple describe in ii) differ only by multiplicative constants on each irreducible component, then the line bundles associated to $\widetilde{L}$ and these descent data are the same.

iv) The descent data are well defined up to a global multiplicative constant.

Let us discuss two examples in which we will be particularly interested.

Example 4.1. If the curve X is of compact type, then the sequence (27) implies that the Picard groups of X and $\widetilde{X}$ are isomorphic. Therefore in this case, we will define line bundles by specifying their restrictions on every irreducible components of X.

Example 4.2. Let us now suppose that the curve X is an irreducible nodal curve with r nodes. Then the sequence (27) gives the sequence

$$1 \to (\mathbb{C}^*)^r \to \mathrm{Pic(X)} \xrightarrow{\alpha^*} \mathrm{Pic}(\widetilde{\mathrm{X}}) \to 1.$$

Hence in this case a line bundle on X is described by a line bundle on $\widetilde{X}$ and a r-tuple of non zero complex numbers.

We now give a description of the limit of a line bundle over a smooth family of generically smooth curves such that the special fibre is of compact type. The proof is given at the beginning of [HM98, Section 5.C].

Theorem 4.3. *Let $f : \mathscr{X} \to \Delta$ be a smooth family such that for every $t \neq 0$, the curve $\mathscr{X}(t)$ is a smooth curve of genus g and $\mathscr{X}(0)$ is a reduced curve of compact type.*

Let $\mathscr{L}$ be a line bundle of relative degree d on $\mathscr{X} \setminus \mathscr{X}(0)$ and

$$\alpha : \mathfrak{Irr}(\mathscr{X}(0)) \to \mathbb{Z}^{|\mathfrak{Irr}(\mathscr{X}(0))|}$$

be any map such that

$$\sum_{X_i \in \mathfrak{Irr}(\mathscr{X}(0))} \alpha(X_i) = d.$$

Then there exists a unique extension $\mathscr{L}_\alpha$ of $\mathscr{L}$ to $\mathscr{X}$ such that

$$\deg(\mathscr{L}_\alpha \otimes \mathcal{O}_{X_i}) = \alpha(X_i)$$

on every irreducible components X_i of $\mathscr{X}(0)$.

Moreover, if N is a node between two irreducible components X_i and X_j, and β is obtained from α by adding 1 to $\alpha(X_i)$ and subtracting 1 from $\alpha(X_j)$, then

$$\mathscr{L}_\beta \otimes \mathcal{O}_{X_i} = \mathscr{L}_\alpha \otimes \mathcal{O}_{X_i}(N), \tag{28}$$

$$\mathscr{L}_\beta \otimes \mathcal{O}_{X_j} = \mathscr{L}_\alpha \otimes \mathcal{O}_{X_j}(-N). \tag{29}$$

If the special fibre is not of compact type, there is not such a precise description. However, the idea at the beginning of [HM98, Section 5.C] remains true for families of curves with more general special fibre.

Theorem 4.4. *Let $f : \mathscr{X} \to \Delta$ be a smooth family such that for every $t \neq 0$, the curve $\mathscr{X}(t)$ is a smooth curve of genus g and $\mathscr{X}(0)$ is a (semi) stable curve.*

Let $\mathscr{L}$ be a line bundle of relative degree d on $\mathscr{X}$ such that the restriction of $\mathscr{L}$ to $\mathscr{X}(0)$ is a line bundle. Let X_i be an irreducible component and let $\{N_{j,k}\}_k$ be the set of nodes between X_i and X_j.

Then, we have the relations

$$\mathscr{L} \otimes \mathcal{O}_{\mathscr{X}}(X_i)|_{X_i} = \mathscr{L}|_{X_i} \otimes \mathcal{O}_{X_i}\left(\sum_{j,k} -N_{j,k}\right), \tag{30}$$

$$\mathscr{L} \otimes \mathcal{O}_{\mathscr{X}}(X_i)|_{X_j} = \mathscr{L}|_{X_j} \otimes \mathcal{O}_{X_j}\left(\sum_{k} N_{j,k}\right). \tag{31}$$

Abstract spin stable curves. Following the article of Cornalba [Cor89], we now extend the notion of spin structure to the case of stable curves. Recall that a spin structure is a pair $(X, \mathcal{L})$, where X is a smooth curve and $\mathcal{L}$ is a theta characteristic on X. The following curves are the base of the construction.

Definition 4.5. Let $\bar{X}$ be a semi stable curve and X its stable model. A *exceptional component* of X is an irreducible component which is contracted by the map $\pi : \bar{X} \to X$. The non-exceptional components of $\bar{X}$ are called the *stable components*.

A *decent curve* is a semi stable curve in which every exceptional component meets precisely two other irreducible components such that two exceptional curves are not allowed to meet. In particular, the exceptional components have no self intersection.

We can think of decent curves as stable curves with some of its nodes blowed up. Now we can define the notion of spin structure on decent curves.

Definition 4.6. A *spin curve* is a triple $(X, \mathcal{L}, \alpha)$, where X is a decent curve, $\mathcal{L}$ is a line bundle of degree $g-1$ on X and α is a map from $\mathcal{L}^{\otimes 2}$ to the dual sheave ω_X, which satisfies the following two properties.

i) The line bundle $\mathcal{L}$ has degree 1 on every exceptional component of X.

ii) The map α is not zero at a general point of every stable component of X.

Now we explain why this is the right generalisation of the notion of smooth spin curves.

First of all it is easy to verify that for smooth curves, this definition coincide with the usual one, since α is uniquely determined by $\mathcal{L}$.

Let X be a curve of compact type and $\mathcal{L}$ a spin structure on it. It follows easily from the definition of spin structures that the restriction of $\mathcal{L}$ to every irreducible component X_i of X of genus $g \geq 1$ is a theta-characteristic on X_i. But the sum of the degrees of these restrictions is the genus of X minus the number of irreducible components of X. To have a line bundle of degree $g-1$, the curve X has to be a decent curve with a projective line at every node.

An expected property of the notion of spin structure is that there exist 2^{2g} isomorphism classes of spin structures on a given decent curve. However, there exist in general infinitely many non isomorphic line bundles $\mathcal{L}$ satisfying the first part of Definition 4.6 (this follows from the exact sequence (27)). The morphism α rigidify this notion and the following proposition shows that it gives the right number of spin structure on a decent curve.

Proposition 4.7. *([Cor89, Paragraph 6]) Let X be a stable curve, then the number of non isomorphic spin structures on (the set of decent curves stably equivalent to) X is 2^{2g}. Moreover, the number of even ones is $2^{g-1}(2^g+1)$ and the number of odd ones is $2^{g-1}(2^g-1)$.*

Before recalling that all these properties are well behaved in families, we discuss a basic but typical example.

Example 4.8. Let X be a curve of genus g, which is the union of X_1 and X_2 of genus i and $g-i$ meeting at a unique point N.

Let us blow up X at N and denote by E the exceptional component. Let $\mathcal{L}$ be a line bundle on $\bar{X}$ such that $\mathcal{L}|_{X_1}$ and $\mathcal{L}|_{X_2}$ are theta characteristics on X_1 and X_2 respectively, and $\mathcal{L}|_E = \mathcal{O}_E(1)$. The degree of $\mathcal{L}$ is $g-1$ on $\bar{X}$. The morphism $\alpha : \mathcal{L}^2 \to \omega_{\bar{X}}$ vanishes on E and is the isomorphism between $\mathcal{L}_i^2$ and ω_{X_i} on X_i.

Moreover, the spin structure $\mathcal{L}$ is odd if the parities of $\mathcal{L}|_{X_1}$ and $\mathcal{L}|_{X_2}$ are distinct, and even otherwise.

Let $\overline{\mathcal{S}_g}$ be the *moduli space of stable spin curves*. It is a natural compactification which projects to $\overline{\mathcal{M}}_g$. Let us recall some important properties of $\overline{\mathcal{S}_g}$.

Proposition 4.9. *([Cor89, Proposition 5.2]) The variety $\overline{\mathcal{S}_g}$ is normal, projective and is the disjoint union of the even part $\overline{\mathcal{S}_g}^+$ and the odd part $\overline{\mathcal{S}_g}^-$. Moreover the forgetful map $\pi : \overline{\mathcal{S}_g} \to \overline{\mathcal{M}}_g$ is a finite map.*

In the rest of this section, we will not precise the morphism α and we will suppose that our spin structures are square roots of the canonical bundle.

Spin structure associated to limit differentials on curves of compact type. In this paragraph we compute the spin structure associated to a limit differential (see Definition 3.2) on a curve of compact type which has only zeros and poles of even orders. But a limit differential of type $(2l_1, \cdots, 2l_n)$ on a stable marked curve of compact type is determined, up to multiplication by constants, by the marked curve (see Corollary 3.5). Hence the invariant that we will construct will only depends on the marked curve, and be well defined for the limit pointed differentials of compact type.

On a smooth curve X, we can associate a spin structure to an Abelian differential with only even orders of zeros by

$$\varphi : \Omega\mathcal{M}_g(2l_1, \cdots, 2l_n) \to \mathcal{S}_g; \quad (X, \omega) \mapsto \left(X, \mathcal{L}_\omega := \mathcal{O}_X\left(\frac{1}{2}\mathrm{div}\,(\omega)\right)\right). \tag{32}$$

We extend this definition to the case of limit differentials on curves of compact type.

Definition 4.10. Let $(X, \omega, Z_1, \cdots, Z_n)$ be a limit differential in the closure of the stratum $\Omega\mathcal{M}^{\mathrm{inc}}_{g,\{n\}}(2l_1, \cdots, 2l_n)$. Let $\pi : \bar{X} \to X$ be the blow-up of X at every node of X. Then the *spin structure $\mathcal{L}_\omega$ associated to* ω is defined by the following restrictions on $\bar{X}$.

- If E is an exceptional component of $\bar{X}$, then $\mathcal{L}_\omega|_E = \mathcal{O}_E(1)$.
- If X_i is an irreducible component of X, then $\mathcal{L}_\omega|_{X_i} = \mathcal{O}_{X_i}\left(\frac{1}{2}\mathrm{div}\,(\omega)\right)$.

We now verify that the line bundle $\mathcal{L}_\omega$ associated to ω is indeed a spin structure in the sense of Definition 4.6.

Proof. Let X_i be an irreducible component of X. The line bundle $\mathcal{L}_\omega|_{X_i}$ is by definition a square root of the canonical bundle of X_i. It remains to check that the degree of $\mathcal{L}_\omega$ is $g-1$. We denote by $\mathcal{N}_e \subset \mathcal{N}_X$ the subset of nodes of X which have been blown up to give the decent curve. At each node $N_1 \sim N_2$, the compatibility condition (7) $\deg_{N_1}(\omega) + \deg_{N_2}(\omega) = -2$ implies that

$$\begin{aligned}\deg(\mathcal{L}_\omega) &= \textstyle\sum_{X_i \in \mathfrak{Irr}(X)} \deg(\mathcal{L}_\omega|_{X_i}) + \#\mathcal{N}_e \\ &= g - 1 - \#\mathcal{N}_X + \#\mathcal{N}_e.\end{aligned}$$

It follows from this equation that $\deg(\mathcal{L}_\omega) = g-1$ if and only if every node of X is blown up. ☐

Of course, this notion could be useful only if it behaves well in families. This is the content of the following lemma.

Lemma 4.11. *Let $(f : \mathscr{X} \to \Delta^*, \mathscr{W}, \mathscr{Z}_1, \cdots, \mathscr{Z}_n)$ be a family of pointed differentials in $\Omega\mathcal{M}^{\mathrm{inc}}_{g,\{n\}}(2l_1, \cdots, 2l_n)$. Let $(f : \mathscr{X} \to \Delta^*, \mathcal{L}_{\mathscr{W}} \to \mathscr{X})$ be the associated family of theta characteristics inside $\mathcal{S}_g$.*

If the stable limit of $\mathscr{X}$ is of compact type, then the spin structure associated to the pointed limit differentials of this family coincides with the restriction to the special curve of the completion of $\mathcal{L}_{\mathscr{W}}$ inside $\overline{\mathcal{S}_g}$.

Proof. Let $(\mathscr{X}, \mathscr{W}, \mathscr{Z}_1, \cdots, \mathscr{Z}_n))$ be a family inside $\Omega\mathcal{M}^{\mathrm{inc}}_{g,\{n\}}(2l_1, \cdots, 2l_n)^\epsilon$, and $(X, \omega, Z_1, \cdots, Z_n)$ be its limit differential. Above Δ^*, the associated theta characteristics are given by the bundle $\mathcal{O}_{\mathscr{X}}(\frac{1}{2}\mathrm{div}\,(\mathscr{W}))$. Let us remark that according to Proposition 4.9, there exists an extension of $\mathcal{L}$ above the decent curve $\bar{X}$ in such a way that $\mathcal{L}|_{\bar{X}}$ is a spin structure on $\bar{X}$. By Theorem 4.3, there exists only one such extension. Since the line bundle defined in Definition 4.10 is such extension, this concludes the proof . □

A direct application of this result, is the fact that the incidence variety compactifications of the even and odd components of $\Omega\mathcal{M}^{\mathrm{inc}}_{g,\{n\}}(2l_1, \cdots, 2l_n)$ remain disjunct above the set of curves of compact type.

Theorem 4.12. *Let $n \geq 3$ and $(X, \omega, Z_1, \cdots, Z_n)$ be a stable differential of compact type in $\Omega\overline{\mathcal{M}}^{\mathrm{inc}}_{g,\{n\}}(2l_1, \cdots, 2l_n)$. Then the parity of the spin structure $\mathcal{L}_\omega$ associated to $(X, \omega, Z_1, \cdots, Z_n)$ is ϵ if and only if $(X, \omega, Z_1, \cdots, Z_n)$ is in $\Omega\overline{\mathcal{M}}^{\mathrm{inc}}_{g,\{n\}}(2l_1, \cdots, 2l_n)^\epsilon$.*

Let us remark that Theorem 4.12 remains true with minor modifications even for $n \leq 2$ zeros. But the fact that in these cases the strata contain three connected components complicates the statement.

Proof. By Corollary 3.5, we can associated a unique (up to multiplicative constants) limit differential to $(X, \omega, Z_1, \cdots, Z_n)$. By Lemma 4.11, this limit differential has parity ϵ if and only if it lies in the closure of $\Omega\mathcal{M}^{\mathrm{inc}}_{g,\{n\}}(2l_1, \cdots, 2l_n)^\epsilon$ □

Let us conclude this paragraph by describing the spin structures associated to the limit differentials of the minimal strata above the generic curves of δ_i for $i \geq 1$.

Proposition 4.13. *Let $X := X_1 \cup X_2/N_1 \sim N_2$ be a curve in δ_i and let $\bar{X} := X_1 \cup E \cup X_2$ the blow-up of X at the node.*

The spin structure $\mathcal{L}$ associated to the limit differential (X, ω, Z) in the boundary of the minimal stratum is given by

$$\mathcal{L}|_{X_i} = \mathcal{O}_{X_i}((g-1)Z - g_j N_i), \quad \mathcal{L}|_{X_j} = \mathcal{O}_{X_j}((g_j - 1)N_j), \quad \mathcal{L}|_E = \mathcal{O}_E(1), \tag{33}$$

where $(i, j) = (1, 2)$ or $(i, j) = (2, 1)$.

Proof. The fact that the point Z is not contained in E has been proved in Corollary 3.19. So we can suppose that $Z \in X_1$. Then ω is a limit differential with a zero of order $2g - 2$ at Z, if it has a pole of order $2g_2$ at N_1. But by Theorem 3.15 the form ω has a zero of order $2g_2 - 2$ at N_2. The description of the restrictions of $\mathcal{L}$ is now given in Definition 4.10. □

4.2 Irreducible Pointed Differentials.

The main purpose of this paragraph is to extend the Arf invariant to the set of irreducible marked curves(see Definition 4.17). This implies that the incidence variety compactifications of the even and odd connected components of every strata remain disjoint above this locus of curves (see Theorem 4.19).

We first recall some basic facts about the Arf invariant of Abelian differentials. It was first investigated in [Joh80] (see also [Zor06]).

Through this paragraph, we will use the following notations. The pair (X,ω) denotes an Abelian differential or an irreducible stable differential with only meromorphic nodes. For a smooth simple closed path $\gamma:[0,1]\to X$, we denote by

$$G(\gamma):[0,1]\to S^1$$

the *Gauss map* associated to γ by the differential ω and by

$$\mathrm{Ind}(\gamma):=\deg(G(\gamma)) \mod 2$$

the *index* of γ.

Definition 4.14. Let (X,ω) be an Abelian differential of genus g and let $(a_1,\cdots,a_g,b_1,\cdots,b_g)$ be a symplectic basis of $H_1(X,\mathbb{Z})$ composed by smooth and simple curves which miss the zeros of ω. The *Arf invariant* of (X,ω) is

$$\mathrm{Arf}(X,\omega):=\sum_{i=1}^{g}(\mathrm{Ind}(a_i)+1)(\mathrm{Ind}(b_i)+1) \mod 2. \tag{34}$$

Johnson has shown that for every differential in $\Omega\mathcal{M}_g(2l_1,\cdots,2l_n)$, the Arf invariant is independent of the choice of the symplectic basis. Moreover, he showed that the Arf invariant coincides with the parity of the theta characteristic associated to the differential ω (see Equation (32)).

We now generalise the Arf invariant in the case of irreducible pointed stable differentials. Note that such differentials have only poles of order one at every node.

First we define the set of curves which generalises the symplectic basis. Let us recall that the normalisation of a nodal curve X is denoted by $\nu:\widetilde{X}\to X$ and the preimages of a node N_i by ν are denoted by $N_{i,1}$ and $N_{i,2}$.

Definition 4.15. Let X be an irreducible stable curve of genus g with k nodes $N_1,\cdots,N_k$. An *admissible symplectic system of curves* $(a_1,\cdots,a_g,b_1,\cdots,b_g)$ on X is an ordered set of simple smooth curves on X satisfying the three following properties.

i) The curves $(\nu^*a_{k+1},\cdots,\nu^*a_g,\nu^*b_{k+1},\cdots,\nu^*b_g)$ form a basis of $H_1(\widetilde{X},\mathbb{Z})$.

ii) For every $i,j\in\{1,\cdots,g\}$ we have

$$a_i\cdot b_j=\delta_{ij},\quad a_i\cdot a_j=0,\ \text{and}\ b_i\cdot b_j=0.$$

iii) For $i\leq k$, we have $\nu^*a_i(0)=N_{i,1}$, $\nu^*a_i(1)=N_{i,2}$ and the limits

$$\lim_{t=0}\frac{\partial\nu^*a_i}{\partial t}(t)\ \text{and}\ \lim_{t=1}\frac{\partial\nu^*a_i}{\partial t}(t)$$

exist.

The curve a_i is called an *admissible path of the node* N_i.

Note that an admissible symplectic system of curves on a smooth curve X is a symplectic basis of $H_1(X, \mathbb{Z})$.

We now describe the behaviour of the Gauss map of the admissible paths.

Lemma 4.16. *Let (X, ω) be an irreducible stable differential with only meromorphic nodes, let N_0 be a node of X and let γ be an admissible path for N_0.*

Then, the limits

$$\lim_{t \to 0} G(\gamma)(t) \text{ and } \lim_{t \to 1} G(\gamma)(t)$$

exist and coincide with the direction of the flat cylinder associated to N_0.

Proof. Since the Gauss map of a smooth path is continuous, there exist limits of $G(\gamma)(t)$ for $t \to 0$ and $t \to 1$. Since the tangent vector of γ has a limit, the path cannot turn around the node infinitely many times. This implies that the limit for the Gauss map is the direction of the flat cylinder associated to this node. □

Lemma 4.16 allows us to define the index of the paths intersecting the nodes in an admissible system of curves.

Definition 4.17. Let (X, ω) be an irreducible differential with meromorphic nodes, N_0 be a node of X and γ be an admissible path for N_0.

The *index* of γ is

$$\mathrm{Ind}(\gamma) := \deg(G(\gamma)) \mod 2.$$

We can now extend the notion of Arf invariant.

Definition 4.18. Let (X, ω) be a stable differential such that X is irreducible and ω has a simple pole at every node of X. Let $(a_1, \cdots, a_g, b_1, \cdots, b_g)$ be an admissible symplectic system of curves for (X, ω).

The *generalised Arf invariant* of (X, ω) is

$$\mathrm{Arf}(X, \omega) := \sum_{i=1}^{g} (\mathrm{Ind}(a_i) + 1)(\mathrm{Ind}(b_i) + 1) \mod 2. \tag{35}$$

We show that the generalised Arf invariant does not depend on the choice of the admissible system.

Theorem 4.19. *Let $(X, \omega, Z_1, \cdots, Z_n) \in \Omega\overline{\mathcal{M}}^{\mathrm{inc}}_{g,\{n\}}(2d_1, \cdots, 2d_n)$ be a stable differential such that X is irreducible with k nodes $N_1, \cdots, N_k$.*

Then the generalised Arf invariant only depends on (X, ω) and $\mathrm{Arf}(X, \omega) = \epsilon$ if and only if (X, ω) is in the closure of a component of $\Omega\mathcal{M}_g(2d_1, \cdots, 2d_n)$ with associated spin structure of parity ϵ.

We prove the result by recurrence on the number of nodes. The main tool for the recurrence step is the Plumbing cylinder construction of Section 3 (see in particular Theorem 3.15).

Proof. If X has no nodes, then the generalised Arf invariant of X coincides with the usual Arf invariant. This implies the result for a smooth differential.

Let us suppose that Theorem 4.19 has been proved in the case of $k-1$ nodes and let $(X,\omega,Z_1,\cdots,Z_n)$ be a differential with k nodes satisfying the hypothesis of Theorem 4.19. Let $(a_1,\cdots,a_g,b_1,\cdots,b_g)$ be an admissible symplectic system for (X,ω).

Let V and W be neighbourhoods of $N_{k,1}$ and $N_{k,2}$ respectively, such that $U:=V\cup W$ and $\omega|_U$ satisfy the hypothesis of Lemma 3.13. Without loss of generality, we can suppose that $U\cap a_i=\emptyset$ for all $i\neq k$ and $U\cap b_j=\emptyset$ for all $i\in\{1,\cdots,g\}$. Moreover, let θ_k be the direction of the cylinders associated to ω at N_k. We can suppose that $G(a_k)(t)\in\left]\theta_k-\frac{\pi}{4},\theta_k+\frac{\pi}{4}\right[$ for every t such that $a_k(t)\in U$. In particular, the path a_k meets only one time the boundaries of V and W.

Since $(X,\omega,Z_1,\cdots,Z_n)$ verifies the hypotheses of Lemma 3.13, we can smooth this differential. In particular, the set U is replaced by a flat cylinder U' and a_k by any simple closed smooth curve which coincide with a_k outside of U'.

By induction, the generalised Arf invariant is well defined on this curve. In particular, it does not depend on the chose of a_k. Hence it remains to show that the index of every curve in the new admissible symplectic system coincide with the index of the corresponding curve in the old admissible system. The indices of every curve distinct from a_k are clearly invariant under the plumbing cylinder construction. It remains to show that the index of a_k and $\widetilde{a_k}$ coincide. But we can choose $\widetilde{a_k}$ such that in U' the Gauss map satisfies $G(\widetilde{a_k})(t)\in\left]\theta-\frac{\pi}{2},\theta+\frac{\pi}{2}\right[$. In particular, it is clear that the index of $\widetilde{a_k}$ coincide with the index of a_k.

This shows that the generalised Arf invariant is a well defined invariant of (X,ω) and coincide with the Arf invariant of any partial smoothing of (X,ω) at a node. By induction these smoothings are in the closure of a component of $\Omega\mathcal{M}_g(2d_1,\cdots,2d_n)$ with associated theta characteristic of parity ϵ. ☐

5 Kodaira Dimension of Some Strata of $\mathbb{P}\Omega\mathcal{M}_g$.

In this section, we compute the Kodaira dimension of some strata of $\mathbb{P}\Omega\mathcal{M}_g$. We show in Theorem 5.10 that the strata which 'impose few conditions on the differentials' (see the theorem loc. cit. for a precise definition) have negative Kodaira dimension. In Theorem 5.7, we compute the dimension of the projection of every connected component of every stratum of $\Omega\mathcal{M}_g$ to $\mathcal{M}_g$. This result implies that the strata $\mathbb{P}\Omega\mathcal{M}_g(k_1,\cdots,k_{g-1})$ different from $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{even}}(2,\cdots,2)$ are of general type when $\mathcal{M}_g$ is of general type (see Theorem 5.4).

The end of this section is devoted to the computation of the Kodaira dimension of other strata. In Proposition 5.13, we show that $\mathbb{P}\Omega\mathcal{M}_g(g-1,1,\cdots,1)$ is of general type when $\mathcal{M}_g$ is of general type. We give the Kodaira dimension of $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{hyp}}(g-1,g-1)$ in Proposition 5.14. Moreover, we give the Kodaira dimension of every odd (see Corollary 5.17) and every even (see Proposition 5.15) component of $\mathbb{P}\Omega\mathcal{M}_g(2,\cdots,2)$.

Generalities. We first recall the definition of the Kodaira dimension of complex varieties Y following [Uen75]. The (complex) dimension of Y will be denoted by $\dim Y$.

Definition 5.1. Let Y be a smooth irreducible compact complex variety. The *Kodaira dimension* $\kappa(Y)$ *of* Y is

$$\kappa(Y) = \begin{cases} -\infty, \text{if } H^0(Y, mK_Y) = 0 \text{ for all } m \geq 0 \\ \min\left\{n \in \mathbb{N} \cup \{0\} : \frac{h^0(Y, mK_Y)}{m^n} \text{ is bounded}\right\}, \text{ otherwise.} \end{cases} \tag{36}$$

The variety Y is of *general type* if $\kappa(Y) = \dim(Y)$.

Since we will be mainly interested in singular non-compact varieties, we extend the notion of Kodaira dimension to singular and non-compact varieties. If Y is a singular compact complex variety, then its *Kodaira dimension* $\kappa(Y)$ is the Kodaira dimension of any non-singular model of Y. If Y is a non-compact complex variety, then its *Kodaira dimension* $\kappa(Y)$ is the Kodaira dimension of any non singular model of any compactification of Y. Let us remark that, as the Kodaira dimension is a birational invariant, the two preceding definitions make sense.

The Kodaira dimension of a given complex variety Y is in general difficult to compute. On the other hand it is easily proved that $\kappa(Y_1 \times Y_2) = \kappa(Y_1) + \kappa(Y_2)$. One could hope that a similar statement holds for more general fibre spaces and for maps $\pi : Y \to Z$ which behave like bundle maps. This is what we explain now.

The first important notion is the one of *fibre space* of complex varieties. This is a proper and surjective morphism $\pi : Y \to Z$ of reduced analytic spaces such that the general fibre of π is connected. Moreover, a meromorphic mapping $\varphi : Y \to Z$ is *generically surjective* if the projection $\pi_\varphi : G_\varphi \to Z$ of the graph of φ to Z induced by the projection of $Y \times Z$ to Z is surjective.

Let us recall, that a fibre space $\pi : Y \to Z$ is *uniruled* if a generic fibre Y_z of π is a projective line. If a space is uniruled, then its Kodaira dimension is negative.

Let us recall that the Kodaira dimension of a fibre space can not be larger than the Kodaira dimension of the base plus the Kodaira dimension of a generic fibre (see [Uen75, Theorem 6.12]).

Theorem 5.2. *Let $\pi : Y \to Z$ be a fibre space of complex varieties. There exists an open dense set $V \subset Z$ such that for any point $z \in V$ the inequality*

$$\kappa(Y) \leq \dim(Z) + \kappa(\pi^{-1}(z)) \tag{37}$$

holds.

In particular, if the Kodaira dimension of a generic fibre or of the basis of a fibre space is negative, then the total space has negative Kodaira dimension.

A very important open problem is to determine determine the best lower bound in the preceding settings.

Conjecture 5.3 (Iitaka conjecture or C_n conjecture.)**.** *Let $\pi : Y \to Z$ be a fibre space of an n-dimensional algebraic manifold Y over an algebraic manifold Z. Then we have*

$$\kappa(Y) \geq \kappa(Z) + \kappa(Y_z), \tag{38}$$

for a generic fibre $Y_z := \pi^{-1}(z)$.

Even if the conjecture is known to be false in general (see [Uen75, Remark 15.3]), it holds in very important cases. The first one is when $\pi : Y \to Z$ is a generically surjective map of complex varieties of the same dimension (see [Uen75, Theorem 6.10]).

Theorem 5.4. *Let $\pi : Y \to Z$ be a generically surjective meromorphic mapping of complex varieties such that $\dim Y = \dim Z$. Then we have the inequality*

$$\kappa(Y) \geq \kappa(Z). \tag{39}$$

The second important case of this conjecture has been proved by Viehweg. He proved that the Iitaka conjecture holds as soon as Z is of general type.

Theorem 5.5 ([Vie82]). *Let $\pi : Y \to Z$ be a generically surjective meromorphic mapping of complex varieties such that $\kappa(Z) = \dim Z$. Then we have the inequality*

$$\kappa(Y) \geq \kappa(Z) + \kappa(Y_z), \tag{40}$$

for a generic fibre $Y_z := \pi^{-1}(z)$

The strata of $\mathbb{P}\Omega\mathcal{M}_g$. The rest of this section is devoted to the computation of the Kodaira dimension of several strata of the moduli space of Abelian differentials.

Let us first remark that the Kodaira dimension of the principal stratum follows directly from Theorem 5.2.

Proposition 5.6. *The Kodaira dimension of the moduli spaces $\mathbb{P}\Omega\mathcal{M}_g$ and the principal strata $\mathbb{P}\Omega\mathcal{M}_g(1, \cdots, 1)$ is $-\infty$.*

Proof. Since $\mathbb{P}\Omega\overline{\mathcal{M}}_g \to \overline{\mathcal{M}}_g$ is a bundle with fibre $\mathbb{P}^{g-1}$, the result follows from Theorem 5.2. Since the closure of the principal stratum is $\mathbb{P}\Omega\overline{\mathcal{M}}_g$, this implies the result for the principal stratum. □

In order to apply the Theorem of Iitaka-Viehweg, we have to determine for which strata the forgetful map $\pi : \Omega\mathcal{M}_g \to \mathcal{M}_g$ is generically surjective. In fact, we compute the dimension of the image of every connected component of the strata of $\Omega\mathcal{M}_g$ via the forgetful map. This theorem greatly generalises a previous result of Chen (see [Che10, Proposition 4.1]).

Theorem 5.7. *Let $g \geq 2$ and S be a connected component of the stratum $\Omega\mathcal{M}_g(k_1, \cdots, k_n)$. The dimension $d_{\pi(S)}$ of the projection of S by the forgetful map $\pi : \Omega\mathcal{M}_g \to \mathcal{M}_g$ is*

$$d_{\pi(S)} = \begin{cases} 2g-1, & \text{if } S = \Omega\mathcal{M}_g(2d,2d)^{\mathrm{hyp}} \\ 3g-4, & \text{if } S = \Omega\mathcal{M}_g(2,\cdots,2)^{\mathrm{even}} \\ 2g-2+n, & \text{if } n < g-1 \text{ and } S \neq \Omega\mathcal{M}_g(2d,2d)^{\mathrm{hyp}} \\ 3g-3, & \text{if } n \geq g-1 \text{ and the parity of } S \text{ is not even.} \end{cases} \tag{41}$$

This theorem is proved by degeneration. The main ingredients are the plumbing cylinder construction of Section 3, the explicit description of the spin structures on the curves of compact type (see Section 4) and the local parametrisation of $\overline{\mathcal{M}}_g$ given by [ACG11, Theorem 3.17].

Before proving the theorem let us introduce the main type of stable curve that we use in the proof.

Definition 5.8. Let $(X_1, N_{1,1})$ and $(X_g, N_{g-1,2})$ be 2 one-marked elliptic curves and let $(X_2, N_{1,2}, N_{2,1}), \cdots, (X_{g-1}, N_{g-2,2}, N_{g-1,1})$ be $g-2$ two-marked elliptic curves. The *snake curve* X defined by these elliptic curves (see Figure 3) is

$$X := \left(\bigcup_{i=1}^{g} X_i \right) / \left(N_{i,1} \sim N_{i,2} \right).$$

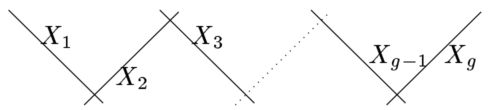


Figure 3: The snake curve X.

Proof. We begin the proof by treating the case of the hyperelliptic strata $\mathcal{H}_g$.

The hyperelliptic strata. The hyperelliptic locus $\mathcal{H}_g \subset \mathcal{M}_g$ of genus g has dimension $2g-1$. Since the projections of each of the hyperelliptic strata $\Omega\mathcal{M}_g(2g-2)^{\mathrm{hyp}}$ and $\Omega\mathcal{M}_g(2d,2d)^{\mathrm{hyp}}$ to $\mathcal{M}_g$ are $\mathcal{H}_g$, they have dimension $2g-1$.

From now on, S will be a an non hyperelliptic connected component of the stratum $\Omega\mathcal{M}_g(k_1, \cdots, k_n)$.

The strata $\Omega\mathcal{M}_g(k_1, \cdots, k_n)$ with $n \geq g$. Let us remark that if $n \geq g$, then the stratum $S' := \Omega\mathcal{M}_g(k_1 + k_n, \cdots, k_{n-1})$ lies in the boundary of S. So if the dimension of the projection of S is d, the dimension of the projection of S' is at least d. This implies that it suffices to prove the theorem for the strata with at most $g-1$ zeros.

From now on, we suppose that $n \leq g-1$.

The connected strata $\Omega\mathcal{M}_g(k_1, \cdots, k_n)$. Let X be the snake curve from above where the points $N_{i-1,2}$ are points of $2(g-i)$-torsion of $(X_i, N_{i,1})$.

Let ω be the differential on X defined by the following restrictions.

- For $i = 1$, let $\omega|_{X_1}$ be a differential on X_1 with a pole of order k_1 at $N_{1,1}$ and a zero Z_1 of order k_1.
- For $i \in \{2, \cdots, n\}$, let $\omega|_{X_i}$ be a differential such that the divisor is

$$\operatorname{div}(\omega_i) = k_i Z_i + \left(\sum_{j<i} k_j - 2(i-1) \right) N_{i-1,2} - \left(\sum_{j \leq i} k_j - 2(i-1) \right) N_{i,1},$$

  where $Z_i \in X_i \setminus \{N_{i-1,2}, N_{i,1}\}$.
- For $i \in \{n+1, \cdots, g-1\}$, the differential $\omega|_{X_i}$ is the differential with divisor

$$\operatorname{div}(\omega_i) = 2(g-i)N_{i-1,2} - 2(g-i)N_{i,1}.$$

- For $i = g$, the differential ω_g is simply the holomorphic differential of X_g.

Let us remark that the differentials $\omega|_{X_i}$ exist and satisfy the Compatibility Condition (7), that is $\mathrm{ord}_{N_{i,1}}(\omega|_{X_i}) = \mathrm{ord}_{N_{i,2}}(\omega|_{X_{i+1}}) = -2$ for every node $N_{i,1} \sim N_{i,2}$. Moreover, the differentials ω_i have no residues, so according to Theorem 3.15, they form a limit differential ω which can be smoothed in the stratum $\Omega\mathcal{M}_g(k_1, \cdots, k_n)$.

We now construct a neighbourhood of X of dimension $2g - 2 + n$ such that every curve in this neighbourhood possesses a limit differential of type $(k_1, \cdots, k_n)$.

Let us first give a parametrisation of a small neighbourhood U of X in $\overline{\mathcal{M}_g}$ (see [ACG11, Theorem 3.17]). Let $(t_1, \cdots, t_{3g-3}) \in \Delta^{3g-3}$ be a parametrisation of U such that the coordinates of X are $(0, \cdots, 0)$ and satisfying the following properties.

- The first g variables $t_1, \cdots, t_g$ parametrise the deformations of the elliptic curves $(X_1, N_{1,1}), \cdots, (X_g, N_{g,1})$.
- The $g - 2$ variables $t_{g+1}, \cdots, t_{2g-2}$ parametrise the deformations of the nodes $N_1, \cdots, N_{g-1}$. Alternatively, they parametrise the deformations of $(X_i, N_{i-1,2}, N_{i,1})$ which leave the curve X_i fixed.
- The $g - 1$ last parameters $t_{2g-1}, \cdots, t_{3g-3}$ parametrise the smoothings of the nodes of X.

Observe that the existence of a limit differential as previously defined does not depend on the normalisation of the elliptic curves. Therefore, we can deform the differential ω above the curves of parameter equal to $(t_1, \cdots, t_g, 0, \cdots, 0)$ in such a way that it remains a limit differential of type $(k_1, \cdots, k_n)$.

Now let us remark that for $i \in \{n + 1, \cdots, g - 1\}$, the points $N_{i-1,2}$ have to be points of $2(g - i)$-torsion of $(X_i, N_{i,1})$. On the other hand, the points $N_{i,2}$ and $N_{i+1,1}$ can move freely on X_i for $i \in \{1, \cdots, n\}$. Hence, using the second characterisation of the deformations of the nodes, this means that only the deformations of the $n - 1$ first nodes of X are allowed.

It follows from Theorem 3.15, that the smoothings of the nodes at the limit differential (X', ω') of parameter $(t_1, \cdots, t_{g+n-1}, 0, \cdots, 0)$ is a differential in S.

Summarising this discussion, we have shown, that every curve with coordinates $(t_1, \cdots, t_{g+n-1}, 0, \cdots, 0, t_{2g-1}, \cdots, t_{3g-3}) \subset \Delta^{3g-3}$ has a limit differential in the closure of S. Since this neighbourhood of X has dimension $2g - 2 + n$, this proves Theorem 5.7 in the case of connected strata.

The non-connected strata. Next, we deal with the non-connected strata of $\Omega\mathcal{M}_g$ determined in [KZ03]. The problem of the last argument is that we do not know a priori in the boundary of which connected component is the limit differential (X, ω) that we have constructed.

Recall from Definition 4.10 that on a curve of compact type X, a spin structure is determined by its restrictions on every irreducible component of X. More precisely, if ω is a limit differential on X with only zeros and poles of even orders, then the theta characteristic on an irreducible component X_i of X is $\mathcal{O}_{X_i}\left(\frac{1}{2}\mathrm{div}\,(\omega|_{X_i})\right)$. Moreover, we have shown in Theorem 4.12 that the parity of a spin structure is given by the sum of the parities of these restrictions and is invariant under deformation.

The components of the strata $\Omega\mathcal{M}_g(2,\cdots,2)$. We first prove that the dimension of the image of $\Omega\mathcal{M}_g^{\mathrm{odd}}(2,\cdots,2)$ under the forgetful map is $3g-3$. The construction of the differential on the snake curve in the case of connected strata can be performed in the case of the strata $\Omega\mathcal{M}_g(2,\cdots,2)$. Hence it suffices to show that this differential has odd parity to prove this case. On the $g-1$ first curves $X_1,\cdots,X_{g-1}$, the theta characteristics are given by the line bundles $\mathcal{O}_{X_i}(Z_i-N_{i,1})$. In particular, they have even parity. On the other hand, the theta characteristic on the curve X_g is $\mathcal{O}_{X_g}$, which has odd parity. Since the parity of ω is given by the sum of the parities, it has odd parity.

We now deal with the case of the component $\Omega\mathcal{M}_g^{\mathrm{even}}(2,\cdots,2)$. Let us remark that the dimension of the projection of this component is at most $3g-4$. Indeed, let $(X,\omega)\in\Omega\mathcal{M}_g^{\mathrm{even}}(2,\cdots,2)$. Then clearly, $\omega\in H^0(X,\frac{1}{2}\mathrm{div}\,(\omega))$. This implies that $h^0(X,\frac{1}{2}\mathrm{div}\,(\omega))\geq 2$. The locus of curves having such theta characteristic is a divisor of $\mathcal{M}_g$ according to [TiB88]. So it remains to show that $\dim(\pi(\Omega\mathcal{M}_g^{\mathrm{even}}(2,\cdots,2)))\geq 3g-4$. We prove this by induction on the genus of the curve.

In genus 3, the even stratum $\Omega\mathcal{M}_3^{\mathrm{even}}(2,2)$ coincides with the hyperelliptic stratum $\Omega\mathcal{M}_3^{\mathrm{hyp}}(2,2)$. So the claim follows from the description of the hyperelliptic strata.

Let us do the induction step. Let $\widetilde{X}$ be generic curve in the image of $\Omega\mathcal{M}_{g-1}^{\mathrm{even}}(2,\cdots,2)$ under the forgetful map. Let $\widetilde{N}\in\widetilde{X}$ be a generic point of $\widetilde{X}$. Let (X_1,N_1) be an elliptic curve. We define the genus g curve X by

$$X:=(\widetilde{X}\cup X_1)/(\widetilde{N}\sim N_1).$$

We now construct a limit differential (X,ω) in the closure of the connected component $\Omega\mathcal{M}_g^{\mathrm{even}}(2,\cdots,2)$. Let $(\widetilde{X},\widetilde{\omega})$ be a differential in the connected component $\Omega\mathcal{M}_{g-1}^{\mathrm{even}}(2,\cdots,2)$. Let ω_1 be a meromorphic differential on X_1 which has a pole of order 2 at N_1 and a zero of order two. The differential ω is given by the differential $\widetilde{\omega}$ on $\widetilde{X}$ and the differential ω_1 on X_1. Since $\widetilde{N}$ is a general point, it is not a zero of $\widetilde{\omega}$. This implies that ω verifies the compatibility condition (7). Hence ω is a limit differential in the closure of the connected component $\Omega\mathcal{M}_g^{\mathrm{even}}(2,\cdots,2)$.

The end of the proof is similar to the case of connected strata. We can parametrise a neighbourhood of X by $(t_1,\cdots,t_{3g-3})\in\Delta^{3g-3}$ such that the locus of nodal curves is given by $t_{3g-3}=0$. Only the deformations of X_{g-1} which stay inside the projection of $\Omega\mathcal{M}_{g-1}^{\mathrm{even}}(2,\cdots,2)$ are allowed. The dimension of such deformations is $3(g-1)-1$ by the induction hypothesis. To conclude, we use a similar deformation-smoothing argument as in the case of connected strata. We can deform the point of attachment on X_{g-1}, the elliptic curve (X_1,N_1) and the node. Thus we deduce by induction, that the dimension of the projection of the component $\Omega\mathcal{M}_g^{\mathrm{even}}(2,\cdots,2)$ is $3g-4$.

The components of the strata $\Omega\mathcal{M}_g(2l_1,\cdots,2l_n)$ for $2\leq n\leq g-2$ and $(2l_1,2l_2)\neq(g-1,g-1)$. Observe that these strata have only two connected components which are determined by the parity of the associated theta characteristics.

Let X be the snake curve defined in the case of connected strata. We show that we can choose a limit differential in two ways, such that one is in the boundary of the odd component and the other in the even component of $\Omega\mathcal{M}_g(2l_1, \cdots, 2l_n)$. Choose a limit differential ω on X, and denote by ω_1 its restriction on X_1. The divisor of the differential ω_1 is $\operatorname{div}(\omega_1) = 2l_1 Z_1 - 2l_1 N_{1,1}$. So the associated theta characteristic is $\mathcal{L}_{\omega_1} := \mathcal{O}_{X_1}(l_1 Z_1 - l_1 N_{1,1})$.

There are two cases to consider: the first one is when $l_1 = 2$ and the second one when $l_1 \geq 3$. If $l_1 = 2$, the theta characteristic $\mathcal{L}_{\omega_1}$ is odd if Z_1 is a 2-torsion of $(X_1, N_{1,1})$ and even if Z_1 is a primitive 4-torsion of $(X_1, N_{1,1})$. If $l_i \geq 3$, the theta characteristic $\mathcal{L}_{\omega_1}$ is even if Z_1 is a 2-torsion of $(X_1, N_{1,1})$ and odd if Z_1 is a primitive l_1-torsion of $(X_1, N_{1,1})$.

The parity of ω is the sum of the parities of the restrictions $\omega|_{X_i}$ on every irreducible curve X_i of X. This implies that fixing ω on the $g-1$ components X_i for $i \geq 2$, we can define a differential ω in the boundary of both components of $\Omega\mathcal{M}_g(2l_1, \cdots, 2l_n)$ by changing the parity of ω_1. The deformation-smoothing argument of the connected strata now implies the claim.

The non-hyperelliptic components of $\Omega\mathcal{M}_g(g-1, g-1)$. Since we have already dealt with the hyperelliptic case, it remains the case of the other connected component if g is even or the two other components if g is odd.

Let $(X_{g-1}, \omega_{g-1}, Z_{g-1}, N_{g-1})$ be a generic pointed differential in the stratum $\Omega\mathcal{M}_{g-1}(g-1, g-3)$ and $(X_1, \omega_1, Z_1, N_1)$ be an elliptic curve with a differential ω_1 such that $\operatorname{div}(\omega_1) = (g-1)Z_1 - (g-1)N_1$. Then the pointed differential $(X, \omega, Z_1, Z_{g-1})$ is a limit differential on the boundary of $\Omega\mathcal{M}_g(g-1, g-1)$. Let us remark that the curve X_{g-1} is not hyperelliptic, because the dimension of the projection of $\Omega\mathcal{M}_{g-1}(g-1, g-3) = 2(g-1)$ is strictly larger than the dimension of the hyperelliptic locus $\mathcal{H}_{g-1}$. In particular, the limit differential $(X, \omega, Z_1, Z_{g-1})$ is not in the boundary of the hyperelliptic component of these strata. Moreover, if $g-1$ is even, then this pointed differential is either in the boundary of the even or in the boundary of the odd strata according to the parity of ω_{g-1}.

The conclusion of this case uses the same deformation-smoothing argument as previously in this proof.

The non-hyperelliptic minimal strata. The zero of a differential (X, ω) in the strata $\Omega\mathcal{M}_g(2g-2)$ is Weierstrass point. Since there exists only finitely many Weierstrass points on a curve, the projection from every component of $\mathbb{P}\Omega\mathcal{M}_g(2g-2)$ to $\mathcal{M}_g$ is finite. It is known that the dimension of $\mathbb{P}\Omega\mathcal{M}_g(2g-2)$ is $2g-2$, so the dimension of its projection has dimension $2g-2$ too.

This concludes the proof of Theorem 5.7. □

As a corollary of Theorem 5.7 we obtain the Kodaira dimension of all the strata $\Omega\mathcal{M}_g(k_1, \cdots, k_{g-1})$ different from $\mathbb{P}\Omega\mathcal{M}_g^{\text{even}}(2, \cdots, 2)$ when $\mathcal{M}_g$ is of general type.

Corollary 5.9. *The strata of the form $\mathbb{P}\Omega\mathcal{M}_g(k_1, \cdots, k_{g-1})$ different from $\mathbb{P}\Omega\mathcal{M}_g^{\text{even}}(2, \cdots, 2)$ are of general type for $g = 22$ and $g \geq 24$.*

Proof. It has been proved that $\mathcal{M}_g$ is of general type for $g \geq 24$ by Harris and Mumford and for $g = 22$ by Farkas. According to Theorem 5.7 and Theorem 5.4 we have

$$\kappa(\mathcal{M}_g) \leq \kappa(\mathbb{P}\Omega\mathcal{M}_g(k_1, \cdots, k_{g-1})) \leq \dim \mathbb{P}\Omega\mathcal{M}_g(k_1, \cdots, k_{g-1}).$$

Since the left and the right term of this inequality are equal to $3g-3$, the inequalities are equalities and the corollary follows. $\square$

Using the subadditivity of the Kodaira dimension (see Theorem 5.2), we can determine the Kodaira dimension of the strata which impose few conditions on the differential.

Theorem 5.10. *For any $g \geq 2$, let $(k_1, \cdots, k_n)$ be a tuple of positive numbers of the form $(k_1, \cdots, k_l, 1, \cdots, 1)$ with $k_i \geq 2$ for $i \leq l$ such that*

$$\sum_{i=1}^{n} k_i = 2g-2 \text{ and } \sum_{i=1}^{l} k_i \leq g-2.$$

Then the Kodaira dimension of the stratum $\mathbb{P}\Omega\mathcal{M}_g(k_1, \cdots, k_n)$ is $-\infty$.

The proof make an essential use of the following space.

Definition 5.11. Let X be a curve of genus g and $i = (i_1, \cdots, i_l) \in \mathbb{N}^l$ be a l-tuple of positive numbers. The *vanishing incidence of order i of X* is

$$I_i(X) := \left\{((Q_1, \cdots, Q_l), \omega) \in X^l \times \mathbb{P}H^0(X, \Omega_X^1) : \operatorname{ord}_{Q_j}(\omega) \geq i_j\right\}.$$

Proof of Theorem 5.10. Let X be a generic curve of genus g. We show that the fibre $\pi^{-1}(X)$ over X by the forgetful map $\pi : \mathbb{P}\Omega\mathcal{M}_g(k_1, \cdots, k_n) \to \mathcal{M}_g$ is connected and has Kodaira dimension $-\infty$. The theorem follows readily from this fact combined with Theorem 5.2.

Recall that by hypothesis the n-tuple $(k_1, \cdots, k_n) = (k_1, \cdots, k_l, 1, \cdots, 1)$, with $k_i \geq 2$ for $i \leq l$. Let us denote $k := (k_1, \cdots, k_l)$ and let $r := \sum_{i=1}^{l} k_i$ be the sum of these orders. We show that the vanishing incidence of order k is an algebraic fibre space with generic fibre $\mathbb{P}^{g-r}$. Indeed, it follows from Riemann-Roch that for any l-tuple of points $(Z_1, \cdots, Z_l) \in X^l$, the vector space

$$H^0\left(X, \Omega_X^1\left(-\sum_{i=1}^{l}(k_i Z_i)\right)\right)$$

is of dimension at least $g-r$. Since X is generic, the space corresponding to differentials having order exactly k_i at Z_i and one otherwise is an open subset of this space. This implies the claim that the vanishing incidence variety of order k is an algebraic fibre space with generic fibre isomorphic to $\mathbb{P}^{g-r-1}$.

Now, the second projection of the vanishing incidence variety of order k to $\mathbb{P}^{g-1}$ is clearly surjective on the closure of $\pi^{-1}(X)$ inside $\mathbb{P}\Omega\overline{\mathcal{M}}_g(k_1, \cdots, k_n)$. Moreover, this map does not factorise through the first projection. This implies that the generic fibre of $\pi : \mathbb{P}\Omega\mathcal{M}_g(k_1, \cdots, k_n) \to \mathcal{M}_g$ is uniruled. Therefore, its Kodaira dimension is $-\infty$. $\square$

Some other strata. We determine the Kodaira dimension of some other strata. Let us remark that if $\mathcal{M}_g$ is of general type and $n \geq g$, it suffices to determine the Kodaira dimension of a generic fibre of the map from the stratum $S := \mathbb{P}\Omega\mathcal{M}_g(k_1, \cdots, k_n)$ to $\mathcal{M}_g$ in order to compute the Kodaira dimension of S. However, this seems to be a quite subtle problem in general.

The strata $\mathbb{P}\Omega\mathcal{M}_g(g-1,1,\cdots,1)$, when $\mathcal{M}_g$ is of general type. According to Theorem 5.7, the generic fibres of the forgetful map

$$\pi : \mathbb{P}\Omega\mathcal{M}_g(g-1,1,\cdots,1) \to \mathcal{M}_g$$

are curves. Let us determine these curves.

Lemma 5.12. *Let X be a generic curve of genus $g \geq 3$. If $g \geq 4$, the closure of the fibre at X by π is a curve isomorphic to X. If $g = 3$, then the closure of the fibre at X by π is a singular curve such that X is its stable model.*

The proof uses the vanishing incidence of order $g-1$ of X that we introduce in Definition 5.11. Let us recall that we denote the set of Weierstrass points of an algebraic curve X by $\mathcal{WP}(X)$.

Proof. Let X be a generic curve in $\mathcal{M}_g$. The preimage of X by the forgetful map $\pi : \mathbb{P}\Omega\mathcal{M}_g(g-1,1,\cdots,1) \to \mathcal{M}_g$ is isomorphic to an open subset of the image of the projection of $I_{g-1}(X \setminus \mathcal{WP}(X))$ into $\mathbb{P}^{g-1}$. The closure of this locus is isomorphic to the projection in $\mathbb{P}^{g-1}$ of the closure of $I_{g-1}(X \setminus \mathcal{WP}(X))$.

Let X be a generic curve of genus 3. Then the fibre of the forgetful map $\pi : \mathbb{P}\Omega\mathcal{M}_3(2,1,1) \to \mathcal{M}_3$ above X is isomorphic to an open subset U of X. The closure of U has 24 cusps (at the Weierstrass points of X) and 28 nodes (at the double tangents of order $(2,2)$).

Let X be a generic curve of genus $g \geq 4$. The fibre of the forgetful map $\pi : \mathbb{P}\Omega\mathcal{M}_g(g-1,1,\cdots,1) \to \mathcal{M}_g$ at X is an open subset U of X. The points of $X \setminus U$ is the union of the Weierstrass points of X together with the points $Q \in X$ such that there exist $\omega \in H^0(X, K_X)$ and $R \in X$ such that

$$\mathrm{div}\,(\omega) \geq (g-1)Q + 2R.$$

The closure of U in $\mathbb{P}^{g-1}$ is also a curve birationally equivalent to X. $\square$

It follows that the generic fibres of the forgetful map π are of general type. Therefore, Theorem 5.5 implies that $\mathbb{P}\Omega\mathcal{M}_g(g-1,1,\cdots,1)$ is of general type when $\mathcal{M}_g$ is of general type:

Proposition 5.13. *The strata $\mathbb{P}\Omega\mathcal{M}_g(g-1,1,\cdots,1)$ are of general type for $g \geq 24$ or $g = 22$.*

The hyperelliptic strata $\mathbb{P}\Omega\mathcal{M}_g(g-1,g-1)$. We show that the hyperelliptic components of the strata $\mathbb{P}\Omega\mathcal{M}_g(g-1,g-1)$ are uniruled.

Proposition 5.14. *The connected component $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{hyp}}(2d,2d)$ is uniruled for every genus $g \geq 2$.*

Proof. The fibre of the morphism $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{hyp}}(2d,2d) \to \mathcal{H}_g$ is a projective line without $2g+2$ points (corresponding to the Weierstrass points). So the closure of the generic fibre is a projective line. The Kodaira dimension of the component $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{hyp}}(2d,2d)$ follows from Theorem 5.2. $\square$

The even connected component of $\mathbb{P}\Omega\mathcal{M}_g(2,\cdots,2)$.

Proposition 5.15. *The connected component $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{even}}(2,\cdots,2)$ is uniruled for every genus $g \geq 2$.*

Proof. Let X be a generic curve in the projection of $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{even}}(2,\cdots,2)$ and ω an even differential on X. By definition, we have $h^0(X, \frac{1}{2}\mathrm{div}\,(\omega)) = 2$. In particular, the fibre of $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{even}}(2,\cdots,2) \to \mathcal{M}_g$ at X is a projective line. This proves the proposition. $\square$

The odd connected component of $\mathbb{P}\Omega\mathcal{M}_g(2,\cdots,2)$. We show that the strata $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{odd}}(2,\cdots,2)$ are birationally equivalent to the moduli space of odd spin structures $\mathcal{S}_g^-$. This allows us to deduce the Kodaira dimensions of these strata using the work of Farkas and Verra [Far12].

Proposition 5.16. *There exists a birational morphism*

$$\begin{aligned}\varphi : \mathbb{P}\Omega\mathcal{M}_g^{\mathrm{odd}}(2,\cdots,2) &\to \mathcal{S}_g^- \\ (X,\omega) &\mapsto \left(X, \mathcal{O}_X\left(\frac{1}{2}\mathrm{div}\,(\omega)\right)\right).\end{aligned}$$

Proof. It is clear that the map φ is well defined. To prove the proposition, we construct a birational inverse for φ.

Let X be a curve in $\mathcal{M}_g$ such that the preimage of the forgetful map $\pi : \mathbb{P}\Omega\mathcal{M}_g^{\mathrm{odd}}(2,\cdots,2) \to \mathcal{M}_g$ is finite, and has no differential in the connected components $\Omega\mathcal{M}_g^{\mathrm{odd}}(2l_1,\cdots,2l_n)$ for $n \leq g-2$ or in any even component $\Omega\mathcal{M}_g^{\mathrm{even}}(2l_1,\cdots,2l_n)$. Moreover, we suppose that every theta characteristic $\mathcal{L}$ on X satisfies $h^0(X,\mathcal{L}) = 1$. According to Theorem 5.7, this set is an open dense set inside $\mathcal{M}_g$. Hence it suffices to give an inverse to φ above this set of curves.

Let $(X,\mathcal{L})$ be an odd theta characteristic on X. It suffices to show that there exists a unique $(g-1)$-tuple $(Q_1,\cdots,Q_{g-1})$ such that

$$2\sum_{i=1}^{g-1} Q_i \sim K_X, \text{ and } \mathcal{L} \sim \mathcal{O}_X\left(\sum_{i=1}^{g-1} Q_i\right).$$

The inverse of φ would then be given by

$$\varphi^{-1}(X,\mathcal{L}) = (X,\omega),$$

where ω is the differential with divisor $\mathrm{div}\,(\omega) = \sum 2Q_i$. Indeed, by hypothesis on X, the differential ω is not in a connected component of the form $\Omega\mathcal{M}_g^{\mathrm{even}}(2l_1,\cdots,2l_n)$ or $\Omega\mathcal{M}_g^{\mathrm{odd}}(2l_1,\cdots,2l_n)$ for $n \leq g-2$. Thus the differential ω is in the stratum $\Omega\mathcal{M}_g^{\mathrm{odd}}(2,\cdots,2)$.

Let us remark that since by definition $h^0(X,\mathcal{L}) \geq 1$, the line bundle $\mathcal{L}$ is effective. Moreover, every effective line bundle of degree $g-1$ on X can be represented by $\mathcal{O}_X\left(\sum Q_i\right)$ for $Q_i \in X$. Since by definition $\mathcal{L}^{\otimes 2} = \mathcal{O}_X(K_X)$ the divisor $2\left(\sum Q_i\right)$ is linearly equivalent to K_X.

To conclude the proof, it suffices to show that this $(g-1)$-tuple is unique up to permutation. Let $(R_1,\cdots,R_{g-1}) \in X^{g-1}$ such that $2\left(\sum R_i\right) \sim K_X$

and $\mathcal{L} = \mathcal{O}_X(\sum R_i)$. Since $\mathcal{L} = \mathcal{O}_X(\sum Q_i) = \mathcal{O}_X(\sum R_i)$, the line bundle $\mathcal{O}_X(\sum Q_i - \sum R_i)$ is the trivial bundle $\mathcal{O}_X$. Applying the Theorem of Riemann-Roch to this line bundle gives:

$$h^0\left(\sum Q_i - \sum R_i\right) - h^0\left(K_X - \sum Q_i + \sum R_i\right) = 1 - g.$$

Now it follows from the linear equivalence $2(\sum Q_i) \sim K_X$ that

$$1 - h^0\left(\sum Q_i + \sum R_i\right) = 1 - g.$$

More explicitly, we have the equation

$$h^0\left(\sum Q_i + \sum R_i\right) = g.$$

In particular, $\mathcal{O}_X(\sum Q_i + \sum R_i)$ is a $\mathfrak{g}^{g-1}_{2g-2}$ on X, so is the canonical line bundle of X. But since $\mathcal{O}_X(K_X - \sum Q_i)$ is a theta characteristic, the dimension of its space of section is by hypothesis

$$h^0\left(K_X - \sum Q_i\right) = 1.$$

In particular, any differential which vanishes at the Q_i is proportional to ω. In particular, the points $R_1, \cdots, R_{g-1}$ coincide with the points $Q_1, \cdots, Q_{g-1}$. $\square$

Therefore, we can deduce the Kodaira dimension of these connected components from the work of Farkas and Verra (see [Far12]).

Corollary 5.17. *The stratum $\mathbb{P}\Omega\mathcal{M}_g(2, \cdots, 2)^{\mathrm{odd}}$ is uniruled if $g \leq 11$ and is of general type for $g \geq 12$.*

6 Hyperelliptic Minimal Strata $\mathbb{P}\Omega\overline{\mathcal{M}}^{\mathrm{inc}}_{g,1}(2g-2)^{\mathrm{hyp}}$.

The main result of this section is Theorem 6.7, where we relate the incidence variety compactification $\mathbb{P}\Omega\overline{\mathcal{M}}^{\mathrm{inc}}_{g,1}(2g-2)^{\mathrm{hyp}}$ of the hyperelliptic minimal strata with the locus $\overline{\mathcal{WP}(\mathcal{H}_g)}$ of Weierstrass points of hyperelliptic curves. We show that the fibres of the forgetful map $\pi : \mathbb{P}\Omega\overline{\mathcal{M}}^{\mathrm{inc}}_{g,1}(2g-2)^{\mathrm{hyp}} \to \overline{\mathcal{WP}(\mathcal{H}_g)}$ are projective spaces.

For sake of concreteness, we describe the hyperelliptic curves with one node in Theorem 6.4 and the closure of the locus of Weierstrass points of hyperelliptic curves in $\overline{\mathcal{M}}_{g,1}$ in Theorem 6.5. Moreover, we describe the pointed differential in the incidence variety compactification of the hyperelliptic minimal strata in the most simple cases in Theorem 6.8 and Theorem 6.9.

Admissible covers. The key tool to study hyperelliptic curves is the theory of admissible covers. Let us quickly recall its definition and relationship with hyperelliptic curves. For more details see [HM98, Section 3.G].

Definition 6.1. Let $(B; Q_1, \cdots, Q_n)$ be a stable n-pointed curve of arithmetic genus zero and $N_1, \cdots, N_k$ the nodes of the curve B. An *admissible cover of the curve B* is a nodal curve X and a regular map $\pi : X \to B$ such that the following two conditions hold.

i) The preimage of the smooth locus of B is the smooth locus of X and the restriction of the map π to this open set is simply branched over the points Q_i and otherwise unramified.

ii) The preimage of the singular locus of B is the singular locus of X and for every node N of B and every node $\widetilde{N}$ of X lying over it, the two branches of X near $\widetilde{N}$ map to the branches of B near N with the same ramification index.

This notion is particularly adapted to describe the closure of the loci of k-gonal curves inside $\overline{\mathcal{M}}_g$.

Definition 6.2. Let X be a stable curve. We say that X is k-gonal if and only if it is a limit of smooth k-gonal curves.

The following theorem allows us to characterise the k-gonal curves (see [HM98, Theorem 3.160]).

Theorem 6.3. *A stable curve X is k-gonal if and only if there exists a k-sheeted admissible cover $X' \to B$ of a stable pointed curve of genus 0 which is stably equivalent to X.*

In particular, since the smooth hyperelliptic curves are exactly the smooth 2-gonal curves, the stable hyperelliptic curves will by given by the 2-sheeted admissible covers.

The hyperelliptic locus $\overline{\mathcal{H}}_g$ in $\overline{\mathcal{M}}_g$. A hyperelliptic curve with one node is described in the following theorem.

Theorem 6.4. *Let $X \in \overline{\mathcal{H}}_g$ be a hyperelliptic curve of genus g with one node.*

- *If X is irreducible, the normalisation $\widetilde{X}$ of X is hyperelliptic and the preimage of the node is a pair of points conjugated by the hyperelliptic involution.*
- *If X is of compact type, the curve X is given by $X_1 \cup X_2/(N_1 \sim N_2)$, where the X_j are hyperelliptic and N_j are Weierstrass points of X_j respectively.*

Let us recall that the *Weierstrass locus* inside $\mathcal{M}_{g,1}$ is defined by

$$\mathcal{WP}(\mathcal{M}_g) := \{(X,W)|\ W \text{ is a Weierstrass point of } X\}.$$

The *hyperelliptic Weierstrass locus* is simply the restriction of this locus above the hyperelliptic locus of $\mathcal{M}_g$:

$$\mathcal{WP}(\mathcal{H}_g) := \{(X,W) \in \mathcal{WP}(\mathcal{M}_g)|\ X \text{ is hyperelliptic}\}.$$

We describe now the marked curves in the closure of $\mathcal{WP}(\mathcal{H}_g)$ which are generic in δ_i.

Theorem 6.5. *Let $(X,W) \in \overline{\mathcal{WP}(\mathcal{H}_g)} \subset \overline{\mathcal{M}}_{g,1}$ be a marked curve in the closure of the hyperelliptic Weierstrass locus, such that X is stably equivalent to a generic curve in δ_i.*

The pair (X,W) is of one of the following form.

- *The curve X is stably equivalent to a curve in δ_0. Then X is either irreducible and W is in $\mathcal{WP}(X)$, or X is the blow-up at the node of an irreducible curve and W is in the exceptional component.*

- *The curve X is generic in the divisor δ_1 and the point W is one of the $2g-1$ smooth Weierstrass points of the curve of genus $g-1$ (or a 2-torsion point if $g=2$) or a 2-torsion point of the elliptic curve.*

- *The curve X is generic in the divisor δ_i for $i \geq 2$ and the points W are smooth Weierstrass points of the irreducible components of X.*

These two theorems are consequences of the theory of admissible covers and in particular, we will use Theorem 6.3 in a crucial way.

Proof of Theorem 6.4 and Theorem 6.5. Let us first suppose that $1 \leq i \leq \left[\frac{g}{2}\right]$ and let X be a hyperelliptic curve in δ_i as given in the theorem. By Theorem 6.3, the curve X is stably equivalent to an admissible cover $\pi : X' \to B$ of degree two above a stable marked curve of genus zero $(B; x_1, \cdots, x_{2g+2})$. Let B_0 be an irreducible component of B which meets only one other component and denote $X_0 := \pi^{-1}(B_0)$. Remark that there exists such a B_0 since B is of compact type. Since $(B; x_1, \cdots, x_{2g+2})$ is a stable marked curve, at least two marked points lie on B_0. Moreover the cardinality of the preimage of the node is one because otherwise X would have a nonseparating node. Let us call this point N_0. It is a ramification point of the map to B_0, so by Riemann-Hurwitz the curve X_0 has genus at least 1. And since X is generic in δ_i, the component X_0 has genus i or $g-i$. We will suppose that X_0 has genus i. Then the curve B_0 has $2i+1$ marked points and the preimages of these points together with N_0 are the Weierstrass points of X_0. Now there is at least one other extremal component and the same argument show that it has genus $g-i$. This concludes the proof of both theorems in the case $1 \leq i \leq \left[\frac{g}{2}\right]$.

The case $i=0$ is similar. Let $\pi : X' \to B$ be an admissible cover of degree two stably equivalent to X. This time, for every irreducible component B_0 of B which meets one other component of B, the preimage of the node contains two distinct points. As in the previous case, the curve B has only two components: one of them contains two marked points and the other the $2g$ reminding ones. The curve X is obtained from X' by blowing down the preimage of the projective line which contains only two marked points. The restriction of the projection to this second component implies that the two preimages of the node are conjugated by the hyperelliptic involution.

Since the Weierstrass points are the ramification points of the map to $\mathbb{P}^1$, their limits are the ramification points of the smooth locus of the admissible cover. $\square$

Let us conclude this paragraph by describing the ramification locus of the forgetful map $\pi : \mathcal{WP}(\overline{\mathcal{H}}_g) \to \overline{\mathcal{H}}_g$ from the hyperelliptic Weierstrass locus to the hyperelliptic locus. This is a direct application of Theorem 6.5.

Corollary 6.6. *The map $\pi : \mathcal{WP}(\overline{\mathcal{H}}_g) \to \overline{\mathcal{H}}_g$ is unramified above the generic locus of the divisors δ_i for $i \geq 1$. On the other hand, above an irreducible curve X with k nodes there are $2g-2-2k$ unramified points and k ramification points of order two.*

The relationship between the hyperelliptic Weierstrass locus and the hyperelliptic minimal strata. We now describe the incidence variety compactification of the hyperelliptic minimal strata. We will describe precisely its relationship with the hyperelliptic Weierstrass locus. Before, let us recall that two irreducible components X_1 and X_2 of X are *polarly related* by a differential ω if $X_1 = X_2$ or ω has simple poles at the nodes between X_1 and X_2 (see Definition 3.22).

Theorem 6.7. *Let $(X,Z) \in \overline{\mathcal{WP}(\mathcal{H}_g)} \subset \overline{\mathcal{M}}_{g,1}$ be a pair consisting of a hyperelliptic curve X together with a Weierstrass point Z.*

Then there exists a stable differential ω on X, such that for every pointed stable differential (X,ω',Z) in $\mathbb{P}\Omega\overline{\mathcal{M}}^{\mathrm{inc}}_{g,1}(2g-2)^{\mathrm{hyp}}$ we have the following two properties.

- *If $\omega \equiv 0$ on an irreducible component X_i, then $\omega' \equiv 0$ on X_i.*
- *There exists $(\alpha_1,\cdots,\alpha_r) \in \mathbb{P}^{r-1}$ such that*

$$\omega|_{\widetilde{X_i}} = \alpha_i \omega'|_{\widetilde{X_i}},$$

where $\left\{\widetilde{X_i}\right\}_{i=1,\cdots,r}$ is the set of polarly related components of the differential (X,ω) such that $\omega|_{\widetilde{X}_i} \not\equiv 0$.

In particular, the fibres of the forgetful map

$$\begin{aligned}\pi : \mathbb{P}\Omega\overline{\mathcal{M}}^{\mathrm{inc}}_{g,1}(2g-2)^{\mathrm{hyp}} &\to \overline{\mathcal{WP}(\mathcal{H}_g)} \\ (X,\omega,Z) &\mapsto (X,Z)\,.\end{aligned}$$

are isomorphic to $\mathbb{P}^{r-1}$.

The proof is similar to the one of corollary 3.5, where we show a related result for curves of compact type. In fact, since hyperelliptic curves are covers of degree two above a curve of compact type, many ideas will work in this case.

Proof. Let (X,Z) be a hyperelliptic curve together with a Weierstrass point of X. There exists a family $(\mathscr{X},\mathscr{Z})$ of hyperelliptic curves with a Weierstrass section which converges to (X,ω). Let $\mathscr{W}$ be a family of differentials on $\mathscr{X}$ such that $\mathscr{W}(t)$ has a zero of order $2g-2$ at $\mathscr{Z}(t)$. It turns out that the limit differential of this family only depends on (X,Z) as we show in the following.

According to Theorem 6.3, there exists a semi stable curve $\bar{X}$ stably equivalent to X such that $\pi : \bar{X} \to B$ is an admissible cover of degree two. Moreover, the point Z is a ramification point of the map π. We will now define a differential on $\bar{X}$ unique up to scaling on the components of $\bar{X}$ such that by contracting the exceptional components we can associate a limit differential on X.

Since B is of compact type, the set of irreducible components of B which meet one other component is not empty. Let us denote this set of irreducible components by $\mathfrak{Irr}_1(B)$. The irreducible components of $\bar{X}$ which map to $\mathfrak{Irr}_1(B)$ are denoted by $\mathfrak{Irr}_1(\bar{X})$. By definition, the irreducible components in $\mathfrak{Irr}_1(\bar{X})$ have at most two nodes. If a component has one node, then it is a Weierstrass point of this component. Otherwise, the two nodes are conjugated by the hyperelliptic involution.

Let X_1 be an irreducible component of genus g_1 in $\mathfrak{Irr}_1(X)$. If X_1 is an exceptional component, then we associate the differential with two simple poles at the nodes and which is holomorphic outside of the nodes. If X_1 is not an exceptional component, there is a unique way (up to scaling) to associate a differential which can be the restriction of a limit differential according to these four cases.

i) If X_1 contains the point Z and has a unique node. Then the differential on X_1 is the differential with a zero of order $2g-2$ at Z and a pole of order $2(g-g_1)$ at the node.

ii) If X_1 contains the point Z and has two nodes. Then the differential on X_1 is the differential with a zero of order $2g-2$ at Z and two poles of order $(g-g_1)$ at both nodes.

iii) If X_1 does not contain the point Z and has a unique node. Then the differential on X_1 is the differential with a zero of order $2g_1-2$ at the node.

iv) If X_1 does not contain the point Z and has two nodes. Then the differential on X_1 is the differential with two zeros of order g_1-1 at both nodes.

Indeed, the only zeros and poles of the differentials are contained in the marked locus. Moreover, the fact that the differential is anti-invariant under the hyperelliptic involution implies that the orders of the differentials have to coincide at a pair of points conjugated by the hyperelliptic involution.

Now we can continue this process in the following way. We remove to the dual graph Γ_B of B the vertices corresponding to $\mathfrak{Irr}_1(B)$ and the edges pointing to them. This new graph is denoted by Γ_B^1. Either Γ_B^1 is empty and we have achieved the construction of the differential. Or Γ_B^1 is a non empty tree. In this case the set of irreducible components $\mathfrak{Irr}_2(B)$ of B corresponding to the leafs of Γ_B^1 is not empty. The irreducible components of $\bar{X}$ mapping to the components of $\mathfrak{Irr}_2(B)$ are denoted by $\mathfrak{Irr}_2(\bar{X})$.

The description of the differential on these components is similar to the previous one. To be more precise, because of the compatibility condition (7), the sum of the degrees of the differentials at the nodes with the components of $\mathfrak{Irr}_1(\bar{X})$ is -2. The only other zeros or poles allowed on an irreducible component are at the marked points and the orders have to be invariant by the hyperelliptic involution.

We continue this process and eventually obtain a differential on the curve $\bar{X}$. Then we can associate a differential $\widetilde{\omega}$ on (X, Z) by contracting the exceptional components of $(\bar{X}, Z)$.

Let us remark that at every pair of points conjugated by the hyperelliptic involution, the residues of $\widetilde{\omega}$ at these points are opposite. This has two consequences. The first one is that nodes corresponding to loops on the dual graph of X satisfy the residue condition. The second consequence is that we can multiply the restrictions on the irreducible components of the form $\widetilde{\omega}$ by constants in such a way that the residue condition is satisfied at every node.

Hence we obtain a unique differential up to multiplicative constants on each polarly related component of $(X, \widetilde{\omega}, Z)$.

To conclude, we obtained a stable differential ω by imposing

$$\omega|_{\widetilde{X}_i} = 0$$

when $\widetilde{\omega}|_{\widetilde{X}_i}$ has a meromorphic node of degree greater or equal to 2 in the polarly component $\widetilde{X}_i$ of $(X, \widetilde{\omega})$, and otherwise

$$\omega|_{X_i} = \widetilde{\omega}|_{X_i}.$$

By an argument similar to the one in Proposition 3.23, we can deduce that there exists a family in $\Omega\mathcal{M}_{g,1}^{\mathrm{inc}}(2g-2)^{\mathrm{hyp}}$ which has (X, ω, Z) as stable limit. Moreover, every other stable differential on (X, Z) in the closure of the connected component $\Omega\mathcal{M}_{g,1}^{\mathrm{inc}}(2g-2)^{\mathrm{hyp}}$ differs only by multiplicative constants on the polarly related components of (X, ω). □

For sake of concreteness, let us describe explicitly the stable differentials inside $\Omega\overline{\mathcal{M}}_{g,1}^{\mathrm{inc}}(2g-2)ol^{\mathrm{hyp}}$ when the curve has at most two irreducible components. First we look at differentials such that the underlying curve is in δ_i for $i \geq 1$.

Theorem 6.8. *Let (X, ω, Z) be a stable differential in $\mathbb{P}\Omega\overline{\mathcal{M}}_{g,1}^{\mathrm{inc}}(2g-2)^{\mathrm{hyp}}$ such that $X := X_1 \cup X_2/(N_1 \sim N_2)$ is in the divisor δ_i. We suppose without lose of generality that $Z \in X_1$.*

Then (X, ω, Z) is characterised by the following three properties.

- *i) The curves X_j are hyperelliptic and the points N_1 and N_2 are Weierstrass points of X_1 and X_2 respectively.*
- *ii) The point Z is a Weierstrass point of X_1.*
- *iii) The differential ω is identically zero on the component of X that contains Z and is the holomorphic differential with a zero of order $2g_2 - 2$ at N_2 on the component X_2.*

Now we look at differentials such that the underlying curve is stably equivalent to a curve in δ_0.

Theorem 6.9. *Let X be either an irreducible curve or an irreducible curve blown up at a node.*

Then (X, ω, Z) is in the incidence variety compactification of the connected component $\mathbb{P}\Omega\mathcal{M}_{g,1}^{\mathrm{hyp}}(2g-2)$ if and only if it is of one of the following two forms.

- *The point Z is in the smooth locus of the irreducible curve X and the differential ω is a section of ω_X which vanishes at Z with order $2g + 2$.*
- *The point Z is in the exceptional divisor coming from the blow-up of a node $N_1 \sim N_2$, and the differential ω is the stable differential with a zero of order $g - 2$ at both N_1 and N_2.*

We omit the proofs of both theorems. They are relatively similar to the proof of Theorem 6.7, and the reader can look at the proofs of the main theorems of Section 7 for similar computations.

7 The Boundary of $\mathbb{P}\Omega\overline{\mathcal{M}}_{3,1}^{\mathrm{inc}}(4)^{\mathrm{odd}}$.

In this section, we give a precise description of the geometry of the pointed differentials which lie in the boundary of the incidence variety compactification of $\mathbb{P}\Omega\mathcal{M}_{3,1}^{\mathrm{odd}}(4)$. Since this description depends in an essential way on the dual graph of the underlying curve, we will restrict ourself to the most simple cases. We define a *generic curve in the divisor* δ_i to be a curve in the divisor δ_i with a single node.

For a generic curve in δ_1, the description of the limit differentials in the boundary of $\mathbb{P}\Omega\mathcal{M}_{3,1}^{\mathrm{odd}}(4)$ is given in Theorem 7.3 and the stable differentials in Corollary 7.5. This description implies (see Corollary 7.6) that the incidence variety compactification of the connected component $\mathbb{P}\Omega\mathcal{M}_{3,1}^{\mathrm{odd}}(4)$ is better than the Deligne-Mumford compactification $\Omega\overline{\mathcal{M}}_3(4)$.

For a curve stably equivalent to a generic curve in δ_0, the description of the limit differentials in the boundary of $\mathbb{P}\Omega\mathcal{M}_{3,1}^{\mathrm{odd}}(4)$ is given in Theorem 7.7 and Theorem 7.8 and the stable differentials in Theorem 7.7 and in Corollary 7.9. In the first theorem we investigate the case where the underlying curve is stable, and in the second only semi stable.

To conclude, we give two examples of families in $\Omega\overline{\mathcal{M}}_{3,1}^{\mathrm{inc}}(4)$. In the first example, the underlying curve is given by a quartic in the projective plane. In the second, we deform the polygonal representation of differentials belonging to $\Omega\overline{\mathcal{M}}_3(4)$.

7.1 The underlying curve is generic in δ_1.

In order to describe the limit differential of $\mathbb{P}\Omega\mathcal{M}_{3,1}^{\mathrm{odd}}(4)$, let us introduce the following definition.

Definition 7.1. Let (X,Q) be an elliptic curve, $k \geq 2$ be an integer and l be a non-trivial divisor of k. The points of $X \setminus Q$ which are k-torsion but not l-torsion of (X,Q) are *primitive k-torsion* of (X,Q).

Moreover, let us give the definition of what we mean with a *generic curve in a divisor.*

Definition 7.2. Let δ_i be a divisor of $\overline{\mathcal{M}}_g$. A *generic curve in δ_i* is a curve in the divisor δ_i with a single node.

In this section X will denote a generic curve in δ_1 and will be given by the union of a curve X_1 of genus one and a curve X_2 of genus two meeting together at $N_1 \in X_1$ and $N_2 \in X_2$.

We now give a precise description of the limit differentials in the boundary of the connected component $\Omega\mathcal{M}_{3,1}^{\mathrm{inc}}(4)^{\mathrm{odd}}$ such that the projection to $\mathcal{M}_3$ is a generic curve of the divisor δ_1.

Theorem 7.3. *Let (X,ω,Z) be a limit differential at the boundary of the odd component of the stratum $\Omega\mathcal{M}_{3,1}(4)$.*

If the curve X is stably-equivalent to a generic curve in the divisor δ_1, then the curve X is a generic curve in δ_1 and (X,ω,Z) is of one of the following two forms.

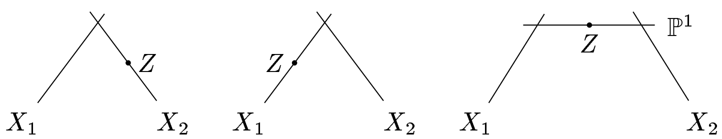
- *The point Z is a primitive 4-torsion point of (X_1,N_1) and the point of attachment $N_2 \in X_2$ is a Weierstrass point of X_2.*

  The restriction of ω to X_1 is the meromorphic differential with a zero of order 4 at Z and a pole of order 4 at N_1. The restriction of ω to X_2 is the Abelian differential with a zero of order 2 at N_2.

- *The point Z is not a Weierstrass point of X_2 and the pair (Z, N_2) satisfies the relation*
$$4Z - 2N_2 \sim K_{X_2}. \tag{42}$$
The restriction of ω to X_1 is an Abelian differential. The restriction of ω to X_2 is the meromorphic differential with a zero of order 4 at Z and a pole of order 2 at N_2.

The main tools of the proof consist of the theory of limits differentials and the spin structure on stable curves.

Proof. Since X is stably-equivalent to a generic curve in δ_1, the marked curve (X, Z) must be of one of the following three forms, where the genus of X_i is i.



The third case does not occur according to Corollary 3.19.

Let us remark that since $\omega|_{X_i}$ has at most one pole, this pole cannot have a residue. Therefore, the limit differentials on the curve X are characterised in Theorem 3.15. In the case at hand, observe that the only relevant condition of Theorem 3.15 is the Compatibility Condition (7)

$$\mathrm{ord}_{N_1}(\omega) + \mathrm{ord}_{N_2}(\omega) = -2,$$

at the node of X.

Let us now treat the case where $Z \in X_1$. Since Z is a limit differential in the boundary of $\Omega\mathcal{M}_{3,1}^{\mathrm{inc}}(4)$ the restriction of ω to X_1 has a zero of order 4 at Z and a pole of the same order at N_1. It follows from the Compatibility Condition (7) that the order of $\omega|_{X_2}$ at N_2 is 2. Thus N_2 is a Weierstrass point of X_2.

It remains to show that Z is a primitive 4-torsion point of (X_1, N_1). By the continuity of the parity of the spin structure (see Theorem 4.12) the parity of the spin structure associated to ω has to be odd. But since the parity of $\omega|_{X_2}$ is odd, the parity of $\omega|_{X_1}$ has to be even. This concludes the first case by observing that for a 4-torsion Z, we have $h^0(X_1, \mathcal{O}_{X_1}(2Z - 2N_1)) = 0$ if Z is primitive and $h^0(X_1, \mathcal{O}_{X_1}(2Z - 2N_1)) = 1$ otherwise.

The case where $Z \in X_2$ is very similar, hence we do not write every detail. Since ω has a zero of order 4 at Z, it has to have a pole of order 2 at N_2. Therefore the points Z and N_2 satisfy Equation (42).

Let us now show that the point Z cannot be a Weierstrass point. First let us remark that in this case, the point N_2 would be a Weierstrass point too. Indeed Equation (42) would be equivalent to

$$2Z \sim 2N_2 \sim K_{X_2},$$

which clearly implies that N_2 is a Weierstrass point. Now the claim follows again from the continuity of the spin structure. Since in this case the restriction of ω to X_1 is odd, the restriction $\omega|_{X_2}$ has to be even. Since the associated theta characteristic is $\mathcal{O}_{X_2}(2Z - N_2)$, it would have exactly one section if Z (and therefore N_2) were a Weierstrass point, contradicting Theorem 4.12. $\square$

Remark 7.4. An interesting fact is that there are only a finite number of points in X_1 which are in the closure of the zero of order 4 of $\Omega\mathcal{M}_{3,1}^{\mathrm{inc}}(4)$. This has to be compared with [HM98, Theorem 5.45] which tells us that when N_2 is a Weierstrass point, then every point of X_1 is in the closure of the Weierstrass locus.

We can characterise the pointed differentials in this case from Theorem 7.3 and Proposition 3.23.

Corollary 7.5. *Let (X,ω,Z) be a stable pointed differential in $\Omega\overline{\mathcal{M}}_{3,1}^{\mathrm{inc}}(4)^{\mathrm{odd}}$.*

If the curve X is stably-equivalent to a generic curve in the divisor δ_1, then X is a stable curve in δ_1 and (X,ω,Z) is of one of the following two forms.

- *The point Z is a primitive 4-torsion point of (X_1,N_1) and N_2 is a Weierstrass point of X_2.*

  The restriction of ω to X_1 vanishes identically. The restriction of ω to X_2 is the Abelian differential with a zero of order 2 at N_2.

- *The point Z is not a Weierstrass point of X_2 and the pair (Z,N_2) satisfies the relation $4Z-2N_2\sim K_{X_2}$.*

  The restriction of ω to X_1 is an holomorphic differential. The restriction of ω to X_2 vanishes identically.

These properties illustrate that the incidence variety compactification of the connected component $\Omega\mathcal{M}_3^{\mathrm{odd}}(4)$ is better than its Deligne-Mumford compactification.

Corollary 7.6. *Let X be a generic curve in δ_1 such that the nodal point of the curve of genus two is a Weierstrass point. Let (X,ω) be a differential in $\Omega\overline{\mathcal{M}}_3(4)$ where ω is of one of the following two kinds.*

i) The restriction of ω is identically zero on X_1 and is a holomorphic differential with a zero of order two at N_2 on X_2.

ii) The restriction of ω is identically zero on X_2 and is holomorphic on X_1.

Then the stable differential (X,ω) lies in the boundary of both connected components of the minimal strata in $\Omega\overline{\mathcal{M}}_3$. However, the closure of the two connected components of $\Omega\overline{\mathcal{M}}_{3,1}^{\mathrm{inc}}(4)$ are disjoint over the generic locus of δ_1.

This corollary follows readily from Theorem 7.3 and the description of the boundary of the closure of the hyperelliptic minimal strata as given in Theorem 6.7.

7.2 The underlying curve is generic in δ_0.

In this section we denote a generic curve in δ_0 by $\widetilde{X}/(N_1\sim N_2)$, where $\widetilde{X}$ is a smooth curve of genus and N_1, N_2 are distinct points of $\widetilde{X}$.

The following two theorems give the description of the limit differentials which lie in the incidence variety compactification $\mathbb{P}\Omega\overline{\mathcal{M}}_{3,1}^{\mathrm{inc}}(4)^{\mathrm{odd}}$ such that the underlying curve is generic in δ_0.

First we give the case where the zero of the differential lies in the smooth part. Observe that in this case the limit differentials in the closure of $\Omega\mathcal{M}_{3,1}^{\mathrm{inc}}(4)^{\mathrm{odd}}$

coincide with the stable differentials in $\Omega\overline{\mathcal{M}}_{3,1}^{\mathrm{inc}}(4)^{\mathrm{odd}}$. In the following theorem, we denote by X the curve $\widetilde{X}/(N_1 \sim N_2)$.

Theorem 7.7. *Let Z be anon Weierstrass point of $\widetilde{X}$. There exists a unique pair of distinct points $(N_1, N_2) \in \widetilde{X}^2$ and a unique differential $\omega \in H^0(X, \omega_X)$ with a zero of order 4 at Z and a simple pole at N_1 and N_2 such that the triple (X, ω, Z) is in $\mathbb{P}\Omega\overline{\mathcal{M}}_{3,1}^{\mathrm{inc}}(4)^{\mathrm{odd}}$.*

The set of triples

$$C := \left\{(N_1, N_2, Z) : (X, Z) \in \pi\left(\Omega\overline{\mathcal{M}}_{3,1}^{\mathrm{inc}}(4)^{\mathrm{odd}}\right)\right\}$$

is a curve in $\widetilde{X}^3$. Moreover, for a given pair among the three points N_1, N_2 and Z from the curve C, there exists exactly one point of $\widetilde{X}$ such that the triple lies in C.

Now we describe the case where the zero of the differential lies on a bridge joining the two points of the node.

Theorem 7.8. *Let (X, ω, Z) be a limit differential at the boundary of the stratum $\Omega\mathcal{M}_{3,1}^{\mathrm{odd}}(4)$ such that X is the union of a smooth curve $\widetilde{X}$ of genus two and a projective line $\mathbb{P}^1$ which meet at two distinct points N_1 and N_2.*

Then the point Z is in the projective line $\mathbb{P}^1$, and (X, ω, Z) is of one of the following two forms.

- *The restriction of ω on $\mathbb{P}^1$ has a zero of order 4 at Z, a pole of order 4 at N_1 and a pole of order 2 at N_2. The restriction of ω to $\widetilde{X}$ is an holomorphic differential with a zero of order two at N_1. In particular, N_1 is a Weierstrass point of $\widetilde{X}$.*
- *The restriction of ω on $\mathbb{P}^1$ has a zero of order 4 at Z and two poles of order 3 at N_1 and N_2. The restriction of ω to $\widetilde{X}$ is a holomorphic differential with two simple zeros at N_1 and N_2. In particular, N_1 and N_2 are conjugated by the hyperelliptic involution of $\widetilde{X}$.*

We can easily deduce the form of the pointed differentials in this case from Theorem 7.8 and Proposition 3.23.

Corollary 7.9. *Let (X, ω, Z) be a stable differential in $\Omega\overline{\mathcal{M}}_{3,1}^{\mathrm{inc}}(4)^{\mathrm{odd}}$ such that the curve X is the union of a smooth curve $\widetilde{X}$ of genus two and a projective line $\mathbb{P}^1$ which meet at two distinct points N_1 and N_2.*

Then the point Z is in the projective line $\mathbb{P}^1$. The restriction of ω on $\mathbb{P}^1$ vanishes everywhere. Either N_1 is a Weierstrass point of $\widetilde{X}$ and the restriction of ω to $\widetilde{X}$ is an holomorphic differential with a zero of order two at N_1 or the points N_1 and N_2 are conjugated by the hyperelliptic involution of $\widetilde{X}$ and the restriction of ω to $\widetilde{X}$ is a holomorphic differential with two simple zeros at N_1 and N_2.

The proofs of Theorem 7.7 and Theorem 7.8 are relatively similar. In particular, the main steps will be the following. The first one is to determine all the possible candidates as triples at the boundary. Then we show that we can smooth them using the plumbing cylinder construction of Section 3. The last step consists of determining the cases such that the smoothing occurs in the odd component and the ones where the smoothing occurs in the hyperelliptic one.

Proof of Theorem 7.7. Let (X, Z) be an irreducible marked curve of genus two. Then the pointed differentials (X, ω, Z) which could appear in the boundary of the stratum $\mathbb{P}\Omega\overline{\mathcal{M}}_{3,1}^{\mathrm{inc}}(4)$ are stable differentials ω with a zero of order 4 at Z and poles at the nodes of X.

Let us now suppose that Z is not a Weierstrass point of $\widetilde{X}$. We want to show that there exists a pair (N_1, N_2) on $\widetilde{X}$ such that $h^0(K_{\widetilde{X}} + N_1 + N_2 - 4Z) = 1$ and moreover that this pair is unique. Since Z is not a Weierstrass point of $\widetilde{X}$, the divisor $4Z - K_{\widetilde{X}}$ is not canonical. Indeed, this would be equivalent to the fact that $2(Z - \iota Z)$ is principal, where ι is the hyperelliptic involution. But this would give the existence of a function with a pole of order two at Z, contradicting the fact that Z is not a Weierstrass point. Now let us consider the locus E inside $\widetilde{X}^{(2)}$ consisting of pairs $(Q, \iota Q)$. Then the Jacobian $\mathcal{J}(\widetilde{X})$ of $\widetilde{X}$ is the quotient $\widetilde{X}^{(2)}/E$. And since $4Z - K_{\widetilde{X}}$ is not canonical, this implies that for each point $Z \notin \mathcal{WP}$ there is a unique pair (N_1, N_2) such that

$$\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}} + N_1 + N_2 - 4Z) = \mathcal{O}_{\widetilde{X}}.$$

It remains to show that the projection of the set of triples (N_1, N_2, Z) to the first coordinate is finite. Since $\widetilde{X}$ is a curve, it is enough to show that there are no pairs $(Q_1, Q_2) \in \widetilde{X}$ such that for an open set of $Q \in \widetilde{X}$ the equality $K_{\widetilde{X}} + Q_1 + Q_2 - 4Q \sim 0$ holds. But this is clearly the case, because the map of $\widetilde{X} \to \mathcal{J}(\widetilde{X})$ is nondegenerate and the pairs are never conjugated by the hyperelliptic involution.

Now using the plumbing cylinder construction of Theorem 3.15, we can smooth every of these differentials, preserving the zero of order four. Moreover, the curves that we obtain are clearly not hyperelliptic since the special fibre is not hyperelliptic.

Suppose now that Z is a Weierstrass point of $\widetilde{X}$. We have to show that every smoothing of such a curve which preserves the zero of order 4 is hyperelliptic. An analogous using the Riemann-Roch Theorem implies that the points N_1 and N_2 are conjugated by the hyperelliptic involution. But then the continuity of the parity of the generalised Arf invariant proved in Theorem 4.19 concludes the proof. □

We now prove Theorem 7.8 following a similar scheme.

Proof of Theorem 7.8. First we prove that it is necessary that the differentials are of the form given in Theorem 7.8.

It is clear that the point Z is on the bridge between N_1 and N_2 since otherwise (X, Z) would not be stable. Moreover, the points which form the node are conjugated by the hyperelliptic involution or one of them is a Weierstrass point. Otherwise, the differential would have a zero at a smooth point of $\widetilde{X}$. But this zero would be preserved by any deformation, contradicting the fact that the differential is in the boundary of $\mathbb{P}\Omega\mathcal{M}_{3,1}(4)$ and that $Z \notin \widetilde{X}$.

Let us suppose that we are in the first case: the restriction of ω to $\widetilde{X}$ has a zero of order two at N_1. Let us take a coordinate z on $\mathbb{P}^1$ such that 0 is identified to N_2 and ∞ to N_1. We define the differential form $\eta := \frac{(z-1)^4}{z^2}\mathrm{d}z$. We want to use the plumbing cylinder construction with parameters (ϵ_1, ϵ_2) at the nodes. By Lemma 3.11, they have to satisfy $\epsilon_1 = \epsilon_2^3 =: c$. We can find a differential η on $\widetilde{X}$ with simple poles at N_1 and N_2 and holomorphic otherwise. By Lemma 3.20, we can plumb the differential and obtain an holomorphic differential with a zero of order 4. Moreover, this differential is not hyperelliptic since the special fibre is not hyperelliptic. This proves the first point.

Let us now suppose that the differential has a single zero at both N_1 and N_2. We can still use Lemma 3.20 to plumb this differential. But this time, there are two distinct ways (up to isomorphisms) to plumb the nodes. Let ϵ_1 be the parameter of the cylinder at the node N_1, then according to Lemma 3.11, the parameter of the cylinder at N_2 has to be of the form $\epsilon_2 = \pm\epsilon_1$. To conclude the proof, it suffices to show that the case $\epsilon_1 = \epsilon_2$ leads to a hyperelliptic curve and that the case $\epsilon_1 = -\epsilon_2$ leads to a non hyperelliptic curve.

From now on, we will use the notations of Lemma 3.13. The hyperelliptic involution ι on X restricts to the hyperelliptic involution on $\widetilde{X}$ and to the involution which fixes Z and permutes N_1 and N_2 on the component $\mathbb{P}^1$. Hence we can suppose that the two open sets U_1 and U_2 and the coordinates z_1, w_1 on U_1 and z_2, w_2 on U_2 are chosen such that $\iota(z_i) = z_j$ and $\iota(w_i) = w_j$ for $i \neq j$.

Let us suppose that the cylinder plumbed at the node N_1 is given by the equation $x_1y_1 = \epsilon_1$ and at the node N_2 it is given by $x_2y_2 = \pm\epsilon_1$. Then on the cylinders, the hyperelliptic involution has to be of the form $\iota(x_1) = x_2$ and $\iota(y_1) = \pm y_2$ in order to coincide with the hyperelliptic involution on the part of the smoothed curve coming from $\widetilde{X}$. But it is easy to verify that this map can be prolonged to a holomorphic map on the whole smoothed curve if and only if the sign is positive. Moreover, in this case one can easily verify that this map is the hyperelliptic involution of the smoothed curve. And in the other case, the uniqueness of the hyperelliptic involution implies that the smoothed curve cannot be hyperelliptic. □

We can deduce from Theorem 7.8 the surprising fact that the odd and hyperelliptic components of the incidence variety compactifications of $\mathbb{P}\Omega\mathcal{M}_{3,1}(4)$ meet at their boundaries.

Corollary 7.10. *Let X be the union of a curve $\widetilde{X}$ of genus two and a projective line glue together at a pair of points of $\widetilde{X}$ conjugated by the hyperelliptic involution. Let $Z \in \mathbb{P}^1$ and ω be a differential which vanishes on $\mathbb{P}^1$ and has two single zeros at the points which form the nodes on $\widetilde{X}$.*

Then the pointed differential (X, ω, Z) is in $\Omega\overline{\mathcal{M}}_{3,1}^{\mathrm{inc}}(4)^{\mathrm{hyp}}$ and $\Omega\overline{\mathcal{M}}_{3,1}^{\mathrm{inc}}(4)^{\mathrm{odd}}$.

Examples. We give two examples of concrete families in $\Omega\mathcal{M}_{3,1}^{\mathrm{odd}}(4)$ which degenerates to a curve stably equivalent to an irreducible curve with one node. The first one is given as family of curves in $\mathbb{P}^2$ with a hyperflex. The second is a family of flat surfaces given as a family of polygons with identifications.

Example 7.11. We define in $\mathbb{P}^2 \times \Delta$ the family of curves given by:

$$P(x, y, z; t) := xyz^2 + y^4 + x^3z + tz^4.$$

Each curve has a hyperflex of order 4 at $(1, 0, 0; t)$, thus the differential corresponding to the line at infinity has a zero of order 4 at this point. The special curve is irreducible with only one node as singularity. Moreover the differential associated to the tangent has a simple pole at the node. Now the Weierstrass form of the normalisation is $y^2 + 4x^5 - 1$ and the preimages of the node are over $x = 0$ and $x = \infty$. In particular, the point which is over $x = \infty$ is a Weierstrass point. We can show that the Igusa invariant of this curve is zero.

More generally, let us consider the family

$$\{xyz^2 + y^4 + a_1x^3z + a_2x^2yz + a_3xy^2z + a_4y^3z + tz^4 = 0\} \subset \mathbb{P}^2 \times \{t\},$$

where the a_i, $i = 1, \cdots, 4$ are complex numbers. This gives us examples where the special curve has any given Igusa invariants.

Let us now take a look at the family given by the equation

$$P(x, y, z; t) := x^2yz + y^4 - x^3z + tz^4.$$

Moreover, the differential associated to the line at infinity has a zero of order 4 at $(1, 0, 0; t)$. The singularity of the special curve is a cusp meeting a smooth branch. It follows from the classification of Kang [Kan00, Corollary 2.5] and the fact that the family is smooth, that the stable limit of this family is an irreducible curve with one node. The limit of the zeros of order 4 is in the node. The limit stable differential has a zero of order two at one of the preimages of the node, which is also a Weierstrass point. In this example, the other preimage of the node is a Weierstrass point of the normalisation.

Let us now give examples using the polygonal representation of the flat surfaces. Since a complete classification of the cylinder decompositions of the flat surfaces in $\mathbb{P}\Omega\mathcal{M}_{3,1}(4)$ is given in [ANW13, Proposition 3.1], these examples could lead to another proof of Theorem 7.8 using degeneration of these diagrams.

Example 7.12. First we give in Figure 4 an example of a curve such that a zero of order two is identified with another point of the curve. In this figure and in the following one, the vertical segment are identified by an horizontal translation. In this example, it is not difficult to see that the second point which forms the node is a Weierstrass point of the curve. However, it is not difficult to construct examples where this point is not a Weierstrass point.

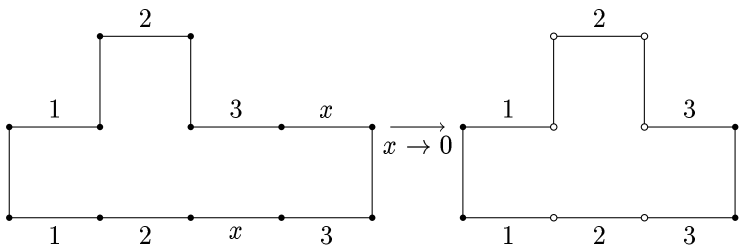


Figure 4: A family of curve in $\Omega\overline{\mathcal{M}}_3(4)$ degenerating to an irreducible curve with one of the points of the node a Weierstrass point.

More interesting is the case where the special curve is irreducible and the nodal points are conjugated by the hyperelliptic involution. In this case, we can produce a smoothing in both connected components of $\Omega\mathcal{M}_{3,1}(4)$. The Figure 5 shows such a smoothing. One can easily verify that the smoothing are in the correct stratum using the Arf invariant of these curves. A consequence of this is that the Arf Invariant of the nodal curve depends on the choice of a basis of the homology.

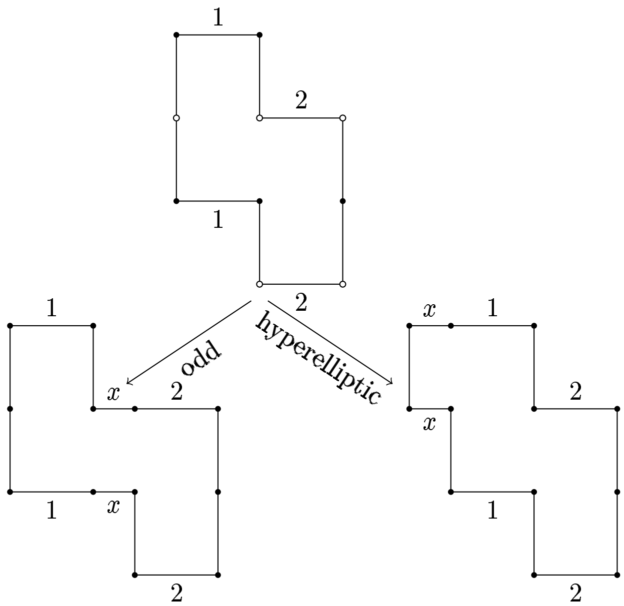


Figure 5: Two smoothings of an irreducible curve with a node of conjugated points, one of them in $\Omega\mathcal{M}_{3,1}^{\mathrm{odd}}(4)$ and the other in $\Omega\mathcal{M}_{3,1}^{\mathrm{hyp}}(4)$.

Acknowledgements.

This work consists of a part of my doctoral dissertation. I am grateful to Martin Möller for his guidance. I would like to thank Sam Grushevsky for helpful suggestions. Dawei Chen has informed the author that he has obtained some of the results in Sections 4 and 7 independently, the methods being in some case related, in some cases disjoint. During my PhD I have been supported by the ERC-StG 257137.

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